

# UNIQUENESS OF SOLUTIONS FOR FOURTH-ORDER NONLOCAL BOUNDARY VALUE PROBLEMS

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Uniqueness implies uniqueness relationships are examined among solutions of the fourth-order ordinary differential equation,  $y^{(4)} = f(x, y, y', y'', y''')$ , satisfying 5-point, 4-point, and 3-point nonlocal boundary conditions.

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## 1. Introduction

We are concerned with uniqueness of solutions of certain nonlocal boundary value problems for the fourth-order ordinary differential equation,

$$y^{(4)} = f(x, y, y', y'', y'''), \quad a < x < b, \quad (1.1)$$

where

(A)  $f : (a, b) \times \mathbb{R}^4 \rightarrow \mathbb{R}$  is continuous,

(B) solutions of initial value problems for (1.1) are unique and exist on all of  $(a, b)$ .

By uniqueness of solutions, our meaning is uniqueness of solutions, when solutions exist.

In particular, we deal with “uniqueness implies uniqueness” relationships among solutions of (1.1) satisfying nonlocal 5-point boundary conditions,

$$\begin{aligned} y(x_1) &= y_1, & y(x_2) &= y_2, \\ y(x_3) &= y_3, & y(x_4) - y(x_5) &= y_4, \end{aligned} \quad (1.2)$$

$$\begin{aligned} y(x_1) - y(x_2) &= y_1, & y(x_3) &= y_2, \\ y(x_4) &= y_3, & y(x_5) &= y_4, \end{aligned} \quad (1.3)$$

where  $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ , with solutions of (1.1) satisfying nonlocal 4-point

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boundary conditions given by

$$y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y(x_2) = y_3, \quad y(x_3) - y(x_4) = y_4, \quad (1.4)$$

$$y(x_1) - y(x_2) = y_1, \quad y(x_3) = y_2, \quad y(x_4) = y_3, \quad y'(x_4) = y_4, \quad (1.5)$$

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_2) = y_3, \quad y(x_3) - y(x_4) = y_4, \quad (1.6)$$

$$y(x_1) - y(x_2) = y_1, \quad y(x_3) = y_2, \quad y'(x_3) = y_3, \quad y(x_4) = y_4, \quad (1.7)$$

where  $a < x_1 < x_2 < x_3 < x_4 < b$ , as well as with solutions of (1.1) satisfying nonlocal 3-point boundary conditions given by

$$y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y''(x_1) = y_3, \quad y(x_2) - y(x_3) = y_4, \quad (1.8)$$

$$y(x_1) - y(x_2) = y_1, \quad y(x_3) = y_2, \quad y'(x_3) = y_3, \quad y''(x_3) = y_4, \quad (1.9)$$

where  $a < x_1 < x_2 < x_3 < b$ , and in each case  $y_1, y_2, y_3, y_4 \in \mathbb{R}$ .

Questions involving “uniqueness implies uniqueness” for solutions of boundary value problems for ordinary differential equations enjoy some history. Jackson’s monumental works [20, 21] dealt with this question for solutions of  $k$ -point conjugate boundary value problems for  $n$ th-order ordinary differential equations. Later, Henderson [12] dealt with this question for  $k$ -point right focal boundary value problems for  $n$ th-order ordinary differential equations. Other uniqueness implies uniqueness results are found in the papers by Clark and Henderson [2], Ehme and Hankerson [4], Henderson and McGwier [17], and Peterson [39].

The questions in this paper involve (i) whether uniqueness of solutions of (1.1), (1.2) implies uniqueness of solutions of (1.1), (1. $j$ ),  $j = 4, 6, 8$ , and (ii) whether uniqueness of solutions of (1.1), (1. $j$ ),  $j = 4, \dots, 9$ , imply uniqueness of solutions (1.1), (1.2) and (1.1), (1.3). A principal reason for considering questions such as (i) or (ii) is that such results often imply the existence of solutions for boundary value problems; see for example [1, 9–11, 13–15, 17, 18, 22, 24, 26, 27].

