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Arbitrary decays for a viscoelastic equation

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Abstract

In this paper, we consider the nonlinear viscoelastic equation $|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + |u|^\rho u = 0$, in a bounded domain with initial conditions and Dirichlet boundary conditions. We prove an arbitrary decay result for a class of kernel function g without setting the function g itself to be of exponential (polynomial) type, which is a necessary condition for the exponential (polynomial) decay of the solution energy for the viscoelastic problem. The key ingredient in the proof is based on the idea of Pata (Q Appl Math 64:499-513, 2006) and the work of Tatar (J Math Phys 52:013502, 2010), with necessary modification imposed by our problem.

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1 Introduction

It is well known that viscoelastic materials have memory effects. These properties are due to the mechanical response influenced by the history of the materials themselves. As these materials have a wide application in the natural sciences, their dynamics are of great importance and interest. From the mathematical point of view, their memory effects are modeled by an integro-differential equations. Hence, questions related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years. Many authors have focused on this problem for the last two decades and several results concerning existence, decay and blow-up have been obtained, see [1-28] and the reference therein.

In [3], Cavalcanti et al. studied the following problem

$$\begin{aligned} &|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma \Delta u_t = 0, \quad \text{in } \Omega \times (0, \infty), \\ &u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \\ &u(x, t) = 0, x \in \partial\Omega, t \geq 0, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$, $\gamma \geq 0$, $0 < \rho \leq \frac{2}{N-2}$ if $N \geq 3$ or $\rho > 0$ if $N = 1, 2$, and the function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-increasing function. This type of equations usually arise in the theory of viscoelasticity when the material density varies according to the velocity. In that paper, they proved a global existence result of weak solutions for $\gamma \geq 0$ and a uniform decay result for $\gamma > 0$.

Precisely, they showed that the solutions goes to zero in an exponential rate for $\gamma > 0$ and g is a positive bounded C^1 -function satisfying

$$1 - \int_0^\infty g(s) \, ds = 1 - l > 0, \tag{1.2}$$

and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \tag{1.3}$$

for all $t \geq 0$ and some positive constants ξ_1 and ξ_2 . Later, this result was extended by Messaoudi and Tatar [15] to a situation where a nonlinear source term is competing with the dissipation terms induced by both the viscoelasticity and the viscosity. Recently Messaoudi and Tatar [14] studied problem (1.1) for the case of $\gamma = 0$, they improved the result in [3] by showing that the solution goes to zero with an exponential or polynomial rate, depending on the decay rate of the relaxation function g .

The assumptions (1.2) and (1.3), on g , are frequently encountered in the linear case ($\rho = 0$), see [1,2,4-6,13,22,23,29-31]. Lately, these conditions have been weakened by some researchers. For instance, instead of (1.3) Furati and Tatar [8] required the functions $e^{\alpha t} g(t)$ and $e^{\alpha t} g'(t)$ to have sufficiently small L^1 -norm on $(0, \infty)$ for some $\alpha > 0$ and they can also have an exponential decay of solutions. In particular, they do not impose a rate of decreasingness for g . Later on Messaoudi and Tatar [21] improved this result further by removing the condition on g' . They established an exponential decay under the conditions $g'(t) \leq 0$ and $e^{\alpha t} g(t) \in L^1(0, \infty)$ for some large $\alpha > 0$. This last condition was shown to be necessary condition for exponential decay [7]. More recently Tatar [25] investigated the asymptotic behavior to problem (1.1) with $\rho = \gamma = 0$ when $h(t)g(t) \in L^1(0, \infty)$ for some nonnegative function $h(t)$. He generalized earlier works to an arbitrary decay not necessary of exponential or polynomial rate.

Motivated by previous works [21,25], in this paper, we consider the initial boundary value problem for the following nonlinear viscoelastic equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s) \, ds + |u|^\rho u = 0, \quad \text{in } \Omega \times (0, \infty), \tag{1.4}$$

with initial conditions

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \tag{1.5}$$

and boundary condition

$$u(x, t) = 0, x \in \partial\Omega, t \geq 0, \tag{1.6}$$

where $\Omega \subset R^N$, $N \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$. Here ρ , $p > 0$ and g represents the kernel of the memory term, with conditions to be stated later [see assumption (A1)-(A3)].

We intend to study the arbitrary decay result for problem (1.4)-(1.6) under the weaker assumption on g , which is not necessarily decaying in an exponential or polynomial fashion. Indeed, our result will be established under the conditions $g'(t) \leq 0$ and $\int_0^\infty \xi(s)g(s) \, ds < \infty$ for some nonnegative function $\xi(t)$. Therefore, our result allows a

larger class of relaxation functions and improves some earlier results concerning the exponential decay or polynomial decay.

