EXISTENCE OF POSITIVE SOLUTION FOR SECOND-ORDER IMPULSIVE BOUNDARY VALUE PROBLEMS ON INFINITY INTERVALS

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We deal with the existence of positive solutions to impulsive second-order differential equations subject to some boundary conditions on the semi-infinity interval.

1. Introduction

In recent years, impulsive differential equations have become a very active area of research and we refer the reader to the monographs [8] and the articles [6, 9, 10, 14, 15], where properties of their solutions are studied and extensive bibliographies are given. In consequence, it is very important to develop a complete basic theory of impulsive differential equations. Also, infinite interval problems have been extensive studied, see [1–5, 11, 12].

In this paper we study the existence of positive solutions for the following boundary value problem (BVP) with impulses:

$$\begin{gathered} y'' + g(t,y,y') = 0, \quad 0 < t < \infty,\ t \neq t_k, \\ \Delta y'(t_k) = b_k y'(t_k), \quad \Delta y(t_k) = a_k y(t_k), \quad k = 1,2,\dots, \\ y(0) = 0, \quad y \text{ bounded on } [0,\infty), \end{gathered} \tag{1.1}$$

where $t_k < t_{k+1}$, $\lim_{k\to\infty} t_k = \infty$, $\Delta y'(t_k) = y'(t_k^+) - y'(t_k^-)$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, and g is continuous except $\{t_k\} \times R \times R$; we assume that for $k \in \mathbb{N}^+ = \{1,2,\dots\}$ and $x, y \in \mathbb{R}$ there exist the limits

$$\lim_{t\to t_k^-} g(t,x,y) = g(t_k,x,y), \qquad \lim_{t\to t_k^+} g(t,x,y). \tag{1.2}$$

The problems of the above type without impulses have been discussed by several authors in the literature, we refer the reader to the pioneer works of Agarwal and O'Regan [1, 2, 4] and Ma [12] and Constantin [11]. But as far as we know the publication on solvability of infinity interval problems with impulses is fewer [15]. In this paper we want to

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fill in this gap and extend the existence results on the case of infinity interval problems with impulses.

Motivated by works of [2, 12], we use the well-known Leray-Schauder continuation theorem [13] to establish new results on finite intervals $[0,n]$ and use a diagonalization argument to get positive solutions on infinity intervals.

Let $J = [0,a]$, a is a constant or $a = +\infty$, in order to define the concept of solution for BVP (1.1), we introduce the following spaces of functions:

$PC(J) = \{u : J \to \mathbb{R},\ u$ is continuous at $t \neq t_k,\ u(t_k^+),\ u(t_k^-)$ exist, and $u(t_k^-) = u(t_k)\}$;

$PC^1(J) = \{u \in PC(J) : u$ is continuously differentiable at $t \neq t_k,\ u'(0^+),\ u'(t_k^+),\ u'(t_k^-)$ exist, and $u'(t_k^-) = u'(t_k)\}$;

$PC^2(J) = \{u \in PC^1(J) : u$ is twice continuously differentiable at $t \neq t_k\}$.

Note that $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|u\|_\infty = \sup\{|u(t)| : t \in J\}, \qquad \|u\|_1 = \max\{\|u\|_\infty, \|u'\|_\infty\}, \tag{1.3}$$

respectively.

Definition 1.1. By a positive solution of BVP (1.1), one means a function $y(t)$ satisfying the following conditions:

(i) $y \in PC^1[0,\infty)$;

(ii) $y(t) > 0$ for $t \in (0,\infty)$ and satisfies boundary condition $y(0) = 0$, y bounded on $[0,\infty)$;

(iii) $y(t)$ satisfies each equality of (1.1).

Definition 1.2. The set $\mathcal{F}$ is said to be quasi-equicontinuous in $[0,c]$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathcal{F}$, $k \in \mathbb{Z}$, $t^*, t^{**} \in (t_{k-1}, t_k] \cap [0,c]$, and $|t^* - t^{**}| < \delta$, then $|x(t^*) - x(t^{**})| < \varepsilon$.