The literature is vast on fourth-order nonlinear boundary value problems, and we cite [3, 5, 23, 28–30, 33, 35, 36, 38, 40] as a list for just a few of these papers dealing with both theoretical issues as well as application models. In addition, nonlocal boundary value problems have received a good deal of research attention. For a brief overview of some research devoted to nonlocal boundary value problems, we suggest the list of papers [6–8, 16, 19, 25, 31, 32, 34, 37, 43, 44].

The motivation for this paper is two-fold. First, it would be the work by Peterson [39] in which he showed that, for the fourth-order equation (1.1), uniqueness of solutions of 4-point “conjugate” boundary value problems is equivalent to uniqueness of both 2-point and 3-point “conjugate” boundary value problems. Second, it would be a recent paper by Clark and Henderson [2] in which they established for “third-order” differential equations, uniqueness of solutions of 4-point nonlocal boundary value problems is equivalent to uniqueness of solutions of both 2-point and 3-point nonlocal boundary value problems.

## 2. Uniqueness results for conjugate problems

In this section, we will state some of the motivational uniqueness results due to Peterson [39] for conjugate boundary value problems for (1.1). In particular, Peterson dealt with relationships among boundary value problems for (1.1) satisfying 4-point conjugate boundary conditions of the form

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3, \quad y(x_4) = y_4, \quad (2.1)$$

$a < x_1 < x_2 < x_3 < x_4 < b$ , along with solutions of (1.1) satisfying 3-point conjugate boundary value problems of the form

$$\begin{aligned} y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y(x_2) = y_3, \quad y(x_3) = y_4, \\ y(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_2) = y_3, \quad y'(x_3) = y_4, \\ y(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3, \quad y'(x_3) = y_4, \end{aligned} \quad (2.2)$$

$a < x_1 < x_2 < x_3 < b$ , as well as with solutions of (1.1) satisfying 2-point conjugate boundary value problems of the form

$$\begin{aligned} y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y''(x_1) = y_3, \quad y(x_2) = y_4, \\ y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y(x_2) = y_3, \quad y'(x_2) = y_4, \\ y(x_1) = y_1, \quad y(x_2) = y_2, \quad y'(x_2) = y_3, \quad y''(x_2) = y_4, \end{aligned} \quad (2.3)$$

$a < x_1 < x_2 < b$ , and in each case  $y_1, y_2, y_3, y_4 \in \mathbb{R}$ .

A major part of Peterson's work dealt with establishing the next result.

**THEOREM 2.1.** *Assume conditions (A) and (B) are satisfied. Let  $k_0 \in \{2, 3, 4\}$  be given, and assume that solutions of  $k_0$ -point conjugate boundary value problems for (1.1) are unique on  $(a, b)$ . Then, for each  $k \in \{2, 3, 4\} \setminus \{k_0\}$ , solutions of  $k$ -point conjugate boundary value problems for (1.1) are unique on  $(a, b)$ .*

It follows, in turn, from a “uniqueness implies existence” result of Hartman [10] and Klaasen [24] for conjugate boundary value problems that, under the hypotheses of Theorem 2.1, solutions of conjugate boundary value problems for (1.1) actually exist.

**THEOREM 2.2.** *Assume the hypotheses of Theorem 2.1. Then for  $k \in \{2, 3, 4\}$ , each  $k$ -point conjugate boundary value problem for (1.1) has a unique solution on  $(a, b)$ .*

## 3. Uniqueness of 5-point implies uniqueness of 4-point and 3-point

In this section, we show that uniqueness of solutions of 5-point nonlocal boundary value problems for (1.1) implies uniqueness of solutions for both 4-point and 3-point nonlocal boundary value problems. In addition to hypotheses (A) and (B), we will draw upon some uniqueness conditions for the 5-point nonlocal problems (1.1), (1.2) and (1.1), (1.3).

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(C) Given  $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ , if  $y(x)$  and  $z(x)$  are two solutions of (1.1) satisfying

$$\begin{aligned} y(x_1) &= z(x_1), & y(x_2) &= z(x_2), & y(x_3) &= z(x_3), \\ y(x_4) - y(x_5) &= z(x_4) - z(x_5), \end{aligned} \quad (3.1)$$

then  $y(x) = z(x)$ ,  $a < x < b$ .