The content of this paper is organized as follows. In Section 2, we give some lemmas and assumptions which will be used later, and we mention the local existence result in Theorem 2.2. In Section 3, we establish the statement and proof of our result related to the arbitrary decay.

2 Preliminary results

In this section, we give some assumptions and lemmas which will be used throughout this work. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual inner products and norms.

Lemma 2.1. (Sobolev-Poincaré inequality) *Let $2 \leq p \leq \frac{2N}{N-2}$, the inequality*

$$\|u\|_p \leq c_s \|\nabla u\|_2 \quad \text{for } u \in H_0^1(\Omega),$$

holds with the optimal positive constant c_s , where $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$.

Assume that ρ satisfies

$$0 < \rho \leq \frac{2}{N-2} \text{ if } N \geq 3 \text{ or } \rho > 0 \text{ if } N = 1, 2. \tag{2.1}$$

With regards to the relaxation function $g(t)$, we assume that it verifies

(A1) $g(t) \geq 0$, for all $t \geq 0$, is a continuous function satisfying

$$0 < \int_0^\infty g(s) \, ds = l < 1. \tag{2.2}$$

(A2) $g'(t) \leq 0$ for almost all $t > 0$.

(A3) There exists a positive nondecreasing function $\zeta(t): [0, \infty) \rightarrow (0, \infty)$ such that

$\frac{\zeta'(t)}{\zeta(t)} = \eta(t)$ is a decreasing function and

$$\int_0^\infty \zeta(s)g(s) \, ds < \infty. \tag{2.3}$$

Now, we state, without a proof, the existence result of the problem (1.4)-(1.6) which can be established by Faedo-Galerkin methods, we refer the reader to [3,5].

Theorem 2.2. *Suppose that (2.1) and (A1) hold, and that $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$.*

Assume $0 < \rho \leq \frac{2}{N-2}$, if $N \geq 3$, $p > 0$, if $N = 1, 2$. Then there exists at least one global solution u of (1.4)-(1.6) satisfying

$$u \in L^\infty([0, \infty); H_0^1(\Omega)), \quad u_t \in L^\infty([0, \infty); H_0^1(\Omega)), \quad u_{tt} \in L^2([0, \infty); L^2(\Omega)).$$

Next, we introduce the modified energy functional for problem (1.4)-(1.6)

$$\begin{aligned} E(t) = & \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2} \left(1 - \int_0^t g(s) \, ds \right) \|\nabla u(t)\|_2^2 + \frac{1}{2} \|\nabla u_t(t)\|_2^2 \\ & + \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{p+2} \|u(t)\|_{p+2}^{p+2}, \end{aligned} \tag{2.4}$$

where

$$(g \circ \nabla u)(t) = \int_0^t \int_{\Omega} g(t-s) |\nabla u(t) - \nabla u(s)|^2 dx ds. \tag{2.5}$$

Lemma 2.3. *Let u be the solution of (1.4)-(1.6), then the modified energy $E(t)$ satisfies*

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t) \|\nabla u(t)\|_2^2 \leq \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \tag{2.6}$$

Proof. Multiplying Eq. (1.4) by u_t and integrating it over Ω , then using integration by parts and the assumption (A1)-(A2), we obtain (2.6).

Remark. It follows from Lemma 2.3 that the energy is uniformly bounded by $E(0)$ and decreasing in t . Besides, from the definition of $E(t)$ and (2, 2), we note that

$$(1-l) \|\nabla u\|_2^2 + \|\nabla u_t(t)\|_2^2 + (g \circ \nabla u)(t) \leq 2E(0), \quad \forall t \geq 0. \tag{2.7}$$

3 Decay of the solution energy

In this section, we shall state and prove our main result. For this purpose, we first define the functional

$$L(t) = E(t) + \sum_{i=1}^3 \lambda_i \Phi_i(t), \tag{3.1}$$

where λ_i are positive constants, $i = 1, 2, 3$ to be specified later and

$$\Phi_1(t) = \frac{1}{\rho+1} \int_{\Omega} |u_t|^\rho u_t u \, dx + \int_{\Omega} \nabla u_t(t) \nabla u(t) \, dx, \tag{3.2}$$

$$\Phi_2(t) = \int_{\Omega} \left(\Delta u_t - \frac{1}{\rho+1} |u_t|^\rho u_t \right) \int_0^t g(t-s) (u(t) - u(s)) \, ds \, dx, \tag{3.3}$$

$$\Phi_3(t) = \int_{\Omega} \int_0^t H(t-s) |\nabla u(s)|^2 \, ds \, dx, \tag{3.4}$$

here

$$H(t) = \xi(t)^{-1} \int_t^\infty g(s) \xi(s) \, ds.$$

Remark. This functional was first introduced by Tatar [25] for the case of $\rho = 0$ and without imposing the dispersion term and forcing term as far as (1.4) is concerned.