Lemma 1.3 (compactness criterion [8]). *The set $\mathcal{F} \subset PC([0,c], R^n)$ is relatively compact if and only if*

(1) *$\mathcal{F}$ is bounded;*

(2) *$\mathcal{F}$ is quasi-equicontinuous in $[0,c]$.*

2. Main results

Theorem 2.1. *Let $g : [0,\infty) \times [0, \in b = 0$, L^{-1} exist and is continuous.*

On the other hand, solving (8) is equivalent to finding a fixed point of

$$L^{-1}Ni : PC(I) \longrightarrow PC(I) \tag{2.1}$$

with $i : PC^1(I) \to PC(I)$ the compact inclusion of $PC^1(I)$ in $PC(I)$. Now, Schauder's fixed point theorem guarantees the existence of at least a fixed point since $L^{-1}Ni$ is continuous and compact.

Next, prove that every solution u of (8) satisfies

$$\alpha(t) \le u(t) \le \beta(t) \quad on\ I. \tag{2.2}$$

By the definition of $p(t,x)$, $\infty)\times[0,\infty)\to[0,\infty)$. *Assume that the following hypothesis hold.*

(A_1) *For any constant* $H>0$, *there exists a function* ψ_H *continuous on* $[0,\infty)$ *and positive on* $(0,\infty)$, *and a constant* γ, $0\le\gamma<1$, *with* $g(t,u,v)\ge\psi_H(t)v^\gamma$ *on* $[0,\infty)\times[0,H]^2$.

(A_2) *There exist functions* $p,r:[0,\infty)\to[0,\infty)$ *such that*

$$\begin{gathered} g(t,u,v)\le p(t)v+r(t)\quad \text{on } [0,\infty)\times[0,\infty)^2, \\ P_1=\int_0^\infty sp(s)ds<\infty,\qquad R_1=\int_0^\infty sr(s)ds<\infty, \\ P=\int_0^\infty p(s)ds<1,\qquad R=\int_0^\infty r(s)ds<\infty. \end{gathered}\tag{2.3}$$

(A_3) $b_k\ge 0$, $a_k\ge -1$ *and* $\sum_{k=1}^\infty |a_k|\le A<1$.
Then BVP (1.1) has at least one solution.

To prove Theorem 2.1, we need the following preliminary lemmas.

Lemma 2.2. *Let* $e(t)\in\mathbb{C}[0,\infty)$, $e(t)\ge 0$, $b_k\ge 0$, $x\in PC^1[0,\infty)\cap PC^2[0,\infty)$ *be such that*

$$\begin{gathered} x''(t)+e(t)=0,\quad t\in(0,b),\ t\ne t_k, \\ \Delta x'(t_k)=b_kx'(t_k), \end{gathered}\tag{2.4}$$

and $x(0)=0$, $x'(b)=0$. *Then*

$$\|x'\|_\infty\le\int_0^b e(s)ds.\tag{2.5}$$

Proof. Since $-x''(t)=e(t)$, $x'(b)=0$, then $x'(t)\ge 0$. Integrating from t to b we obtain

$$x'(t)=\int_t^b e(s)ds-\sum_{t<t_k<b}b_kx'(t_k)\le\int_t^b e(s)ds\le\int_0^b e(s)ds.\tag{2.6}$$

□

Lemma 2.3. *Let* $g:[0,\infty)\times[0,\infty)\times[0,\infty)\to[0,\infty)$ *and conditions* (A_1)–(A_3) *hold. Let* n *be a positive integer and consider the boundary value problem*

$$\begin{gathered} y''+g(t,y,y')=0,\quad 0<t<n,\ t\ne t_k, \\ \Delta y'(t_k)=b_ky'(t_k),\qquad \Delta y(t_k)=a_ky(t_k), \\ y(0)=0,\qquad y'(n)=0. \end{gathered}\tag{2.2$_n$}$$

