BOUNDARY REGULARITY OF WEAK SOLUTIONS TO NONLINEAR ELLIPTIC OBSTACLE PROBLEMS

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We study the boundary regularity of weak solutions to nonlinear obstacle problem with $C^{1,\beta}$-obstacle function, and obtain the ${C_{\mathrm{loc}}}^{1,\alpha}$ boundary regularity.

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1. Introduction

We consider the following variational inequality:

$$\begin{aligned} u \in \mathfrak{R} : \int_\Omega A(x,\nabla u)\cdot(\nabla v - \nabla u)dx \\ \geq \int_\Omega H(x,u,\nabla u)(v-u)dx + \int_\Omega F(x,u)\cdot(\nabla v - \nabla u)dx \end{aligned} \tag{1.1}$$

for all $v \in \mathfrak{R} = \{v \in W_0^{1,p}(\Omega),\ v \geq \psi \text{ a.e. in } \Omega\}$. Here Ω is a bounded domain in $R^N\,(N \geq 2)$ with Lipschitz boundary, $2 \leq p \leq N$.

$A(x,\xi) : \Omega \times R^N \to R^N$ satisfies the following conditions:

(i) A is a vector valued function, the mapping $x \mapsto A(x,\xi)$ is measurable for all $\xi \in R^N$, $\xi \mapsto A(x,\xi)$ is continuous for a.e. $x \in \Omega$;
(ii) the homogeneity condition: $A(x,t\xi) = t|t|^{p-2}A(x,\xi)$, $t \in R$, $t \neq 0$;
(iii) the monotone inequality: $(A(x,\xi) - A(x,\zeta))(\xi - \zeta) \geq a|\xi - \zeta|^p$;
(iv) $|h||a^{ij}| + |\partial A^i(x,h)/\partial x_j| \leq \tau_1|h|^{p-1}$;
(v) $\sum_{i,j=1}^N a^{ij}\xi_i\xi_j \geq \tau_2|h|^{p-2}|\xi|^2$;
(vi) $|A(x,\xi) - A(y,\xi)| \leq b_1(1+|\xi|^{p-1})|x-y|^{\alpha_0}$;
(vii) $|A(x,\xi) - A(x,\eta)| \leq b_2||\xi|^{p-2}\xi - |\eta|^{p-2}\eta|$;

where $a^{ij} = \partial A^i/\partial h_j$, a, b_1, b_2, τ_1, τ_2 are positive constants.

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We assume that $H(x,u,\lambda)$, $F(x,u) = \{F_i(x,u)\}_{1\le i\le N}$ in (1.1) are of the form:

$$|H(x,u,\nabla u)| \le c(|\nabla u|^{p/r'} + |u|^{r-1} + g(x)), \tag{1.2}$$

$$|F(x,u)| \le c(|u|^{q/p'} + h(x)), \tag{1.3}$$

where $p < q < r$, $r' = r/(r-1)$, $p' = p/(p-1)$, and if $2 \le p < N$, $r = Np/(N-p)$, while if $p = N$, then r can be some sufficiently large positive number.

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$$I(u) = \inf\left\{ \int_\Omega |\nabla u|^p dx : u \in K(\psi)\right\}, \tag{1.4}$$

where

$$K(\psi) = \{v \in W^{1,p}(\Omega) : v \ge \psi \text{ a.e.}\}, \tag{1.5}$$

has been studied by various authors. In the case when ψ is assumed to have only minimal regularity properties, it was shown by [8, 11] that the solution of (1.1) is continuous. In particular, if $\psi \in C^{0,\alpha}(\Omega)$, then the solution u is also an element of $C^{0,\alpha'}(\Omega)$. In the case when $\psi \in C^2(\Omega)$, papers [4, 6, 10, 12] employed different techniques to prove interior $C^{1,\alpha}(\Omega)$ regularity for the solution u to (1.4). Reference [1] gave an interesting result: the condition for $\mathfrak{K}$ to be nonempty is just that ψ should have finite capacity. This implies, among other things, that $\psi^+ = \max(\psi,0)$ must vanish on $\partial\Omega$, C—almost everywhere. This condition is important for the existence of weak solutions to obstacle problem.

When ψ is smooth (say $C^{1,\alpha}(\Omega)$), the interior regularity of weak solutions to problem (1.1) has been studied extensively by many authors ([3, 13, 14]).

In view of De Giorgi class, paper [2] obtained $C^{0,\alpha}$ interior regularity for solutions of nonlinear elliptic obstacle problem with natural growth in the gradient by taking appropriate test function.

The main concern of these papers is the question of the regularity of the solution u in terms of the given regularity properties of the obstacle ψ and relevant data. This is especially interesting in view of the fact that there is a limit to the amount of regularity that u can inherit from ψ: it is possible for ψ to be real analytic, but u will be at best $C^{1,1}$, that is, have bounded second derivatives.

This paper obtains $C^{1,\alpha}_{\text{loc}}$ boundary regularity of weak solutions to the obstacle problem with $C^{1,\beta}$-obstacle function under controllable growth condition (1.2). We present a new proof to a useful comparison principle.

2. Notations and preliminaries

Ω is an open bounded subset of R^N, $N \ge 2$; $\partial\Omega$ is the boundary of Ω. If $z \in R^N$, we put

$$\begin{aligned} B_R(z) &= \{x \in R^N : |x-z| < R\}, & \Gamma_R(z) &= \{x \in B_R(z) : x_n = 0\}, \\ B_R^+(z) &= \{x \in B_R(z) : x_n > 0\}, & B_R^-(z) &= \{x \in B_R(z) : x_n < 0\}. \end{aligned} \tag{2.1}$$

We denote by B, B^+, B^-, Γ, respectively, $B_1(0)$, $B_1^+(0)$, $B_1^-(0)$, $\Gamma_1(0)$. For every set E we denote by $\bar{E}$ its closure, and by $|E|$ its Lebesgue measure. $(f)_R = (1/|B_R|)\int_{B_R} f(x)dx$. The letter c is used throughout to denote a positive constant, not necessarily the same at each occurrence.