The following Lemma tells us that $L(t)$ and $E(t) + \Phi_3(t)$ are equivalent.

Lemma 3.1. *There exists two positive constants β_1 and β_2 such that the relation*

$$\beta_1(E(t) + \Phi_3(t)) \leq L(t) \leq \beta_2(E(t) + \Phi_3(t)), \tag{3.5}$$

holds for all $t \geq 0$ and λ_i small, $i = 1, 2$.

Proof. By Hölder inequality Young's inequality Lemma 2.1, (2.7) and (2.2), we deduce that

$$\begin{aligned} \left| \int_{\Omega} |u_t|^\rho u_t u \, dx \right| &\leq \frac{1}{2} \|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{2} \|u\|_2^2 \\ &\leq \frac{c_s^{2(\rho+1)}}{2} \|\nabla u_t\|_2^{2(\rho+1)} + \frac{c_s^2}{2} \|\nabla u\|_2^2 \\ &\leq \frac{\alpha_1}{2} \|\nabla u_t\|_2^2 + \frac{c_s^2}{2} \|\nabla u\|_2^2, \\ \left| \int_{\Omega} \nabla u_t(t) \nabla u(t) \, dx \right| &\leq \frac{1}{2} (\|\nabla u_t\|_2^2 + \|\nabla u\|_2^2), \end{aligned}$$

$$\begin{aligned} &\left| \int_{\Omega} \nabla u_t \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \right| \\ &\leq \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \int_{\Omega} \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) \right)^2 \, ds \, dx \\ &\leq \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{l}{2} (g \circ \nabla u)(t), \end{aligned}$$

and

$$\begin{aligned} &\left| \int_{\Omega} |u_t|^\rho u_t \int_0^t g(t-s)(u(t) - u(s)) \, ds \, dx \right| \\ &\leq \frac{1}{2} \|u_t\|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{2} \int_{\Omega} \left(\int_0^t g(t-s)(u(t) - u(s)) \right)^2 \, ds \, dx \\ &\leq \frac{\alpha_1}{2} \|\nabla u_t\|_2^2 + \frac{lc_s^2}{2} (g \circ \nabla u)(t), \end{aligned}$$

where $\alpha_1 = c_s^{2(\rho+1)}(2E(0))^\rho$. Therefore, from above estimates, the definition of $E(t)$ by (2.4) and (2.2), we have

$$\begin{aligned} L(t) &= E(t) + \sum_{i=1}^3 \lambda_i \Phi_i(t) \\ &\leq E(t) + c_1 \|\nabla u\|_2^2 + c_2 \|\nabla u_t\|_2^2 + c_3 (g \circ \nabla u)(t) + \lambda_3 \Phi_3(t) \end{aligned}$$

and

$$\begin{aligned}
 L(t) &\geq E(t) - c_1 \|\nabla u\|_2^2 - c_2 \|\nabla u_t\|_2^2 - c_3 (g \circ \nabla u)(t) + \lambda_3 \Phi_3(t) \\
 &\geq \frac{1}{\rho + 2} \|u_t\|_{\rho+2}^{\rho+2} + \left(\frac{1}{2}(1-l) - c_1\right) \|\nabla u(t)\|_2^2 \\
 &\quad + \left(\frac{1}{2} - c_2\right) \|\nabla u_t(t)\|_2^2 + \left(\frac{1}{2} - c_3\right) (g \circ \nabla u)(t) \\
 &\quad + \frac{1}{\rho + 2} \|u(t)\|_{\rho+2}^{\rho+2} + \lambda_3 \Phi_3(t),
 \end{aligned}$$

where $c_1 = \frac{\lambda_1(c_s^2 + \rho + 1)}{2(\rho + 1)}$, $c_2 = \frac{(\lambda_1 + \lambda_2)(\alpha_1 + \rho + 1)}{2(\rho + 1)}$, and $c_3 = \frac{l(\rho + 1 + c_s^2)\lambda_2}{2(\rho + 1)}$. Hence, selecting λ_i , $i = 1, 2$ such that

$$\begin{aligned}
 \lambda_1 &< \min \left\{ \frac{(\rho + 1)(1-l)}{c_s^2 + \rho + 1}, \frac{\rho + 1}{\alpha_1 + \rho + 1} \right\}, \\
 \lambda_2 &< \min \left\{ \frac{\rho + 1}{l(c_s^2 + \rho + 1)}, \frac{\rho + 1}{\alpha_1 + \rho + 1} - \lambda_1 \right\},
 \end{aligned}$$

and again from the definition of $E(t)$, there exist two positive constants β_1 and β_2 such that

$$\beta_1(E(t) + \Phi_3(t)) \leq L(t) \leq \beta_2(E(t) + \Phi_3(t)), \quad t \geq 0.$$

To obtain a better estimate for $\int_{\Omega} \nabla u \int_0^t g(t-s) \nabla u(s) \, ds \, dx$, we need the following Lemma which repeats Lemma 2 in [25].