Then (2.2_n) *has at least one positive solution* $y_n\in PC^1[0,n]$ *and there is a constant* $M>0$

independent of n such that

$$\left((1-\gamma)\int_t^n \prod_{t<t_k<s}(1+b_k)^{\gamma-1}\psi_M(s)ds\right)^{1/(1-\gamma)} \le y_n'(t) \le M, \quad t\in[0,n], \tag{2.7}$$

$$\int_0^t \prod_{s<t_k<t}(1+a_k)\left((1-\gamma)\int_s^n \prod_{s<t_k<\tau}(1+b_k)^{\gamma-1}\psi_M(\tau)d\tau\right)^{1/(1-\gamma)} ds \le y_n(t) \le M, \quad t\in[0,n]. \tag{2.8}$$

Proof. Let $n\in\mathbb{N}^+$ be fixed and $Y=X=PC^1[0,n]$. We first show that

$$\begin{gathered} y''+g^*(t,y,y')=0, \quad 0<t<n,\ t\neq t_k, \\ \Delta y'(t_k)=b_k y'(t_k), \qquad \Delta y(t_k)=a_k y(t_k), \\ y(0)=0, \qquad y'(n)=0 \end{gathered} \tag{2.9}$$

has at least one solution, here

$$g^*(t,y,v)=\begin{cases} g(t,y,v), & y\ge 0,\ v\ge 0,\\ g(t,y,0), & y\ge 0,\ v<0,\\ g(t,0,v), & y<0,\ v\ge 0,\\ g(t,0,0), & y<0,\ v<0. \end{cases} \tag{2.10}$$

Define a linear operator $L_n : D(L_n)\subset X\to Y$ by setting

$$D(L_n)=\{x\in PC^2[0,n] : x(0)=x'(n)=0\}, \tag{2.11}$$

and for $y\in D(L_n) : L_n y=(-y'',\Delta y'(t_k),\Delta y(t_k))$. We also define a nonlinear mapping $F : X\to Y$ by setting

$$(Fy)(t)=(g^*(t,y(t),y'(t)),b_k y'(t_k),a_k y(t_k)). \tag{2.12}$$

From the assumption of g, we see that F is a bounded mapping from X to Y. Next, it is easy to see that $L_n : D(L_n)\to Y$ is one-to-one mapping. Moreover, it follows easily using Lemma 1.3 that $(L_n)^{-1}F : X\to X$ is a compact mapping.

We note that $y\in PC^1[0,n]$ is a solution of (2.9) if and only if y is a fixed point of the equation

$$y=(L_n)^{-1}Fy. \tag{2.13}$$

We apply the Leray-Schauder continuation theorem to obtain the existence of a solution for $y=(L_n)^{-1}Fy$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$y'' + \lambda g^*(t, y, y') = 0, \quad 0 < t < n,\ t \neq t_k,$$
$$\Delta y'(t_k) = \lambda b_k y'(t_k), \qquad \Delta y(t_k) = \lambda a_k y(t_k), \tag{2.5$_\lambda$}$$
$$y(0) = y'(n) = 0$$

is a prior bounded in $PC^1[0,n]$ by a constant independent of $0 < \lambda < 1$.

Let $y \in PC^1[0,n]$ be any solutions of (2.5$_\lambda$), then $y' \geq 0$ and $y \geq 0$ on $[0,n]$. Applying Lemma 2.2 and using (2.5$_\lambda$), we can get that

$$y'(t) \leq \int_0^n g^*(s, y(s), y'(s))\,ds \leq \int_0^n p(s)y'(s)\,ds + \int_0^n r(s)\,ds \leq P\|y'\|_\infty + R, \tag{2.14}$$

so

$$\|y'\|_\infty \leq \frac{R}{1-P} := M_1. \tag{2.15}$$

From (2.5$_\lambda$) and $b_k \geq 0$, we have

$$y'(t) = \lambda \int_t^n g^*(s, y(s), y'(s))\,ds - \lambda \sum_{t<t_n<n} b_k y'(t_k) \leq \int_t^n g^*(s, y(s), y'(s))\,ds. \tag{2.16}$$