Since $\overline{\Omega}$ is compact, $\partial\Omega$ can be covered by a finite number of neighbourhoods V of its points. It is enough to prove the better regularity of u holds true in $V\cap\Omega$. Since $\partial\Omega$ is a Lipschitz boundary, one can find T which is an invertible Lipschitz mapping such that

$$T(V)=B,\qquad T(V\cap\Omega)=B^+,\qquad T(V\backslash\overline{\Omega})=B^-,\qquad T(V\cap\partial\Omega)=\Gamma. \tag{2.2}$$

Under the mapping T the variational inequality in Ω is transformed to a variational inequality of the same form in B^+, for $\bar{u}=u\circ T^{-1}$ which satisfies

$$\begin{aligned}&\int_{B^+}\bar{A}(x,\nabla\bar{u})\cdot(\nabla v-\nabla\bar{u})dx\\&\quad\geq\int_{B^+}\bar{H}(x,\bar{u},\nabla\bar{u})(v-\bar{u})dx+\int_{B^+}\bar{F}(x,\bar{u})\cdot(\nabla v-\nabla\bar{u})dx,\quad\forall v\in\bar{\mathfrak{K}},\end{aligned} \tag{2.3}$$

where $\bar{\mathfrak{K}}=\{v\in W_0^{1,p}(B^+),\ v\geq\psi,\ \text{a.e. in } B^+\}$, $\bar{A}$, $\bar{H}$, $\bar{F}$ satisfy assumptions of type (i)–(vii), (1.2), (1.3) with different constants.

In order to simplify the notations, we still denote $\bar{u}$, $\bar{\mathfrak{K}}\bar{A}$, $\bar{H}$, $\bar{F}$ by u, $\mathfrak{K}$, A, H, F, respectively.

Since the original $u\in W_0^{1,p}(\Omega)$, we define then

$$u(x):=\begin{cases}u(x), & \text{if } x\in B^+,\\ -3u(x_1,\dots,x_{n-1},-x_n)+4u\left(x_1,\dots,x_{n-1},-\dfrac{x_n}{2}\right), & \text{if } x\in B^-.\end{cases} \tag{2.4}$$

In light of Extension theorem [5, page 254], we only need to prove a better regularity of u in B^+.

Definition 2.1. The function $u\in\mathfrak{K}$ that satisfies (2.3) for all $v\in\mathfrak{K}$ is called a weak solution to the obstacle problem with obstacle ψ.

Definition 2.2. Call $f\in C^{0,\alpha}(\Gamma)$, if for all $x\in\Gamma$, there exists $B_r(x)$ (a ball centered at x of radius r), $r>0$, such that $f\in C^{0,\alpha}(B_r(x))$.

In the sequel, we will abbreviate $B^+\cap B_R(y_0)=B_R^+$, $B^+\cap B_\rho(y_0)=B_\rho^+$, for $0<\rho<R\leq 1$, the point $y_0\in\Gamma$ to be understood.

In the following, we will use some lemmas which we state below.

LEMMA 2.3. *Let $w\in W^{1,p}(B_R^+)$ be a solution of the Dirichlet problem*

$$\begin{gathered}\int_{B_R^+}A(x,\nabla w)\nabla\phi\,dx=0\quad\forall\phi\in W_0^{1,p}(B_R^+),\\ w-u\in W_0^{1,p}(B_R^+).\end{gathered} \tag{2.5}$$

For $0<\rho<R/2$, $\sigma\in(0,1)$,

$$\int_{B_R^+}|\nabla w|^p dx \le c\int_{B_R^+}|\nabla u|^p dx, \tag{2.6}$$

$$\int_{B_\rho^+}|\nabla w|^p dx \le c\left(\frac{\rho}{R}\right)^N\int_{B_R^+}|\nabla w|^p dx, \tag{2.7}$$

$$\int_{B_\rho^+}\left|\nabla w-(\nabla w)_\rho\right|^p dx \le c\left(\frac{\rho}{R}\right)^{N+\sigma}\int_{B_R^+}\left|\nabla w-(\nabla w)_R\right|^p dx. \tag{2.8}$$

Proof. We can easily get (2.6) by inserting $\phi = w-u$ in (2.5).

An argument similar to the one in [15, Lemma 2.2] shows that (2.7) hold.

The proof of (2.8) is similar to that of [9, Theorem 1.7]. □

Lemma 2.4. *Let* $v\in W^{1,p}(B_R^+)$ *be a solution of the Dirichlet problem*

$$\begin{gathered}\int_{B_R^+}A(x,\nabla v)\nabla\phi\,dx=\int_{B_R^+}A(x,\nabla\psi)\nabla\phi\,dx,\\ w-v\in W_0^{1,p}(B_R^+),\quad \forall\phi\in W_0^{1,p}(B_R^+),\end{gathered} \tag{2.9}$$

then

$$\int_{B_R^+}|\nabla w|^p dx \le c\int_{B_R^+}|\nabla v|^p dx, \tag{2.10}$$

$$\int_{B_R^+}|\nabla v-\nabla w|^p dx \le c\int_{B_R^+}|\nabla \psi|^p dx. \tag{2.11}$$

Proof. Formula (2.10) follows immediately from taking $\phi = w-v$ in (2.5).