Lemma 3.2. *For $t \geq 0$, we have*

$$\begin{aligned}
 \int_{\Omega} \nabla u \int_0^t g(t-s) \nabla u(s) \, ds \, dx &= \frac{1}{2} \left(\int_0^t g(s) \, ds \right) \|\nabla u\|_2^2 + \frac{1}{2} \int_0^t g(t-s) \|\nabla u(s)\|_2^2 \, ds \\
 &\quad - \frac{1}{2} (g \circ \nabla u)(t).
 \end{aligned} \tag{3.6}$$

Proof. Straightforward computations yield this identity.

Now, we are ready to state and prove our result. First, we introduce the following notations as in [24,25]. For every measurable set $A \subset R^+$, we define the probability measure \widehat{g} by

$$\widehat{g}(A) = \frac{1}{l} \int_A g(s) \, ds.$$

The flatness set and the flatness rate of g are defined by

$$F_g = \{s \in R^+ \mid g(s) > 0 \text{ and } g'(s) = 0\} \tag{3.7}$$

and

$$R_g = \widehat{g}(F_g) = \frac{1}{l} \int_{F_g} g(s) \, ds. \tag{3.8}$$

Before proceeding, we note that there exists $t_0 > 0$ such that

$$\int_0^t g(s) \, ds \geq \int_0^{t_0} g(s) \, ds = g_* > 0, \quad \forall t \geq t_0, \tag{3.9}$$

since g is nonnegative and continuous.

Theorem 3.3. *Let $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be given. Suppose that (A1)-(A3), (2, 1) and the hypothesis on p hold. Assume further that $R_g < \frac{g_* \alpha}{l(2\alpha + 2c_s^2)}$, $H(0) < \frac{g_*(4+l)-3l}{4}$ and $g_* > \frac{3l}{4+l}$ with*

$$\alpha = \frac{(g_*(4+l) - 3l)(1-l)^p}{8c_s^{2(p+1)}(2E(0))^p}.$$

Then the solution energy of (1.4)-(1.6) satisfies

$$E(t) \leq K\xi(t)^{-\mu}, \quad t \geq 0,$$

where μ and K are positive constants.

Proof. In order to obtain the decay result of $E(t)$, it suffices to prove that of $L(t)$. To this end, we need to estimate the derivative of $L(t)$. It follows from (3.2) and Eq. (1.4) that

$$\begin{aligned} \Phi_1'(t) &= \frac{1}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \|\nabla u\|_2^2 \\ &\quad + \int_{\Omega} \nabla u \int_0^t g(t-s) \nabla u(s) \, ds \, dx - \|u\|_{\rho+2}^{\rho+2}, \end{aligned}$$

which together with the identity (3.6) and (2.2) gives

$$\begin{aligned} \Phi_1'(t) &\leq \frac{1}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - \left(1 - \frac{l}{2}\right) \|\nabla u\|_2^2 \\ &\quad + \frac{1}{2} \int_0^t g(t-s) \|\nabla u(s)\|_2^2 \, ds - \frac{1}{2} (g \circ \nabla u)(t) - \|u\|_{\rho+2}^{\rho+2}. \end{aligned} \tag{3.10}$$

Next, we would like to estimate $\Phi_2'(t)$. Taking a derivative of Φ_2 in (3.3) and using Eq. (1.4) to get

$$\begin{aligned} \Phi_2'(t) &= \left(1 - \int_0^t g(s) \, ds\right) \int_{\Omega} \nabla u(t) \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\ &\quad + \int_{\Omega} \left(\int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds\right)^2 \, dx - \frac{1}{\rho+1} \left(\int_0^t g(s) \, ds\right) \|u_t\|_{\rho+2}^{\rho+2} \\ &\quad - \left(\int_0^t g(s) \, ds\right) \|\nabla u_t\|_2^2 - \int_{\Omega} \nabla u_t(t) \int_0^t g'(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\ &\quad - \frac{1}{\rho+1} \int_{\Omega} |u_t|^{\rho} u_t \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, dx \\ &\quad + \int_{\Omega} |u|^{\rho} u \int_0^t g(t-s) (u(t) - u(s)) \, ds \, dx. \end{aligned} \tag{3.11}$$