Integrate (2.16) from 0 to t to obtain

$$\begin{aligned}
y(t) &\leq t\int_t^n g^*(s, y(s), y'(s))\,ds + \int_0^t s g^*(s, y(s), y'(s))\,ds + \lambda \sum_{0<t_k<t} \Delta y(t_k) \\
&\leq \int_t^n s g^*(s, y(s), y'(s))\,ds + \int_0^t s g^*(s, y(s), y'(s))\,ds + \lambda \sum_{0<t_k<t} a_k y(t_k) \\
&\leq \|y'\|_\infty \int_0^n s p(s)\,ds + \int_0^n s r(s)\,ds + \|y\|_\infty \sum_{0<t_k<t} |a_k| \\
&\leq P_1 M_1 + R_1 + A\|y\|_\infty.
\end{aligned} \tag{2.17}$$

Hence we have

$$\|y\|_\infty \leq \frac{PM_1 + R_1}{1-A} := M_2. \tag{2.18}$$

Let

$$M = \max\{M_1, M_2\}, \tag{2.19}$$

it follows that

$$\|y\|_1 \le M. \tag{2.20}$$

Note that M is independent of λ.

Therefore (2.20) implies that (2.5_λ) has a solution y_n with $\|y_n\|_1 \le M$. In fact,

$$0 \le y_n(t) \le M, \quad 0 \le y_n'(t) \le M \quad \text{for } t \in [0,n], \tag{2.21}$$

and y_n satisfies (2.2_n).

Finally, it is easy to see from (2.19) that M is independent of $n \in \mathbb{N}^+$. Now (A_1) guarantees the existence of a function $\psi_M(t)$ continuous on $[0,\infty)$ and positive on $(0,\infty)$, a constant $\gamma \in [0,1)$, with $g(t,y_n(t),y_n'(t)) \ge \psi_M(t)(y_n'(t))^\gamma$ for $(t,y_n(t),y_n'(t)) \in [0,n] \times [0,M]^2$.

From (2.2_n) we have

$$-y_n''(t) \ge \psi_M(t)\left(y_n'(t)\right)^\gamma, \tag{2.22}$$

integrate the above inequality from t to n to obtain

$$y_n'(t) \ge \left((1-\gamma)\int_t^n \prod_{t<t_k<s}(1+b_k)^{\gamma-1}\psi_M(s)\,ds\right)^{1/(1-\gamma)}, \quad t \in [0,n], \tag{2.23}$$

and so

$$y_n(t) \ge \int_0^t \prod_{s<t_k<t}(1+a_k)\left((1-\gamma)\int_s^n \prod_{s<t_k<\tau}(1+b_k)^{\gamma-1}\psi_M(\tau)\,d\tau\right)^{1/(1-\gamma)} ds, \quad t \in [0,n], \tag{2.24}$$

which completes the proof. □

Proof of Theorem 2.1. From (2.2_n) and (2.21), we know that

$$0 \le -y_n'' \le \phi(t), \quad t \in [0,n], \tag{2.25}$$

where $\phi(t) := p(t)M + r(t)$, and M is given by (2.19). In addition, we have by $b_k \ge 0$ that

$$y_n'(t) \le \int_t^n \phi(s)\,ds \le \int_t^\infty \phi(s)\,ds \quad \text{for } t \in [0,n]. \tag{2.26}$$

To show that BVP (1.1) has a solution, we will apply the diagonalization argument. Let

$$u_n(t) = \begin{cases} y_n(t), & t \in [0,n], \\ y_n(n), & t \in [n,\infty). \end{cases} \tag{2.27}$$

Notice that $u_n \in PC^1[0,\infty)$ with

$$0 \le u_n(t) \le M, \quad 0 \le u_n'(t) \le M \quad \text{for } t \in [0,\infty). \tag{2.28}$$