(D) Given  $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ , if  $y(x)$  and  $z(x)$  are two solutions of (1.1) satisfying

$$\begin{aligned} y(x_1) - y(x_2) &= z(x_1) - z(x_2), & y(x_3) &= z(x_3), \\ y(x_4) &= z(x_4), & y(x_5) &= z(x_5), \end{aligned} \quad (3.2)$$

then  $y(x) = z(x)$ ,  $a < x < b$ .

*Remarks 3.1.* (a) We note that, under either assumption (C) or (D), solutions of 4-point “conjugate” boundary value problems for (1.1) are unique, when they exist. That is, if  $y(x)$  and  $z(x)$  are both solutions of (1.1) such that, for some points  $a < t_1 < t_2 < t_3 < t_4 < b$ ,  $y(t_i) = z(t_i)$ ,  $i = 1, 2, 3, 4$ , then by the intermediate value theorem, there exist  $t_1 < \tau_1 < \tau_2 < t_2 < t_3 < \sigma_1 < \sigma_2 < t_4$  such that, both  $y(\tau_1) - y(\tau_2) = z(\tau_1) - z(\tau_2)$ ,  $y(t_i) = z(t_i)$ ,  $i = 2, 3, 4$ , and  $y(t_i) = z(t_i)$ ,  $i = 1, 2, 3$ ,  $y(\sigma_1) - y(\sigma_2) = z(\sigma_1) - z(\sigma_2)$ . Namely, if either (C) or (D) holds, then  $y(x) = z(x)$ .

(b) As a consequence, if either (A), (B), and (C), or (A), (B), and (D) are assumed, then Theorem 2.2 implies that each  $k$ -point “conjugate” boundary value problem for (1.1),  $k = 2, 3, 4$ , has a unique solution.

Behind the uniqueness results of this section is the role of continuous dependence of solutions on boundary conditions. This continuous dependence arises somewhat from applications of the Brouwer theorem on invariance of domain [41] in conjunction with continuous dependence of solutions on initial conditions. We present our first such continuous dependence result. The proof is rather standard in the context of uniqueness properties on solutions with respect to both initial conditions and boundary conditions. So we will omit the details of the proof, but we suggest [2, 21] as good references for typical arguments used in the proof.

**THEOREM 3.2.** *Assume (A), (B), and (C), and let  $z(x)$  be an arbitrary solution of (1.1). Then, for any  $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$  and  $a < c < x_1$ , and  $x_5 < d < b$ , and given any  $\epsilon > 0$ , there exists  $\delta(\epsilon, [c, d]) > 0$ , so that  $|x_i - t_i| < \delta$ ,  $1 \leq i \leq 5$ ,  $|z(x_i) - y_i| < \delta$ ,  $i = 1, 2, 3$ , and  $|z(x_4) - z(x_5) - y_4| < \delta$  imply that (1.1) has a solution  $y(x)$  with*

$$\begin{aligned} y(t_i) &= y_i, & i &= 1, 2, 3, \\ y(t_4) - y(t_5) &= y_4, \end{aligned} \quad (3.3)$$

and  $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$  on  $[c, d]$ ,  $i = 1, 2, 3, 4$ .

We now proceed to establish a sequence of theorems exhibiting that uniqueness of solutions of (1.1), (1.2) implies uniqueness of solutions of (1.1), (1.j),  $j = 4, 6, 8$ .

**THEOREM 3.3.** *Assume (A), (B), and (C) are satisfied. Then solutions of (1.1), (1.4) are unique when they exist.*

*Proof.* Suppose (1.1), (1.4) has two solutions  $y(x)$  and  $z(x)$ , and let us say

$$\begin{aligned} z(x_1) &= y(x_1), & z'(x_1) &= y'(x_1), \\ z(x_2) &= y(x_2), & z(x_3) - z(x_4) &= y(x_3) - y(x_4), \end{aligned} \tag{3.4}$$

for some  $a < x_1 < x_2 < x_3 < x_4 < b$ . By uniqueness of 2-point conjugate boundary value problems for (1.1),  $z''(x_1) \neq y''(x_1)$  and  $z'(x_2) \neq y'(x_2)$ .