We now estimate the first two terms on the right-hand side of (3.11) as in [25].
 Indeed, for all measure set A and F such that $A = R^+ - F$, we have

$$\begin{aligned}
 & \int_{\Omega} \nabla u(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
 &= \int_{\Omega} \nabla u(t) \int_{A \cap [0,t]} g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
 & \quad + \int_{\Omega} \nabla u(t) \int_{F \cap [0,t]} g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
 & \leq \int_{\Omega} \nabla u(t) \int_{A \cap [0,t]} g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
 & \quad + \left(\int_{F \cap [0,t]} g(s) \, ds \right) \|\nabla u\|_2^2 - \int_{\Omega} \nabla u(t) \int_{F \cap [0,t]} g(t-s) \nabla u(s) \, ds \, dx.
 \end{aligned} \tag{3.12}$$

To simplify notations, we denote

$$A_t = A \cap [0, t] \text{ and } F_t = F \cap [0, t].$$

Using Hölder inequality Young's inequality and (2.2), we see that, for $\delta_1 > 0$,

$$\begin{aligned}
 & \int_{\Omega} \nabla u(t) \int_{A_t} g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
 & \leq \delta_1 \|\nabla u\|_2^2 + \frac{l}{4\delta_1} \int_{\Omega} \int_{A_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\Omega} \nabla u(t) \int_{F_t} g(t-s) \nabla u(s) \, ds \, dx \\
 & \leq \frac{1}{2} \left(\int_{F_t} g(s) \, ds \right) \|\nabla u\|_2^2 + \frac{1}{2} \int_{F_t} g(t-s) \|\nabla u(s)\|_2^2 \, ds.
 \end{aligned}$$

Thus, from the definition of $\widehat{g}(F)$ by (3.8), (3.12) becomes

$$\begin{aligned}
 & \int_{\Omega} \nabla u(t) \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \\
 & \leq \left(\delta_1 + \frac{3}{2} \widehat{g}(F) \right) \|\nabla u\|_2^2 + \frac{l}{4\delta_1} \int_{\Omega} \int_{A_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\
 & \quad + \frac{1}{2} \int_{F_t} g(t-s) \|\nabla u(s)\|_2^2 \, ds.
 \end{aligned} \tag{3.13}$$

The second term on the right-hand side of (3.11) can be estimated as follows (see [25]), for $\delta_2 > 0$,

$$\begin{aligned}
 & \int_{\Omega} \left(\int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right)^2 dx \\
 &= \int_{\Omega} \left(\int_{A_t} g(t-s) (\nabla u(t) - \nabla u(s)) \, ds + \int_{F_t} g(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right)^2 dx \\
 &\leq \left(1 + \frac{1}{\delta_2} \right) l \int_{\Omega} \int_{A_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\
 &\quad + (1 + \delta_2) l \widehat{g}(F) \int_{\Omega} \int_{F_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx.
 \end{aligned} \tag{3.14}$$

Using Hölder inequality Young's inequality and (A2) to deal with the fifth term, for $\delta_3 > 0$,

$$\begin{aligned}
 & \left| \int_{\Omega} \nabla u_t(t) \int_0^t g'(t-s) (\nabla u(t) - \nabla u(s)) \, ds \, dx \right| \\
 &\leq \delta_3 \| \nabla u_t \|_2^2 + \frac{1}{4\delta_3} \int_{\Omega} \left(\int_0^t g'(t-s) (\nabla u(t) - \nabla u(s)) \, ds \right)^2 dx \\
 &\leq \delta_3 \| \nabla u_t \|_2^2 - \frac{g(0)}{4\delta_3} (g' \circ \nabla u)(t).
 \end{aligned} \tag{3.15}$$

Exploiting Hölder inequality Young's inequality Lemma 2.1 and (A2) to estimate the sixth term, for $\delta_4 > 0$,

$$\begin{aligned}
 & \left| \frac{1}{\rho+1} \int_{\Omega} |u_t|^{\rho} u_t \int_0^t g'(t-s) (u(t) - u(s)) \, ds \, dx \right| \\
 &\leq \frac{1}{\rho+1} \left(\delta_4 \| u_t \|_{2(\rho+1)}^{2(\rho+1)} + \frac{1}{4\delta_4} \int_{\Omega} \left(\int_0^t g'(t-s) (u(t) - u(s)) \, ds \right)^2 dx \right) \\
 &\leq \frac{1}{\rho+1} \left(\delta_4 c_s^{2(\rho+1)} \| \nabla u_t \|_2^{2(\rho+1)} - \frac{g(0)c_s^2}{4\delta_4} (g' \circ \nabla u)(t) \right) \\
 &\leq \frac{1}{\rho+1} \left(\alpha_1 \delta_4 \| \nabla u_t \|_2^2 - \frac{g(0)c_s^2}{4\delta_4} (g' \circ \nabla u)(t) \right).
 \end{aligned} \tag{3.16}$$