Inserting $\phi = v-w$ in (2.5) and (2.9), by monotone inequality (iii) and Hölder's inequality, we have

$$\begin{aligned}\int_{B_R^+}|\nabla v-\nabla w|^p dx &\le c\int_{B_R^+}\left(A(x,\nabla v)-A(x,\nabla w)\right)\cdot(\nabla v-\nabla w)dx\\ &= c\int_{B_R^+}A(x,\nabla\psi)\cdot(\nabla v-\nabla w)dx\\ &\le c\int_{B_R^+}|\nabla\psi|^{p-1}|\nabla v-\nabla w|dx\\ &\le c\left(\int_{B_R^+}|\nabla\psi|^p dx\right)^{(p-1)/p}\left(\int_{B_R^+}|\nabla v-\nabla w|^p dx\right)^{1/p}\end{aligned} \tag{2.12}$$

from which we get (2.11). □

Lemma 2.5. *If* $v\in W^{1,p}(B_R^+)$ *is a solution of the Dirichlet problem (2.9), then* $v\ge\psi$ *in* B_R^+.

Proof. It follows from $v=u$ on ∂B_R^+, $u\in\mathfrak{R}$, that $v\ge\psi$ on ∂B_R^+. Let $\xi=\min(v,\psi)$, $\xi=\psi$ on ∂B_R^+, $\xi-\psi\in W_0^{1,p}(B_R^+)$. As test functions in (2.9) we take $\phi=\xi-\psi$, from (2.9) and monotony inequality (iii), we have

$$\begin{aligned}
0 &= \int_{B_R^+} A(x,\nabla v) - A(x,\nabla\psi)\cdot\nabla(\xi-\psi)dx \\
&= \int_{B_R^+\bigcap\{x,v(x)\le\psi(x)\}} A(x,\nabla v) - A(x,\nabla\psi)\cdot\nabla(v-\psi)dx \\
&\ge a\int_{B_R^+\bigcap\{x,v(x)\le\psi(x)\}} |\nabla v-\nabla\psi|^p dx \\
&= a\int_{B_R^+} |\nabla\xi-\nabla\psi|^p dx
\end{aligned} \tag{2.13}$$

therefore $\xi=\psi$ a.e. in B_R^+, that is, $v\ge\psi$ a.e. in B_R^+. □

This lemma is a useful comparison principle, it can be used to obtain the existence or regularity of solutions to elliptic equation or variational inequality.

We extend v to B^+ by setting $v=u$ on $B^+\backslash B_R^+$, and hence $v\in\mathfrak{R}$. We have the following corollary.

Corollary 2.6. *Suppose u is a weak solution to the obstacle problem (2.3), $v\in W^{1,p}(B_R^+)$ is a solution of the Dirichlet problem (2.9), then $v\in\mathfrak{R}$ satisfies the variational inequality*

$$\int_{B^+} A(x,\nabla u)\cdot(\nabla v-\nabla u)dx \ge \int_{B^+} H(x,u,\nabla u)(v-u)dx + \int_{B^+} F(x,u)\cdot(\nabla v-\nabla u)dx. \tag{2.14}$$

Lemma 2.7. *Assume u is a weak solution to the obstacle problem (2.3), where H, F verify (1.2), (1.3), respectively, $g\in L^t(B^+)$ with $t>N/p$, $h\in L^s(B^+)$ with $s>p'$, v satisfies (2.9), then*

$$\int_{B_R^+}|\nabla v|^p dx \le c\left[\int_{B_R^+}|\nabla u|^p dx + \int_{B_R^+}|\nabla\psi|^p dx\right], \tag{2.15}$$

$$\begin{aligned}
\int_{B_R^+}|\nabla u-\nabla v|^p dx \le c\Bigg\{&\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)dx\right]^{1+\delta} \\
&+\left(\int_{B_R^+}|\nabla u|^p dx\right)^{q/p}+\left(\int_{B_R^+}|u|^r dx\right)^{q/p} \\
&+R^{Np(1-1/r-1/t)/(p-1)}+R^{N(1-p'/s)}+R^{N(1-p/m)}\Bigg\},
\end{aligned} \tag{2.16}$$

where $\delta=(r-p)/r(p-1)>0$.

Proof. By inserting $\phi=v-u$ in (2.9), an application of Hölder's inequality and Young's inequality yields

$$\begin{aligned}
\int_{B_R^+}|\nabla v|^p dx &\le c\int_{B_R^+} A(x,\nabla v)\cdot\nabla v\,dx \\
&= c\left[\int_{B_R^+} A(x,\nabla v)\cdot\nabla u\,dx + \int_{B_R^+} A(x,\nabla\psi)\cdot\nabla(v-u)dx\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq c\Big[\int_{B_R^+}|\nabla v|^{p-1}|\nabla u|dx+\int_{B_R^+}|\nabla\psi|^{p-1}|\nabla u-\nabla v|dx\Big]\\
&\leq c\bigg\{\Big(\int_{B_R^+}|\nabla v|^{p}dx\Big)^{(p-1)/p}\Big(\int_{B_R^+}|\nabla u|^{p}dx\Big)^{1/p}\\
&\qquad+\Big(\int_{B_R^+}|\nabla\psi|^{p}dx\Big)^{(p-1)/p}\Big(\int_{B_R^+}|\nabla u-\nabla v|^{p}dx\Big)^{1/p}\bigg\}\\
&\leq c\epsilon_1\int_{B_R^+}|\nabla v|^{p}dx+c(\epsilon_1,p)\int_{B_R^+}|\nabla u|^{p}dx+c\epsilon_2\int_{B_R^+}|\nabla u-\nabla v|^{p}dx\\
&\quad+c(\epsilon_2,p)\int_{B_R^+}|\nabla\psi|^{p}dx\\
&\leq(c\epsilon_1+c\epsilon_2)\int_{B_R^+}|\nabla v|^{p}dx+(c(\epsilon_1,p)+c\epsilon_2)\int_{B_R^+}|\nabla u|^{p}dx\\
&\quad+c(\epsilon_2,p)\int_{B_R^+}|\nabla\psi|^{p}dx
\end{aligned}
\tag{2.17}
$$

for $(c\epsilon_1+c\epsilon_2)$ sufficiently small $(c\epsilon_1+c\epsilon_2<1)$, we can get (2.15). □