Without loss of generality, we assume  $y(x) > z(x)$  on  $(a, x_2) \setminus \{x_1\}$ . Then  $y(x) < z(x)$  on  $(x_2, b)$ . Fix  $a < \tau < x_1$ . By Theorem 3.2, for  $\epsilon > 0$  sufficiently small, there exist a  $\delta > 0$  and a solution  $z_\delta(x)$  of (1.1) satisfying

$$\begin{aligned} z_\delta(\tau) &= z(\tau), & z_\delta(x_1) &= z(x_1) + \delta, \\ z_\delta(x_2) &= z(x_2) = y(x_2), \\ z_\delta(x_3) - z_\delta(x_4) &= z(x_3) - z(x_4) = y(x_3) - y(x_4), \end{aligned} \tag{3.5}$$

and  $|z_\delta^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ ,  $i = 1, 2, 3, 4$ , on  $[\tau, x_4]$ . For  $\epsilon$  small, there exists  $\tau < \sigma_1 < x_1 < \sigma_2 < x_2$  so that

$$\begin{aligned} z_\delta(\sigma_1) &= y(\sigma_1), & z_\delta(\sigma_2) &= y(\sigma_2), \\ z_\delta(x_2) &= y(x_2), & z_\delta(x_3) - z_\delta(x_4) &= y(x_3) - y(x_4). \end{aligned} \tag{3.6}$$

By assumption (C),  $z_\delta(x) = y(x)$  on  $(a, b)$ . However,  $z_\delta(x_1) = z(x_1) + \delta = y(x_1) + \delta > y(x_1)$ , which is a contradiction.

So solutions of (1.1), (1.4) are unique. □

*Remark 3.4.* In view of Theorem 3.3, we remark that, as in Theorem 3.2, solutions of the nonlocal problem (1.1), (1.4) depend continuously on 4-point nonlocal boundary conditions. This type of remark will hold true following each of the subsequent uniqueness results.

**THEOREM 3.5.** *Assume (A), (B), and (C) are satisfied. Then solutions of (1.1), (1.6) are unique when they exist.*

*Proof.* Suppose (1.1), (1.6) has two solutions  $y(x)$  and  $z(x)$ , and let us say

$$\begin{aligned} z(x_1) &= y(x_1), & z(x_2) &= y(x_2), \\ z'(x_2) &= y'(x_2), & z(x_3) - z(x_4) &= y(x_3) - y(x_4), \end{aligned} \tag{3.7}$$

for some  $a < x_1 < x_2 < x_3 < x_4 < b$ . By uniqueness of solutions of 2-point conjugate boundary value problems for (1.1),  $z'(x_1) \neq y'(x_1)$  and  $z''(x_2) \neq y''(x_2)$ .

Without loss of generality, we assume  $y(x) > z(x)$  on  $(x_1, b) \setminus \{x_2\}$ . Then  $y(x) < z(x)$  on  $(a, x_1)$ . Fix  $x_1 < \tau < x_2$ . By Theorem 3.2, for  $\epsilon > 0$  sufficiently small, there exists a  $\delta > 0$

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and a solution  $z_\delta(x)$  of (1.1) satisfying

$$\begin{aligned} z_\delta(x_1) = z(x_1) = y(x_1), \quad z_\delta(\tau) = z(\tau), \quad z_\delta(x_2) = z(x_2) + \delta, \\ z_\delta(x_3) - z_\delta(x_4) = z(x_3) - z(x_4) = y(x_3) - y(x_4), \end{aligned} \quad (3.8)$$

and  $|z_\delta^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ ,  $i = 1, 2, 3, 4$ , on  $[\tau, x_4]$ . For  $\epsilon$  small, there exists  $x_1 < \sigma_1 < x_2 < \sigma_2 < x_4$  so that

$$\begin{aligned} z_\delta(x_1) = y(x_1), \quad z_\delta(\sigma_1) = y(\sigma_1), \\ z_\delta(\sigma_2) = y(\sigma_2), \quad z_\delta(x_3) - z_\delta(x_4) = y(x_3) - y(x_4). \end{aligned} \quad (3.9)$$

By assumption (C),  $z_\delta(x) = y(x)$  on  $(a, b)$ . However,  $z_\delta(x_2) = z(x_2) + \delta = y(x_2) + \delta > y(x_2)$ , which is a contradiction.