For the last term, thanks to Hölder inequality Young's inequality Lemma 2.1, (2.7), (2.2) and (3.8), we have, for $\delta_5 > 0$,

$$\begin{aligned}
 & \left| \int_{\Omega} |u|^{\rho} u \int_0^t g(t-s) (u(t) - u(s)) \, ds \, dx \right| \\
 &\leq \delta_5 \int_{\Omega} |u|^{2(\rho+1)} \, dx + \frac{1}{4\delta_5} \int_{\Omega} \left(\int_0^t g(t-s) (u(t) - u(s)) \, ds \right)^2 dx \\
 &\leq \delta_5 \alpha_2 \| \nabla u \|_2^2 + \frac{lc_s^2}{2\delta_5} \int_{\Omega} \int_{A_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\
 &\quad + \frac{lc_s^2}{2\delta_5} \widehat{g}(F) \int_{\Omega} \int_{F_t} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx,
 \end{aligned} \tag{3.17}$$

where $\alpha_2 = c_s^{2(\rho+1)} \left(\frac{2E(0)}{1-l} \right)^{\rho}$. Thus, gathering these estimates (3.13)-(3.17) and using (3.9), we obtain, for $t \geq t_0$,

$$\begin{aligned} \Phi'_2(t) &\leq \left((1 - g_*) \left(\delta_1 + \frac{3}{2} \widehat{lg}(F) \right) + \delta_5 \alpha_2 \right) \| \nabla u \|_2^2 \\ &\quad - \left(\frac{g(0)}{4\delta_3} + \frac{g(0)c_s}{4(\rho+1)\delta_4} \right) (g' \circ \nabla u)(t) + \frac{1 - g_*}{2} \int_{F_t} g(t-s) \| \nabla u(s) \|_2^2 \, ds \\ &\quad + \left(\delta_3 + \frac{\alpha_1 \delta_4}{\rho+1} - g_* \right) \| \nabla u_t \|_2^2 - \frac{g_*}{(\rho+1)} \| u_t \|_{\rho+2}^{\rho+2} \\ &\quad + l \left(1 + \frac{1}{\delta_2} + \frac{c_s^2}{2\delta_5} + \frac{1 - g_*}{4\delta_1} \right) \int_{\Omega} \int_{A_t} g(t-s) | \nabla u(t) - \nabla u(s) |^2 \, ds \, dx \\ &\quad + \left(1 + \delta_2 + \frac{c_s^2}{2\delta_5} \right) \widehat{lg}(F) \int_{\Omega} \int_{F_t} g(t-s) | \nabla u(t) - \nabla u(s) |^2 \, ds \, dx. \end{aligned} \tag{3.18}$$

Further, taking a derivative of $\Phi_3(t)$, using the fact that $\frac{\xi'(t)}{\xi(t)} = \eta(t)$ is a decreasing function and the definition of $\Phi_3(t)$ by (3.4), we derive that (see [25])

$$\begin{aligned} \Phi'_3(t) &= H(0) \| \nabla u \|_2^2 - \int_0^t \frac{\xi'(t-s)}{\xi(t-s)} H(t-s) \| \nabla u(s) \|_2^2 \, ds - \int_0^t g(t-s) \| \nabla u(s) \|_2^2 \, ds \\ &\leq H(0) \| \nabla u \|_2^2 - \eta(t) \int_0^t H(t-s) \| \nabla u(s) \|_2^2 \, ds - \int_0^t g(t-s) \| \nabla u(s) \|_2^2 \, ds \\ &= H(0) \| \nabla u \|_2^2 - \eta(t) \Phi_3(t) - \int_0^t g(t-s) \| \nabla u(s) \|_2^2 \, ds. \end{aligned} \tag{3.19}$$

Hence, we conclude from (2.6), (3.10), (3.18) and (3.19) that for any $t \geq t_0 > 0$,