By $\psi\in W^{1,m}(\Omega)$, $m>N$, we have

$$
\int_{B_R}|\nabla\psi|^{p}dx\leq c\|\nabla\psi\|_m^pR^{N(1-p/m)}.
\tag{2.18}
$$

Combining monotone inequality (iii), (2.9), (2.14), and (1.2) and using Poincare's inequality, Hölder's inequality, we have

$$
\begin{aligned}
&\int_{B_R^+}|\nabla u-\nabla v|^{p}dx\\
&\qquad\leq c\int_{B_R^+}[A(x,\nabla u)-A(x,\nabla v)]\cdot(\nabla u-\nabla v)dx\\
&\qquad\leq c\int_{B_R^+}[H(x,u,\nabla u)(u-v)+(F(x,u)-A(x,\nabla\psi))\cdot(\nabla u-\nabla v)]dx\\
&\qquad\leq c\int_{B_R^+}(|\nabla u|^{p(1-1/r)}+|u|^{r-1}+|g|)|u-v|dx\\
&\qquad\quad+c\int_{B_R^+}(|u|^{q/p'}+h(x))|\nabla u-\nabla v|dx+c\int_{B_R^+}|\nabla\psi|^{p-1}\cdot|\nabla u-\nabla v|dx\\
&\qquad\leq c\Big[\int_{B_R^+}(|\nabla u|^{p}+|u|^{r}+|g|^{r/(r-1)})dx\Big]^{1-1/r}\Big(\int_{B_R^+}|u-v|^{r}dx\Big)^{1/r}\\
&\qquad\quad+\Big(\int_{B_R^+}(|u|^{q}+|h|^{p'})dx\Big)^{1/p'}\Big(\int_{B_R^+}|\nabla u-\nabla v|^{p}dx\Big)^{1/p}\\
&\qquad\quad+\int_{B_R^+}|\nabla\psi|^{p-1}\cdot|\nabla u-\nabla v|dx
\end{aligned}
$$

$$
\begin{aligned}
&\le c\left\{\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\right]^{1-1/r}+\|g\|_t R^{N(1-1/r-1/t)}\right\}\\
&\quad\times R^{1-N(1/p-1/r)}\left(\int_{B_R^+}|\nabla u-\nabla v|^p\,dx\right)^{1/p}\\
&\quad+c\left\{\left(\int_{B_R^+}(|u|^p+|\nabla u|^p)\,dx\right)^{q/pp'}+\|h\|_s R^{N(1/p'-1/s)}+\left\||\nabla\psi|^{p-1}\right\|_{p/(p-1)}\right\}\\
&\quad\times\left(\int_{B_R^+}|\nabla u-\nabla v|^p\,dx\right)^{1/p}
\end{aligned}
\tag{2.19}
$$

since $0<R\le 1$, by (2.18), Hölder inequality, Young inequality, we have

$$
\begin{aligned}
&\int_{B_R^+}|\nabla u-\nabla v|^p\,dx\\
&\quad\le c\left\{\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\right]^{1+\delta}+\left(\int_{B_R^+}|\nabla u|^p\,dx\right)^{q/p}\right.\\
&\qquad+\left(\int_{B_R^+}|u|^p\,dx\right)^{q/p}+\|g\|_t^{p/(p-1)}R^{Np(1-1/r-1/t)/(p-1)}\\
&\qquad\left.+\|h\|_s^{p/(p-1)}R^{N(1-p'/s)}+\int_{B_R^+}|\nabla\psi|^p\,dx\right\}\\
&\quad\le c\left\{\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\right]^{1+\delta}+\left(\int_{B_R^+}|\nabla u|^p\,dx\right)^{q/p}\right.\\
&\qquad\left.+\left(\int_{B_R^+}|u|^p\,dx\right)^{q/p}+R^{Np(1-1/r-1/t)/(p-1)}+R^{N(1-p'/s)}+R^{N(1-p/m)}\right\}\\
&\quad\le c\left\{\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\right]^{1+\delta}+\left(\int_{B_R^+}|\nabla u|^p\,dx\right)^{q/p}\right.\\
&\qquad\left.+\left(\int_{B_R^+}|u|^r\,dx\right)^{q/r}|B_R|^{(q/p)(1-p/r)}+R^{Np(1-1/r-1/t)/(p-1)}+R^{N(1-p'/s)}+R^{N(1-p/m)}\right\}\\
&\quad\le c\left\{\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\right]^{1+\delta}+\left(\int_{B_R^+}|\nabla u|^p\,dx\right)^{q/p}\right.\\
&\qquad\left.+\left(\int_{B_R^+}|u|^r\,dx\right)^{q/p}+R^{N(q/p)}+R^{Np(1-1/r-1/t)/(p-1)}+R^{N(1-p'/s)}+R^{N(1-p/m)}\right\}\\
&\quad\le c\left\{\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\right]^{1+\delta}+\left(\int_{B_R^+}|\nabla u|^p\,dx\right)^{q/p}\right.\\
&\qquad\left.+\left(\int_{B_R^+}|u|^r\,dx\right)^{q/p}+R^{Np(1-1/r-1/t)/(p-1)}+R^{N(1-p'/s)}+R^{N(1-p/m)}\right\}
\end{aligned}
\tag{2.20}
$$

which implies (2.16).