So solutions of (1.1), (1.6) are unique.  $\square$

**THEOREM 3.6.** *Assume (A), (B), and (C) are satisfied. Then solutions of (1.1), (1.8) are unique when they exist.*

*Proof.* Suppose (1.1), (1.8) has two solutions  $y(x)$  and  $z(x)$  satisfying

$$\begin{aligned} y(x_1) = z(x_1), \quad y'(x_1) = z'(x_1), \\ y''(x_1) = z''(x_1), \quad y(x_2) - y(x_3) = z(x_2) - z(x_3), \end{aligned} \quad (3.10)$$

for some  $a < x_1 < x_2 < x_3 < b$ . Now  $y'''(x_1) \neq z'''(x_1)$ , and we may assume  $y'''(x_1) > z'''(x_1)$ .

By the last remark above, solutions of (1.1), (1.4) depend continuously on their boundary conditions. Fix  $x_1 < \rho < x_2$ . For  $\epsilon > 0$  small, there is a  $\delta > 0$  and a solution  $z_\delta(x)$  satisfying

$$\begin{aligned} z_\delta(x_1) = z(x_1) = y(x_1), \quad z'_\delta(x_1) = z'(x_1) + \delta, \quad z_\delta(\rho) = z(\rho), \\ z_\delta(x_2) - z_\delta(x_3) = z(x_2) - z(x_3) = y(x_2) - y(x_3), \end{aligned} \quad (3.11)$$

and  $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ ,  $i = 1, 2, 3, 4$ , on  $[x_1, x_3]$ . For  $\epsilon$  sufficiently small, there exist points  $a < \tau_1 < x_1 < \tau_2 < \rho$ , which are in a neighborhood of  $x_1$ , such that  $y(x)$  and  $z_\delta(x)$  both satisfy

$$\begin{aligned} z_\delta(\tau_1) = y(\tau_1), \quad z_\delta(x_1) = y(x_1), \\ z_\delta(\tau_2) = y(\tau_2), \quad z_\delta(x_2) - z_\delta(x_3) = y(x_2) - y(x_3). \end{aligned} \quad (3.12)$$

So we have  $z_\delta(x) = y(x)$  on  $(a, b)$  by hypothesis (C). But

$$z'_\delta(x_1) = z'(x_1) + \delta = y'(x_1) + \delta > y'(x_1). \quad (3.13)$$

This is a contradiction. So (1.1), (1.8) has at most one solution.  $\square$

Of course, in terms of the uniqueness condition (D), there are dual uniqueness results, which we now state as one theorem.

**THEOREM 3.7.** *Assume (A), (B), and (D) are satisfied. Then solutions of (1.1), (1.j),  $j = 5, 7, 9$ , are unique when they exist.*

#### 4. Uniqueness of 4-point and 3-point implies uniqueness of 5-point

In this section, our consideration is with a question converse to the uniqueness results of Section 3. In particular, we assume that solutions of 4-point and 3-point nonlocal boundary value problems for (1.1) are unique. It is then established that solutions of both (1.1), (1.2) and (1.1), (1.3) are also unique. Fundamental to our arguments is a Kamke type of convergence result for boundary value problems due to Vidossich [42], as well as a pre-compactness condition on bounded sequences of solutions of (1.1) due to Jackson and Schrader; see Agarwal [1]. We state both of those results at the outset of the section.

**THEOREM 4.1 (Vidossich).** *For each  $n > 0$ , let  $g_n : [c, d] \times \mathbb{R}^N \rightarrow \mathbb{R}$  be continuous, let  $L_n : C([c, d] \times \mathbb{R}^N, \mathbb{R}) \rightarrow \mathbb{R}^N$  be continuous, and let  $r_n \in \mathbb{R}^N$ . Assume that*

- (a)  $\lim_n r_n = r_0$ ,
- (b)  $\lim_n g_n = g_0$  and  $\lim_n L_n = L_0$  uniformly on compact subsets of  $[c, d] \times \mathbb{R}^N$ , respectively,
- (c) each initial value problem,

$$x' = g_n(t, x), \quad x(a) = u, \tag{4.1}$$

has at most one local solution for  $u \in \mathbb{R}^N$ ,

- (d) the functional boundary value problem,

$$x' = g_0(t, x), \quad L_0(x) = r, \tag{4.2}$$

has at most one solution for each  $r \in \mathbb{R}^N$ .