$$\begin{aligned} L'(t) &= E'(t) + \lambda_1 \Phi'_1(t) + \lambda_2 \Phi'_2(t) + \lambda_3 \Phi'_3(t) \\ &\leq \left(\frac{1}{2} - \lambda_2 \left(\frac{g(0)}{4\delta_3} + \frac{g(0)c_s^2}{4(\rho+1)\delta_4} \right) \right) (g' \circ \nabla u)(t) \\ &\quad + \frac{1}{\rho+1} (\lambda_1 - \lambda_2 g_*) \| u_t \|_{\rho+2}^{\rho+2} - \lambda_1 \| u \|_{\rho+2}^{\rho+2} - \frac{\lambda_1}{2} (g \circ \nabla u)(t) \\ &\quad + \left(\lambda_1 + \lambda_2 \left(\delta_3 + \frac{\alpha_1 \delta_4}{\rho+1} - g_* \right) \right) \| \nabla u_t \|_2^2 + \left(\frac{\lambda_1}{2} - \lambda_3 \right) \int_0^t g(t-s) \| \nabla u(s) \|_2^2 \, ds \\ &\quad + \left(\lambda_2 (1 - g_*) \left(\delta_1 + \frac{3}{2} \widehat{lg}(F) \right) + \lambda_2 \delta_5 \alpha_2 - \lambda_1 \left(1 - \frac{l}{2} \right) + \lambda_3 H(0) \right) \| \nabla u \|_2^2 \\ &\quad + \lambda_2 l \left(1 + \frac{1}{\delta_2} + \frac{c_s^2}{2\delta_5} + \frac{1 - g_*}{4\delta_1} \right) \int_{\Omega} \int_{A_t} g(t-s) | \nabla u(t) - \nabla u(s) |^2 \, ds \, dx \\ &\quad + \lambda_2 \left(1 + \delta_2 + \frac{c_s^2}{2\delta_5} \right) \widehat{lg}(F) \int_{\Omega} \int_{F_t} g(t-s) | \nabla u(t) - \nabla u(s) |^2 \, ds \, dx \\ &\quad + \lambda_2 \left(\frac{1 - g_*}{2} \right) \int_{F_t} g(t-s) \| \nabla u(s) \|_2^2 \, ds - \lambda_3 \eta(t) \Phi_3(t). \end{aligned} \tag{3.20}$$

For $n \in \mathcal{N}$, we consider the sets (see [24,25])

$$A_n = \{s \in R^+ \mid ng'(s) + g(s) \leq 0\}$$

and observe that

$$\bigcup_n A_n = R^+ - \{F_g \cup N_g\},$$

where F_g is given in (3.7) and N_g is the null set where g' is not defined. In addition, denoting $F_n = R^+ - A_n$, then

$$\lim_{n \rightarrow \infty} \widehat{g}(F_n) = \widehat{g}(F_g),$$

because A_n are increasingly nested. Thus, choosing $A = A_n$, $F = F_n$ and $\lambda_1 = (g^* - \varepsilon)$ λ_2 for some $\varepsilon > 0$ in (3.20), we obtain

$$\begin{aligned} L'(t) \leq & \left(\frac{1}{2} - \lambda_2 \left(\frac{g(0)}{4\delta_3} + \frac{g(0)c_s^2}{4(\rho+1)\delta_4} \right) \right) (g' \circ \nabla u)(t) \\ & - \frac{\lambda_2 \varepsilon}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} + \lambda_2 \left(\delta_3 + \frac{\alpha_1 \delta_4}{\rho+1} - \varepsilon \right) \|\nabla u_t\|_2^2 - (g^* - \varepsilon) \lambda_2 \|u\|_{\rho+2}^{\rho+2} \\ & - \frac{(g^* - \varepsilon)\lambda_2}{2} (g \circ \nabla u)(t) + \left(\frac{(g^* - \varepsilon)\lambda_2}{2} - \lambda_3 \right) \int_0^t g(t-s) \|\nabla u(s)\|_2^2 \, ds \\ & + \left(\lambda_2(1 - g^*) \left(\delta_1 + \frac{3}{2} \widehat{g}(F_n) \right) + \lambda_2 \delta_5 \alpha_2 - \lambda_2 (g^* - \varepsilon) \left(1 - \frac{1}{2} \right) + \lambda_3 H(0) \right) \|\nabla u\|_2^2 \quad (3.21) \\ & + \lambda_2 l \left(1 + \frac{1}{\delta_2} + \frac{c_s^2}{2\delta_5} + \frac{1 - g^*}{4\delta_1} \right) \int_{\Omega} \int_{A_n} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\ & + \lambda_2 \left(1 + \delta_2 + \frac{c_s^2}{2\delta_5} \right) \widehat{g}(F_n) \int_{\Omega} \int_{F_n} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\ & + \lambda_2 \left(\frac{1 - g^*}{2} \right) \int_{F_n} g(t-s) \|\nabla u(s)\|_2^2 \, ds - \lambda_3 \eta(t) \Phi_3(t). \end{aligned}$$