From the definition of u_n, we get for $s_1, s_2 \in (t_k, t_{k+1}]$ that

$$|u_n'(s_1) - u_n'(s_2)| \le \left| \int_{s_1}^{s_2} \phi(s)ds \right|. \tag{2.29}$$

In addition

$$u_n'(t) \le \int_t^\infty \phi(s)ds \quad \text{for } t \in [0,\infty), \tag{2.30}$$

$$u_n(t) \ge \int_0^t \prod_{s<t_k<t} (1+a_k) \left((1-\gamma) \int_s^n \prod_{s<t_k<\tau} (1+b_k)^{\gamma-1} \psi_M(\tau) d\tau \right)^{1/(1-\gamma)} ds, \quad t \in [0,n]. \tag{2.31}$$

In particular

$$\begin{aligned} u_n(t) &\ge \int_0^t \prod_{s<t_k<t} (1+a_k) \left((1-\gamma) \int_s^1 \prod_{s<t_k<\tau} (1+b_k)^{\gamma-1} \psi_M(\tau) d\tau \right)^{1/(1-\gamma)} ds \\ &\equiv a_1(t), \quad t \in [0,1]. \end{aligned} \tag{2.32}$$

Lemma 1.3 guarantees the existence of a subsequence N_1 of $\mathbb{N}^+$ and a function $z_1 \in PC^1[0,1]$ with $u_n^{(j)}$ converging uniformly on $[0,1]$ to $z_1^{(j)}$ as $n \to \infty$ through N_1, here $j = 0,1$. Also from (2.32), $z_1(t) \ge a_1(t)$ for $t \in [0,1]$ (in particular, $z_1 > 0$ on (0, 1]).

Let $N_1^+ = N_1 \backslash \{1\}$, notice from (2.31) that

$$\begin{aligned} u_n(t) &\ge \int_0^t \prod_{s<t_k<t} (1+a_k) \left((1-\gamma) \int_s^2 \prod_{s<t_k<\tau} (1+b_k)^{\gamma-1} \psi_M(\tau) d\tau \right)^{1/(1-\gamma)} ds \\ &\equiv a_2(t), \quad t \in [0,2]. \end{aligned} \tag{2.33}$$

Lemma 1.3 guarantees the existence of a subsequence N_2 of N_1^+ and a function $z_2 \in PC^1[0,2]$ with $u_n^{(j)}$ converging uniformly on $[0,2]$ to $z_2^{(j)}$ as $n \to \infty$ through N_2, here $j = 0,1$. Also from (2.41), $z_2(t) \ge a_2(t)$ for $t \in [0,2]$ (in particular, $z_2 > 0$ on $(0,2]$). Note that $z_2 = z_1$ on [0, 1], since $N_2 \subset N_1^+$. Let $N_2^+ = N_2 \backslash \{2\}$, proceed inductively to obtain for $k = 1,2,\ldots$, a subsequence N_k of N_{k-1}^+ and a function $z_k \in PC^1[0,k]$ with $u_n^{(j)}$ converging uniformly on $[0,k]$ to $z_k^{(j)}$ as $n \to \infty$ through N_k, here $j = 0,1$. Also

$$\begin{aligned} z_k(t) &\ge a_k(t) \\ &\equiv \int_0^t \prod_{s<t_k<t} (1+a_k) \left((1-\gamma) \int_s^k \prod_{s<t_k<\tau} (1+b_k)^{\gamma-1} \psi_M(\tau) d\tau \right)^{1/(1-\gamma)} ds, \quad t \in [0,k] \end{aligned} \tag{2.34}$$

(so in particular, $z_k > 0$ on $(0,k]$). Note that $z_k = z_{k-1}$ on $[0,k-1]$.