Let  $x_0$  be the solution to  $x' = g_0(t, x)$ ,  $L_0(x) = r_0$ . Then for each  $\epsilon > 0$ , there exists  $n_\epsilon$  such that the functional boundary value problem,

$$x' = g_n(t, x), \quad L_n(x) = r_n, \tag{4.3}$$

has a solution  $x_n$ , for  $n > n_\epsilon$ , satisfying the condition

$$\|x_0 - x_n\|_\infty < \epsilon. \tag{4.4}$$

**THEOREM 4.2 (Jackson-Schrader).** *Assume that, with respect to (1.1), conditions (A) and (B) hold. In addition, assume that solutions of 4-point conjugate boundary value problems are unique. If  $\{y_k(x)\}$  is a sequence of solutions of (1.1) for which there exists an interval  $[c, d] \subset (a, b)$  and there exists an  $M > 0$  such that  $|y_k(x)| < M$ , for all  $x \in [c, d]$  and for all  $k \in \mathbb{N}$ , then there exists a subsequence  $\{y_{k_j}(x)\}$  such that, for  $i = 0, 1, 2, 3$ ,  $\{y_{k_j}^{(i)}(x)\}$  converges uniformly on each compact subinterval of  $(a, b)$ .*

**Remark 4.3.** We remark that if solutions of (1.1) satisfying each of the nonlocal boundary conditions (1.j),  $j = 4, \dots, 9$ , are unique, when they exist, then solutions of 2-point and

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3-point conjugate boundary value problems for (1.1) are unique. As a consequence of Theorems 2.1 and 2.2, it would follow that if (A) and (B) are also assumed, then each  $k$ -point conjugate boundary value problem for (1.1) has a solution which is unique,  $k = 2, 3, 4$ .

We now provide a type of converse to the results of Section 3.

**THEOREM 4.4.** *Assume (A) and (B) are satisfied. Assume solutions of (1.1) satisfying any of (1.j),  $j = 4, \dots, 9$  are unique when they exist. Then solutions of both (1.1), (1.2) and (1.1), (1.3) are unique when they exist.*

*Proof.* We establish the result for only (1.1), (1.2). Suppose (1.1), (1.2) has two distinct solutions  $y(x)$  and  $z(x)$ , for some  $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$  and some  $y_1, y_2, y_3, y_4 \in \mathbb{R}$ . That is,

$$\begin{aligned} y(x_i) &= z(x_i), \quad i = 1, 2, 3, \\ y(x_4) - y(x_5) &= z(x_4) - z(x_5). \end{aligned} \tag{4.5}$$

By assumptions (A) and (B) and uniqueness of solutions of 4-point and 3-point non-local boundary value problems, we know from the remark preceding the proof of this theorem that solutions of all conjugate boundary problems for (1.1) exist and are unique.

For each  $n \geq 1$ , let  $y_n(x)$  be the solution of the boundary value problems for (1.1) satisfying the 3-point conjugate boundary conditions:

$$\begin{aligned} y_n(x_3) &= y(x_3) = z(x_3), & y'_n(x_3) &= y'(x_3) - n, \\ y_n(x_4) &= y(x_4), & y_n(x_5) &= y(x_5). \end{aligned} \tag{4.6}$$

It follows from uniqueness of solutions of 4-point conjugate problems that, for  $n \geq 1$ ,

$$y(x) < y_n(x) < y_{n+1}(x) \tag{4.7}$$

on  $(a, x_3)$ .

For each  $n \geq 1$ , let

$$E_n = \{x : x_1 \leq x \leq x_2 \mid \text{where } y_n(x) \leq z(x)\}. \tag{4.8}$$

We claim that  $E_n \neq \emptyset$ , for each  $n \geq 1$ . In that direction, suppose there exists  $n_0$  so that  $E_{n_0} = \emptyset$ . Then  $y_{n_0}(x) > z(x)$  on  $[x_1, x_2]$ .

Next, for all  $\epsilon \geq 0$ , let  $y_\epsilon$  be the solution of (1.1) satisfying the 3-point conjugate boundary conditions:

$$\begin{aligned} y_\epsilon(x_3) &= y(x_3) = z(x_3), & y'_\epsilon(x_3) &= y'(x_3) - \epsilon, \\ y_\epsilon(x_4) &= y(x_4), & y_\epsilon(x_5) &= y(x_5). \end{aligned} \tag{4.9}$$

Note when  $\epsilon = 0$ ,  $y_\epsilon(x) = y(x)$ .