3. $C^{0,\lambda}$ regularity

Theorem 3.1. *Assume that $H(x,u,\nabla u)$ satisfies (1.2), $g \in L^t(B^+)$ with $t > N/p$, $F(x,u)$ satisfies (1.3), $h \in L^s(B^+)$ with $s > N/(p-1)$, and $\psi \in W^{1,m}(B^+)$ with $m > N$. If $u \in \mathfrak{R}$ makes (2.3) hold, then $u \in C^{0,\lambda}(\Gamma)$ with $\lambda = \min\{1 - N(1/t + 1/r - 1/p)/(p-1), 1 - N/s(p-1), 1 - N/m\}$.*

Before proceeding with the formal proof, we make an important observation. It is a well-known result.

Proposition 3.2. *If $f \in W^{1,p}(\Omega)$, then for all constants $k \in R^N$,*

$$\int_{B_\rho} |\nabla f - (\nabla f)_{\rho,x_0}|^p dx \le C(p) \int_{B_\rho} |\nabla f - k|^p dx \tag{3.1}$$

for every ρ for which $B_\rho(x_0) \subset \Omega$.

Proof. By elementary inequality, we have

$$\int_{B_\rho} |\nabla f - (\nabla f)_{\rho,x_0}|^p dx \le C(p) \left[\int_{B_\rho} |\nabla f - k|^p dx + \int_{B_\rho} |k - (\nabla f)_{\rho,x_0}|^p dx \right]. \tag{3.2}$$

Moreover

$$\begin{aligned}\int_{B_\rho} |k - (\nabla f)_{\rho,x_0}|^p dx &= |B_\rho| \, |k - (\nabla f)_{\rho,x_0}|^p = |B_\rho| \left| k - \frac{1}{|B_\rho|} \int_{B_\rho} \nabla f \, dx \right|^p \\ &= |B_\rho| \left| \frac{1}{|B_\rho|} \int_{B_\rho} (k - \nabla f) dx \right|^p = |B_\rho|^{1-p} \left| \int_{B_\rho} (k - \nabla f) dx \right|^p \\ &\le |B_\rho|^{1-p} \int_{B_\rho} |\nabla f - k|^p dx \, |B_\rho|^{p(1-1/p)} = \int_{B_\rho} |\nabla f - k|^p dx.\end{aligned} \tag{3.3}$$

Therefore (3.1) holds for any $k \in R^N$. □

Proof of Theorem 3.1. To get the regularity, we need to prove the following inequality:

$$\int_{B_\rho^+} |\nabla u|^p dx \le c\rho^{N-p+p\lambda}. \tag{3.4}$$

Let us consider three different situations.

(1) If $B_{2R}(y_0) \subset B^+$, inequality (3.4)–(4.1) has been proved in [13], since it is related to interior regularity.

(2) If $B_R(y_0) \subset B^-$, by Extension theorem [5, page 254], if we can get $C^{1,\alpha}$ regularity of u in B^+, we can deduce the same result for u in B^-, so we need not care about this situation.

(3) If $B_R(y_0)\cap B^+\neq\emptyset$, we also give three different situations as follows:

(a) $y_0\in\Gamma$,

(b) $y_0\in B^-$,

(c) $y_0\in B^+$.

We only prove the situation (a), since the others can be transformed into the situation (a) or the interior regularity situation by applying the finitely covered theorem, see [13].

Assume $h\in L^s(B^+,R^N)$ with $s>N/(p-1)$, $\psi\in W^{1,m}(B^+)$ with $m>N$, we see that

$$\begin{gathered}\int_{B_R^+}|h|^{p/(p-1)}\leq\|h\|_s^{p/(p-1)}R^{N[1-p/s(p-1)]},\\ \int_{B_R^+}\left||\nabla\psi|^{p-2}\nabla\psi-\left(|\nabla\psi|^{p-2}\nabla\psi\right)_R\right|^{p/(p-1)}dx\leq c\int_{B_R^+}|\nabla\psi|^p dx\leq c\|\nabla\psi\|_m^p R^{N(1-p/m)}.\end{gathered}\tag{3.5}$$

Combining (2.7), (2.10), (2.11), (2.16), and (3.5), we have

$$\begin{aligned}\int_{B_\rho^+}&|\nabla u|^p dx\\ &\leq c\int_{B_\rho^+}|\nabla w|^p dx+c\int_{B_\rho^+}|\nabla u-\nabla w|^p dx\\ &\leq c\left(\frac{\rho}{R}\right)^N\int_{B_R^+}|\nabla u|^p dx+c\int_{B_R^+}|\nabla u-\nabla v|^p dx+c\int_{B_R^+}|\nabla v-\nabla w|^p dx\\ &\leq c\left(\frac{\rho}{R}\right)^N\int_{B_R^+}|\nabla u|^p dx\\ &\quad+c\bigg\{\left[\int_{B_R^+}\left(|\nabla u|^p+|u|^r\right)dx\right]^{1+\delta}+\left(\int_{B_R^+}|\nabla u|^p dx\right)^{q/p}+\left(\int_{B_R^+}|u|^r dx\right)^{q/p}\\ &\qquad+R^{Np(1-1/r-1/t)/(p-1)}+R^{N[1-p/s(p-1)]}+R^{N(1-p/m)}\bigg\}\\ &\leq c\left(\frac{\rho}{R}\right)^N\int_{B_R^+}|\nabla u|^p dx+c\left[\int_{B_R^+}\left(|\nabla u|^p+|u|^r\right)dx\right]^{1+\delta}\\ &\quad+c\left(\int_{B_R^+}|\nabla u|^p dx\right)^{q/p}+c\left(\int_{B_R^+}|u|^r dx\right)^{q/p}+cR^{N-p+p\lambda},\end{aligned}\tag{3.6}$$

where $\lambda=\min\{1-N(1/t+1/r-1/p)/(p-1),1-N/s(p-1),1-N/m\}$.