Define a function y as follows: fix $t \in (0,\infty)$ and let $k \in N^+$ with $t < k$. Define $y(t) = z_k(t)$. Note that y is well defined and $y(t) = z_k(t) > 0$, we can do this for each $t \in (0,\infty)$ and so $y \in PC^1[0,\infty)$. In addition, $0 \le y(t) \le M$, $0 \le y'(t) \le M$, and

$$y'(t) \le \int_t^{\infty} \phi(s)ds \quad \text{for } t \in [0,\infty). \tag{2.35}$$

Fix $x \in [0,\infty)$ and choose $k \ge x$, $k \in N^+$. Then for each $n \in N_k^+ = N_k \setminus \{k\}$, we have

$$\begin{aligned} y_n(x) = y_n'(k)x + \int_0^x \int_s^k g(\tau, y_n(\tau), y_n'(\tau))d\tau\, ds - \sum_{0<t_i<k} b_i y_n'(t_i)x \\ + \sum_{0<t_i\le x} b_i y_n'(t_i)(x-t_i) + \sum_{0<t_i<x} a_i y_n(t_i). \end{aligned} \tag{2.36}$$

Let $n \to \infty$ through N_k^+ to obtain

$$\begin{aligned} z_k(x) = z_k'(k)x + \int_0^x \int_s^k g(\tau, z_k(\tau), z_k'(\tau))d\tau\, ds \\ - \sum_{0<t_i<k} b_i z_k'(t_i)x + \sum_{0<t_i\le x} b_i z_k'(t_i)(x-t_i) + \sum_{0<t_i<x} a_i z_k(t_i). \end{aligned} \tag{2.37}$$

Thus

$$\begin{aligned} y(x) = y'(k)x + \int_0^x \int_s^k g(\tau, y(\tau), y'(\tau))d\tau\, ds \\ - \sum_{0<t_i<k} b_i y'(t_i)x + \sum_{0<t_i\le x} b_i y'(t_i)(x-t_i) + \sum_{0<t_i<x} a_i y(t_i). \end{aligned} \tag{2.38}$$

Consequently $y \in PC^2(0,\infty)$ with

$$\begin{aligned} y''(t) + g(t, y(t), y'(t)) = 0, \quad 0 < t < \infty,\ t \ne t_k, \\ \Delta y'(t_k) = b_k y'(t_k), \qquad \Delta y(t_k) = a_k y(t_k). \end{aligned} \tag{2.39}$$

Thus y is a solution of (1.1) with $y > 0$ on $(0,\infty)$. The proof is complete. □

THEOREM 2.4. *Let* $g : [0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$. *Assume that* ($A_1$), ($A_3$) *of Theorem 2.1 and the following condition hold.*

(B_1) $g(t,x,v) \le q(t)w(\max\{x,v\})$ *on* $[0,\infty)\times[0,\infty)\times[0,\infty)$ *with* $w > 0$ *continuous and nondecreasing on* $[0,\infty)$, $q(t) \in \mathbb{C}[0,\infty)$.

(B_2)

$$\begin{aligned} Q = \int_0^{\infty} q(s)ds < \infty, \qquad Q_1 = \int_0^{\infty} sq(s)ds < \infty, \\ \sup_{c\ge 0} \frac{c}{w(c)} > T = \max\left\{\frac{Q_1}{1-A}, Q\right\}. \end{aligned} \tag{2.40}$$

Then BVP (1.1) has at least one positive solution.

Proof. Choose $M > 0$ with

$$\frac{M}{w(M)} > T. \tag{2.41}$$

We first show that (2.9) has at least one solution. To the end, we consider the operator

$$y = \lambda (L_n)^{-1} F y, \quad \lambda \in (0,1), \tag{2.42}$$

which is equivalent to (2.5_λ). Let $y \in PC^1[0,n]$ be any solution of (2.5_λ), then $y \geq 0$, $y' \geq 0$ on $[0,n]$. From (B_1) we have

$$-y''(t) \leq q(t) w(\|y\|_1) \quad \text{for } t \in [0,n]. \tag{2.43}$$

Integrate (2.43) from t to n to obtain

$$y'(t) \leq w(\|y\|_1) \int_t^n q(s)ds - \sum_{t<t_k<n} b_k y'(t_k) \leq w(\|y\|_1) \int_t^n q(s)ds \tag{2.44}$$

so

$$y'(t) \leq Q w(\|y\|_1). \tag{2.45}$$

Integrate (2.44) from 0 to t to obtain

$$y(t) \leq w(\|y\|_1) \int_0^t \int_s^n q(\tau) d\tau\, ds + \sum_{0<t_k<t} a_k y(t_k) \leq w(\|y\|_1) \int_0^t s q(s) ds + A \|y\|_\infty. \tag{2.46}$$