Define

$$S = \{\epsilon \geq 0 \mid \text{for some } x_1 \leq x \leq x_2, y_\epsilon(x) \leq z(x)\}, \tag{4.10}$$

$S \neq \emptyset$  since  $0 \in S$ . Now since  $E_{n_0} = \emptyset$ ,  $S$  is bounded above.

Let  $\epsilon_0 = \sup S$ , and consider the solution  $y_{\epsilon_0}(x)$  of (1.1). We claim that there exists  $\tau \in (x_1, x_2)$  so that  $y_{\epsilon_0}(\tau) \leq z(\tau)$ . If not, then  $y_{\epsilon_0}(x) > z(x)$ , for all  $x_1 \leq x \leq x_2$ . By continuous dependence of solutions of (1.1) on 3-point conjugate boundary conditions, there exists  $0 < \epsilon_1 < \epsilon_0$ , so that  $y_{\epsilon_1}(x) > z(x)$  for all  $x_1 \leq x \leq x_2$ . Therefore  $\epsilon_1$  is an upper bound of  $S$ . But by assumption  $\epsilon_0 = \sup S$ , whereas  $0 < \epsilon_1 < \epsilon_0$ . This is a contradiction. Therefore there exists  $\tau \in (x_1, x_2)$  so that  $y_{\epsilon_0}(\tau) \leq z(\tau)$ .

Next, if  $y_{\epsilon_0}(\tau) < z(\tau)$ , then by continuity, there exists an interval  $[\tau - \rho, \tau + \rho]$  so that  $y_{\epsilon_0}(x) < z(x)$  on  $[\tau - \rho, \tau + \rho]$ . So there exists  $\epsilon_0 < \epsilon_2$  so that  $y_{\epsilon_2}(x) \leq z(x)$  on some interval  $[\tau - \eta, \tau + \eta] \subset [\tau - \rho, \tau + \rho] \subset [x_1, x_2]$ . So  $\epsilon_2 \in S$ . But  $\epsilon_2 > \epsilon_0$ , and so we contradict that  $\epsilon_0$  is the least upper bound of  $S$ .

Now for this  $\tau \in (x_1, x_2)$ ,  $y_{\epsilon_0}(\tau) = z(\tau)$ , and  $y_{\epsilon_0}(x) \geq z(x)$  for all  $x \in [x_1, x_2] \setminus \{\tau\}$ .

In particular,

$$\begin{aligned} y_{\epsilon_0}(\tau) &= z(\tau), & y'_{\epsilon_0}(\tau) &= z'(\tau), \\ y_{\epsilon_0}(x_3) &= z(x_3), & y_{\epsilon_0}(x_4) - y_{\epsilon_0}(x_5) &= z(x_4) - z(x_5). \end{aligned} \tag{4.11}$$

By the uniqueness of solutions of 4-point nonlocal boundary value problems, we reach a contradiction. So  $E_n \neq \emptyset$ , for all  $n \geq 1$ .

Thus,  $E_{n+1} \subset E_n \subset (x_1, x_2)$ , for each  $n \geq 1$ , and each  $E_n$  is also compact. Hence,

$$\bigcap_{n=1}^{\infty} E_n := E \neq \emptyset. \tag{4.12}$$

Next, we observe that the set  $E$  consists of a single point  $\{x_0\}$  with  $x_1 < x_0 < x_2$ . To see this, suppose there are points  $t_1, t_2 \in E$  with  $x_1 < t_1 < t_2 < x_2$ .