By $t>N/p$, we have the following.

(i) If $2\leq p<N$, then $1/t<p/N$, $1/t+1/r-1/p<p/N+(N-p)/Np-1/p=(p-1)/N$, $N(1/t+1/r-1/p)/(p-1)<1$.

(ii) If $p=N$, by $t>1$, we can assume that r is a positive number sufficiently large, such that: $1/t+1/r<1$, so $N(1/t+1/r-1/N)/(N-1)<(N/(N-1))(1-1/N)=1$.

Hence, if $2\leq p\leq N$, we always have $N(1/t+1/r-1/p)/(p-1)<1$.

Using $s>N/(p-1)$, $m>N$, by the definition of λ, we see that: $0<\lambda<1$.

In the meantime, by Poincare's inequality and Hölder's inequality, we also have

$$\begin{aligned}\int_{B_\rho^+}|u|^r dx &\le c\int_{B_\rho^+}|u_R|^r dx + c\int_{B_R^+}|u-u_R|^r dx\\ &\le c\left(\frac{\rho}{R}\right)^N\int_{B_R^+}|u|^r dx + cR^{r[1-N(1/p-1/r)]}\left(\int_{B_R^+}|\nabla u|^p dx\right)^{r/p},\end{aligned}\tag{3.7}$$

where

$$r\left[1-N\left(\frac{1}{p}-\frac{1}{r}\right)\right]=\begin{cases}\dfrac{Np}{N-p}\left[1-N\left(\dfrac{1}{p}-\dfrac{N-p}{Np}\right)\right]=0, & \text{if } 2\le p<N;\\ r\left[1-N\left(\dfrac{1}{N}-\dfrac{1}{r}\right)\right]=N, & \text{if } p=N.\end{cases}\tag{3.8}$$

Adding (3.7) to (3.6) and setting

$$\phi(R)=\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx,\tag{3.9}$$

we obtain

$$\phi(\rho)\le c\left[\left(\frac{\rho}{R}\right)^N+\chi(R)\right]\phi(R)+cR^{N-p+p\lambda},\tag{3.10}$$

where

$$\begin{aligned}\chi(R)=&\left[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\right]^\delta\\ &+\left(\int_{B_R^+}|\nabla u|^p dx\right)^{(q-p)/p}+\left(\int_{B_R^+}|u|^r dx\right)^{(q-p)/p}\\ &+\begin{cases}\left(\displaystyle\int_{B_R^+}|\nabla u|^p dx\right)^{(r-p)/p}, & \text{if } 2\le p<N;\\ R^N\left(\displaystyle\int_{B_R^+}|\nabla u|^p dx\right)^{(r-p)/p}, & \text{if } p=N.\end{cases}\end{aligned}\tag{3.11}$$

We can always get $\chi(R)\to 0$ as $R\to 0^+$. Applying [7, page 86, Lemma 2.1], we deduce that for ρ sufficiently small,

$$\int_{B_\rho^+}|\nabla u|^p dx\le\phi(\rho)\le c\rho^{N-p+p\lambda}.\tag{3.12}$$

□

By Dirichlet growth theorem (see [7, page 64, Theorem 1.1]), $u\in C^{0,\lambda}_{\mathrm{loc}}(\Gamma)$.

4. C^{1,α_1} regularity

Theorem 4.1. *Assume that* $H(x,u,\nabla u)$ *satisfies (1.2),* $g \in L^t(B^+)$ *with* $t > N$, $F \in C^{0,\beta}(B^+,R)$ *with* $\beta > 0$, $\psi \in C^{1,\gamma}(B^+)$ *for some* $\gamma > 0$. *If* $u \in \mathfrak{R}$ *makes (2.3) hold, then* $u \in C^{1,\alpha_1}(\Gamma)$ *for some* $0 < \alpha_1 < \sigma/p$.

To get our result, we need to prove the following inequality:

$$\int_{B_\rho^+} |\nabla u - (\nabla u)_\rho|^p dx \le c\rho^{N+p\alpha_1}. \tag{4.1}$$

It is easy to see that $|\nabla\psi|^{p-2}\nabla\psi \in C^{0,\gamma}(B^+)$ if $\psi \in C^{1,\gamma}(B^+)$ and $2 \le p \le N$.
Utilizing the conditions of Theorem 4.1, we see that:

$$\int_{B_R^+} |F - F_R|^{p/(p-1)} dx \le \|F\|^{p/(p-1)}_{C^{0,\beta}(B^+)} R^{N+\beta p/(p-1)}, \tag{4.2}$$

$$\begin{aligned} &|\nabla\psi(x)|^{p-2}\nabla\psi(x) - |\nabla\psi(y)|^{p-2}|\nabla\psi(y)| \\ &\quad \le \big\||\nabla\psi|^{p-2}\nabla\psi\big\|_{C^{0,\gamma}(B^+)}|x-y|^\gamma, \quad \forall x,y \in B^+. \end{aligned} \tag{4.3}$$