Combine (2.45) and (2.46) to find

$$\|y\|_1 \leq T w(\|y\|_1). \tag{2.47}$$

Now (2.41) together with (2.47) implies $\|y\|_1 \neq M$. Set

$$U = \{u \in PC^1[0,n] : \|u\|_1 < M\}, \qquad K = E = PC^1[0,n]. \tag{2.48}$$

Now the nonlinear alternative of Leray-Schauder type [7] guarantees that $(L_n)^{-1}N$ has a fixed point, that is, (2.9) has a solution $y_n \in PC^1[0,n]$, and

$$0 \leq y_n \leq M, \qquad 0 \leq y_n' \leq M. \tag{2.49}$$

The other proof is similar to the proof of Theorem 2.1, here we omit it. □

3. Examples

Example 3.1. Consider the boundary value problem

$$\begin{gathered} y'' + \eta(y')^{\beta} e^{-t} + \mu e^{-t} = 0, \quad 0 < t < \infty, \\ \Delta y'(t_k) = \frac{1}{k} y'(t_k), \quad \Delta y(t_k) = \frac{2}{3k(k+1)} y(t_k), \quad k = 1, 2, \dots, \\ y(0) = 0, \quad y \text{ bounded on } [0,\infty) \end{gathered} \tag{3.1}$$

with $\beta \in [0,1)$, $\eta \in (0,1)$, $\mu > 0$. Set $g(t,u,v) = \eta e^{-t}(y')^{\beta} + \mu e^{-t}$. Take $p(t) = \eta e^{-t}$, $r(t) = \mu e^{-t}$, then g satisfies (A_2) and $P = \eta < 1$. For each $H > 0$, take $\psi_H(t) = \eta e^{-t}$ and $\gamma = \beta$, then (A_1) is satisfied. Furthermore,

$$b_k = \frac{1}{k} > 0, \qquad \sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{2}{3k(k+1)} = \frac{2}{3} < 1. \tag{3.2}$$

Therefore, Theorem 2.1 now guarantees that (3.1) has a solution $y \in PC^1[0,\infty)$ with $y > 0$ on $(0,\infty)$.

Example 3.2. Consider the boundary value problem

$$\begin{gathered} y'' + (y^{\alpha} + (y')^{\beta}) e^{-t} + \mu e^{-t} = 0, \quad 0 < t < \infty, \\ \Delta y'(t_k) = y'(t_k), \quad \Delta y(t_k) = \frac{1}{(k+1)^2} y(t_k), \quad k = 1, 2, \dots, \\ y(0) = 0, \quad y \text{ bounded on } [0,\infty) \end{gathered} \tag{3.3}$$

with $\alpha \in [0,1)$, $\beta \in [0,1)$, $\mu > 0$. We will apply Theorem 2.4 with $q(t) = e^{-t}$, $w(s) = s^{\alpha} + s^{\beta} + \mu$. Clearly (A_1), (A_3), and (B_1) hold. Also,

$$\sup_{c \geq 0} \frac{c}{w(c)} = \sup_{c \geq 0} \frac{c}{c^{\alpha} + c^{\beta} + \mu} = \infty, \tag{3.4}$$

so (B_2) is true. Theorem 2.4 shows that (3.3) has a solution $y \in PC^1[0,\infty)$ with $y > 0$ on $(0,\infty)$.

Remark 3.3. We cannot apply the results of [12] even if (3.3) has no impulses, since [12, condition (2.3) of Theorem 2.1] is not satisfied.

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