We claim that the interval  $[t_1, t_2] \subseteq E$ . Suppose to the contrary that there exists  $\tau \in (t_1, t_2)$  such that  $\tau \notin E$ . Then, there exists an  $N \in \mathbb{N}$  such that, for each  $n \geq N$ ,  $y_n(\tau) > z(\tau)$ . By continuity, there exists a  $\lambda > 0$  such that, for each  $n \geq N$ ,

$$z(x) < y_n(x) < y_{n+1}(x), \quad x \in [\tau - \lambda, \tau + \lambda]. \tag{4.13}$$

With the solution  $y_\epsilon(x)$  of (1.1) as defined above, we define a new set:

$$S' = \{\epsilon \geq 0 \mid \text{for some } \tau - \lambda \leq x \leq \tau + \lambda, y_\epsilon(x) \leq z(x)\}. \tag{4.14}$$

Again  $0 \in S'$ , and so  $S' \neq \emptyset$ . In this case  $N$  is an upper bound of  $S'$ . We reach the same contradiction as above in showing the foregoing sets  $E_n$  are nonnull. We conclude that the interval  $[t_1, t_2] \subseteq E$ , and the claim is verified.

However,  $[t_1, t_2] \subseteq E$  implies that the sequence  $\{y_n(x)\}$  is uniformly bounded on  $[t_1, t_2]$ . It follows from Theorem 4.2 that there is a subsequence  $\{y_{n_j}(x)\}$  such that for each

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$i = 0, 1, 2, 3$ ,  $\{y_{n_j}^{(i)}(x)\}$  converges uniformly on each compact subinterval of  $(a, b)$ . However,

$$\lim_{j \rightarrow \infty} y'_{n_j}(x_3) = \lim_{j \rightarrow \infty} y'(x_3) - n_j = -\infty; \quad (4.15)$$

this is a contraction.

Thus we conclude that

$$E = \{x_0\}, \quad (4.16)$$

with  $x_1 < x_0 < x_2$ , and we also have

$$\lim_{n \rightarrow \infty} y_n(x_0) \leq z(x_0). \quad (4.17)$$

Now, let  $y_0(x)$  be the solution of the 4-point conjugate boundary value problem for (1.1) satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} y_n(x_0) = y_0(x_0), \quad y_0(x_3) = y(x_3) = z(x_3), \\ y_0(x_4) = y(x_4), \quad y_0(x_5) = y(x_5). \end{aligned} \quad (4.18)$$

By Theorem 4.1,  $\{y_n^{(i)}(x)\}$  converges to  $y_0^{(i)}(x)$ ,  $i = 0, 1, 2, 3$ , on each compact subinterval of  $(a, b)$ .

So  $y_0(x_0) \leq z(x_0)$ , which we claim that it leads to contradictions. There are two cases to resolve. First, assume  $y_0(x_0) = z(x_0)$ . Then we have two solutions  $y_0(x)$  and  $z(x)$  of (1.1) satisfying

$$\begin{aligned} y_0(x_0) = z(x_0), \quad y'_0(x_0) = z'(x_0), \quad y_0(x_3) = z(x_3), \\ y_0(x_4) - y_0(x_5) = y(x_4) - y(x_5) = z(x_4) - z(x_5), \end{aligned} \quad (4.19)$$

and so by uniqueness of solutions 4-point nonlocal boundary value problems (1.1), (1.4),  $y_0(x) \equiv z(x)$  on  $(a, b)$ . This is a contradiction. So  $\lim_{n \rightarrow \infty} y_n(x_0) \neq z(x_0)$ .

The remaining case is that  $y_0(x_0) < z(x_0)$ . In this case, by the continuity of  $y_0(x)$ , there exists  $\delta > 0$  with  $[x_0 - \delta, x_0 + \delta] \subset (x_1, x_2)$  on which  $y_0(x) < z(x)$ . Since  $\lim_n y(x) = y_0(x)$  uniformly on each compact subinterval of  $(a, b)$ , it follows that  $[x_0 - \delta, x_0 + \delta] \subset E$ . This is a contradiction.

From this final contradiction, we conclude that  $y_0(x_0) \leq z(x_0)$  is impossible. This resolves all situations, and we conclude that solutions of (1.1), (1.2) are unique. Of course, completely symmetric arguments yield that solutions of (1.1), (1.3) are also unique.  $\square$

As a final statement, we present a theorem summarizing the results of this paper.

**THEOREM 4.5.** *Assume conditions (A) and (B) are satisfied. Then solutions of both (1.1), (1.2) and (1.1), (1.3) are unique when they exist, if and only if solutions of (1.1) satisfying each of (1.j),  $j = 4, \dots, 9$ , are unique when they exist.*

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