By $\psi \in C^{1,\gamma}(B^+)$, we can get $|\nabla\psi|^{p-1} \le c$, so combining condition (vi) we have

$$|A(x,\nabla\psi) - A(y,\nabla\psi)| \le c|x-y|^{\alpha_0}. \tag{4.4}$$

By condition (vii), (4.3), and (4.4), we see that for all $\phi \in W_0^{1,p}(B_R^+)$, there holds

$$\begin{aligned} &\int_{B_R^+} A(x,\nabla\psi)\cdot\nabla\phi\, dx \\ &\quad = \int_{B_R^+} \Big(A(x,\nabla\psi) - \big(A(x,\nabla\psi)\big)_R\Big)\cdot\nabla\phi\, dx \\ &\quad = \int_{B_R^+} \fint_{B_R} \big[A(x,\nabla\psi(x)) - A(y,\nabla\psi(y))\big]\cdot\nabla\phi(x)\,dy\,dx \\ &\quad = \int_{B_R^+} \fint_{B_R} \big[A(x,\nabla\psi(x)) - A(x,\nabla\psi(y)) \\ &\qquad\qquad + A(x,\nabla\psi(y)) - A(y,\nabla\psi(y))\big]\cdot\nabla\phi(x)\,dy\,dx \\ &\quad \le c\int_{B_R^+} \fint_{B_R} \Big(|x-y|^{\alpha_0} + \Big||\nabla\psi(x)|^{p-2}\nabla\psi(x) - |\nabla\psi(y)|^{p-2}\nabla\psi(y)\Big|\Big)|\nabla\phi(x)|\,dy\,dx \\ &\quad \le c\int_{B_R^+} (R^{\alpha_0} + R^\gamma)|\nabla\phi(x)|\,dx. \end{aligned} \tag{4.5}$$

From last formula, we see that

$$\int_{B_R^+} A(x,\nabla\psi)\cdot\nabla\phi\, dx \le c\int_{B_R^+} (R^{\alpha_0} + R^\gamma)|\nabla\phi(x)|\,dx. \tag{4.6}$$

In the following, we give two lemmas which will be used in the proof of Theorem 4.1.

Lemma 4.2. *Assume that* $\psi \in C^{1,\gamma}(B^+)$, $w, v \in W^{1,p}(B_R^+)$ *solve the Dirichlet problem (2.5), (2.9), respectively, then there holds*

$$\int_{B_R^+} |\nabla v - \nabla w|^p dx \le c\Big(R^{N+\gamma p/(p-1)} + R^{N+\alpha_0 p/(p-1)}\Big). \tag{4.7}$$

Proof. Inserting $\phi = v - w$ in (2.5) and (2.9), by (4.6), monotone inequality (iii), and Hölder inequality, we have

$$\begin{aligned}\int_{B_R^+} |\nabla v - \nabla w|^p dx &\le c\int_{B_R^+} \big(A(x,\nabla v) - A(x,\nabla w)\big)\cdot(\nabla v - \nabla w)dx \\ &= c\int_{B_R^+} A(x,\nabla\psi)\cdot(\nabla v - \nabla w)dx \\ &\le c\int_{B_R^+} (R^{\alpha_0} + R^{\gamma})|\nabla v - \nabla w|dx \\ &\le c(R^{\gamma+N(p-1)/p} + R^{\alpha_0+N(p-1)/p})\left(\int_{B_R^+} |\nabla v - \nabla w|^p dx\right)^{1/p}.\end{aligned} \tag{4.8}$$

From last formula, we get (4.7). □

Lemma 4.3. *Assume that* $A(x,\xi)$ *satisfies condition (i)–(vii),* u *is a weak solution to obstacle problem (1.1), where* H *verifies (1.2),* $g \in L^t(B^+)$, $t > N$; $F \in C^{0,\beta}(B^+)$, $\beta > 0$; $\psi \in C^{1,\gamma}(B^+)$, $\gamma > 0$. $v \in W^{1,p}(B_R^+)$ *solves the Dirichlet problem (2.9), then there holds*

$$\begin{aligned}\int_{B_R^+} |\nabla u - \nabla v|^p dx \le c\Big\{&\Big[\int_{B_R^+} (|\nabla u|^p + |u|^r)dx\Big]^{1+\delta} + R^{Np(1-1/r-1/t)/(p-1)} \\ &+ R^{N+\beta p/(p-1)} + R^{N+\gamma p/(p-1)} + R^{N+\alpha_0 p/(p-1)}\Big\},\end{aligned} \tag{4.9}$$

where $\delta = (r-p)/r(p-1) > 0$.

Proof. Combining monotone inequality (iii), and (1.2), (2.14), (2.18), (4.6), Hölder inequality, we have

$$\begin{aligned}\int_{B_R^+} |\nabla u - \nabla v|^p dx &\le c\int_{B_R^+} [A(x,\nabla u) - A(x,\nabla v)]\cdot(\nabla u - \nabla v)dx \\ &\le c\int_{B_R^+} \big[H(x,u,\nabla u)(u - v) + \big(F(x,u) - A(x,\nabla\psi)\big)\cdot(\nabla u - \nabla v)\big]dx \\ &\le c\int_{B_R^+} (|\nabla u|^{p(1-1/r)} + |u|^{r-1} + |g|)|u - v|dx \\ &\quad + c\int_{B_R^+} |F - F_R||\nabla u - \nabla v|dx + c\int_{B_R^+} (R^{\alpha_0} + R^{\gamma})|\nabla u - \nabla v|dx\end{aligned}$$

$$
\begin{aligned}
&\leq c\Big\{\Big(\int_{B_R^+}(|\nabla u|^p+|u|^r+|g|^{r/(r-1)})\,dx\Big)^{1-1/r}\Big(\int_{B_R^+}|u-v|^r dx\Big)^{1/r}\\
&\quad+\Big(\int_{B_R^+}|F-F_R|^{p'}dx\Big)^{1/p'}\Big(\int_{B_R^+}|\nabla u-\nabla v|^p dx\Big)^{1/p}\\
&\quad+\int_{B_R^+}(R^{\alpha_0}+R^{\gamma})|\nabla u-\nabla v|\,dx\Big\}\\
&\leq c\Big\{\Big[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\Big]^{1-1/r}+\|g\|_t R^{N(1-1/r-1/t)}\Big\}\\
&\quad\times R^{1-N(1/p-1/r)}\Big(\int_{B_R^+}|\nabla u-\nabla v|^p dx\Big)^{1/p}\\
&\quad+c\Big\{\Big(\int_{B_R^+}|F-F_R|^{p'}dx\Big)^{1/p'}+R^{\gamma+N(p-1)/p}+R^{\alpha_0+N(p-1)/p}\Big\}\\
&\quad\times\Big(\int_{B_R^+}|\nabla u-\nabla v|^p dx\Big)^{1/p}.
\end{aligned}
\tag{4.10}
$$

Since $0<R\leq 1$, by (4.2), and last formula, we have

$$
\begin{aligned}
\int_{B_R^+}|\nabla u-\nabla v|^p dx\leq c\Big\{&\Big[\int_{B_R^+}(|\nabla u|^p+|u|^r)\,dx\Big]^{1+\delta}+R^{Np(1-1/r-1/t)/(p-1)}\\
&+R^{N+\beta p/(p-1)}+R^{N+\gamma p/(p-1)}+R^{N+\alpha_0\cdot p/(p-1)}\Big\}.
\end{aligned}
\tag{4.11}
$$

Hence (4.9) holds. □

Similar to the proof of (3.12), we get for any $\mu\in(0,1)$

$$
\int_{B_\rho^+}|\nabla u|^p dx,\qquad \int_{B_\rho^+}|\nabla v|^p dx,\qquad \int_{B_\rho^+}|u|^r dx\leq c\rho^{N-p+p\mu}.
\tag{4.12}
$$

In view of $t>N$, we have the following.

(i) If $2\leq p<N$, $1-1/r-1/t>1-1/r-1/N=(p-1)/p$, from which we get $Np(1-1/r-1/t)/(p-1)>N$.

(ii) If $p=N$, we can assume that r is a positive number large such that $1/t+1/r<1/N$, $Np(1-1/r-1/t)/(p-1)>N^2(1-1/N)/(N-1)=N$.

Hence,we always have $Np(1-1/r-1/t)/(p-1)>N$ when $2\leq p\leq N$.

In the following, we prove Theorem 4.1.

By (4.2), (4.3), (4.9), Lemmas 2.3, 4.2, and 4.3, we have

$$\begin{aligned}
&\int_{B_\rho^+} |\nabla u-(\nabla u)_\rho|^p dx \\
&\quad\le c\int_{B_\rho^+} |\nabla u-(\nabla w)_\rho|^p dx \\
&\quad\le c\int_{B_\rho^+} |\nabla w-(\nabla w)_\rho|^p dx + c\int_{B_\rho^+} |\nabla u-\nabla v|^p dx + c\int_{B_\rho^+} |\nabla v-\nabla w|^p dx \\
&\quad\le c\int_{B_\rho^+} |\nabla w-(\nabla w)_\rho|^p dx \\
&\qquad + c\bigg\{\bigg[\int_{B_R^+} (|\nabla u|^p+|u|^r)dx\bigg]^{1+\delta} \\
&\qquad\qquad + R^{Np(1-1/r-1/t)/(p-1)} + R^{N+\beta p/(p-1)} + R^{N+\gamma p/(p-1)} + R^{N+\alpha_0 p/(p-1)}\bigg\} \\
&\qquad + c\Big(R^{N+\gamma p/(p-1)} + R^{N+\alpha_0 p/(p-1)}\Big) \\
&\quad\le c\Big(\frac{\rho}{R}\Big)^{N+\sigma}\int_{B_R^+} |\nabla u-(\nabla u)_R|^p dx \\
&\qquad + c\bigg\{\bigg[\int_{B_R^+} (|\nabla u|^p+|u|^r)dx\bigg]^{1+\delta} \\
&\qquad\qquad + R^{Np(1-1/r-1/t)/(p-1)} + R^{N+\beta p/(p-1)} + R^{N+\gamma p/(P-1)} + R^{N+\alpha_0 p/(p-1)}\bigg\} \\
&\quad\le c\Big(\frac{\rho}{R}\Big)^{N+\sigma}\int_{B_R^+} |\nabla u-(\nabla u)_R|^p dx \\
&\qquad + c\Big(R^{(1+\delta)(N-p+p\mu)} + R^{Np(1-1/r-1/t)/(p-1)} + R^{N+\beta p/(p-1)} \\
&\qquad\qquad + R^{N+\gamma p/(P-1)} + R^{N+\alpha_0 p/(p-1)}\Big).
\end{aligned} \tag{4.13}$$

We can select μ sufficiently close to 1, such that $(1+\delta)(N-p+p\mu)>N$. Hence we get

$$\int_{B_\rho^+} |\nabla u-(\nabla u)_\rho|^p dx \le c\Big(\frac{\rho}{R}\Big)^{N+\sigma}\int_{B_R} |\nabla u-(\nabla u)_R|^p dx + cR^{N+p\alpha_1} \tag{4.14}$$

for some $0<\alpha_1<\sigma/p$.

Applying [7, page 86, Lemma 2.1], we get

$$\int_{B_\rho^+} |\nabla u-(\nabla u)_\rho|^p dx \le c\rho^{N+p\alpha_1} \tag{4.15}$$

for ρ sufficiently small. By [7, page 72, Theorem 1.3], we obtain that $u\in C^{1,\alpha_1}_{\mathrm{loc}}(\Gamma)$.

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