At this point, we take $\delta_3 = \delta_4 < \frac{(\rho+1)\varepsilon}{2(\rho+1+\alpha_1)}$ and select λ_2 so that

$$\frac{1}{2} - \lambda_2 \left(\frac{g(0)}{4\delta_3} + \frac{g(0)c_s^2}{4(\rho+1)\delta_4} \right) \geq \frac{1}{4},$$

then (3.21) becomes

$$\begin{aligned} L'(t) \leq & - \frac{\lambda_2 \varepsilon}{\rho+1} \|u_t\|_{\rho+2}^{\rho+2} - \frac{\lambda_2 \varepsilon}{2} \|\nabla u_t\|_2^2 - (g^* - \varepsilon) \lambda_2 \|u\|_{\rho+2}^{\rho+2} \\ & - \frac{(g^* - \varepsilon)\lambda_2}{2} (g \circ \nabla u)(t) + \left(\frac{(g^* - \varepsilon)\lambda_2}{2} - \lambda_3 \right) \int_0^t g(t-s) \|\nabla u(s)\|_2^2 \, ds \\ & + \left(\lambda_2(1 - g^*) \left(\delta_1 + \frac{3}{2} \widehat{g}(F_n) \right) + \lambda_2 \delta_5 \alpha_2 - \lambda_2 (g^* - \varepsilon) \left(1 - \frac{1}{2} \right) + \lambda_3 H(0) \right) \|\nabla u\|_2^2 \\ & + \left(\lambda_2 l \left(1 + \frac{1}{\delta_2} + \frac{c_s^2}{2\delta_5} + \frac{1 - g^*}{4\delta_1} \right) - \frac{1}{4n} \right) \int_{\Omega} \int_{A_n} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\ & + \lambda_2 \left(1 + \delta_2 + \frac{c_s^2}{2\delta_5} \right) \widehat{g}(F_n) \int_{\Omega} \int_{F_n} g(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\ & + \lambda_2 \left(\frac{1 - g^*}{2} \right) \int_{F_n} g(t-s) \|\nabla u(s)\|_2^2 \, ds - \lambda_3 \eta(t) \Phi_3(t). \end{aligned}$$

For ε, δ_2 small enough and large value of n and t_0 , we see that if

$$\delta_5 = \alpha \text{ and } \widehat{g}(F_n) < \frac{g_*\alpha}{l(2\alpha + 2c_s^2)},$$

then

$$\left((1 + \delta_2) + \frac{c_s^2}{2\delta_5} \right) l \widehat{g}(F_n) - \frac{g_* - \varepsilon}{2} \leq 0$$

and

$$\frac{3}{2}(1 - g_*)l \widehat{g}(F_n) < \delta(g_* - \varepsilon) \left(1 - \frac{l}{2} \right), \tag{3.22}$$

where

$$\alpha = \frac{g_*(4+l) - 3l}{8\alpha_2} = \frac{(g_*(4+l) - 3l)(1-l)^p}{8c_s^{2(p+1)}(2E(0))^p}$$

and

$$\delta = \frac{3(1 - g_*)l}{g_*(4 - 2l)}.$$

Note that $\alpha > 0$ and $0 < \delta < 1$ due to $g_* > \frac{3l}{4+l}$. Furthermore, we require λ_2 and λ_3 satisfying

$$\lambda_2 l \left(1 + \frac{1}{\delta_2} + \frac{c_s^2}{2\delta_5} + \frac{1 - g_*}{4\delta_1} \right) - \frac{1}{4n} < 0$$

and

$$\frac{\lambda_2}{2} < \lambda_3 < \frac{\lambda_2(g_*(4+l) - 3l)}{8H(0)},$$

this is possible because of $H(0) < \frac{g_*(4+l)-3l}{4}$. Then, letting δ_1 be small enough and using (3.22), we see that

$$\lambda_2(1 - g_*) \left(\delta_1 + \frac{3}{2} l \widehat{g}(F_n) \right) + \lambda_2 \delta_5 \alpha_2 - \lambda_2(g_* - \varepsilon) \left(1 - \frac{l}{2} \right) + \lambda_3 H(0) \leq 0.$$

Hence, from the definition of $E(t)$ by (2.4), we have, for all $t \geq t_0$,

$$L'(t) \leq -c_4 E(t) - \lambda_3 \eta(t) \Phi_3(t),$$

for some positive constant c_4 . As $\eta(t)$ is decreasing, we have $\eta(t) \leq c_4$ after some $t_* \geq t_0$. Hence, with the help of the right hand side inequality in (3.5), we find

$$L'(t) \leq -c_5 \eta(t) L(t), \quad \forall t \geq t_*, \tag{3.23}$$

for some positive constant $c_5 > 0$. An integration of (3.23) over (t_*, t) gives

$$L(t) \leq L(t_*) e^{-c_5 \int_{t_*}^t \eta(s) ds}, \quad \forall t \geq t_*.$$

Then using the left hand side inequality in (3.5) leads to

$$\beta_1(E(t) + \Phi_3(t)) \leq L(t_*)e^{-c_5 \int_{t_*}^t \eta(s) ds}, \quad \forall t \geq t_*.$$

Therefore, by virtue of the continuity and boundedness of $E(t)$ and $\zeta(t)$ on the interval $[0, t_*]$, we infer that

$$E(t) \leq K\xi(t)^{-\mu}, \quad t \geq 0,$$

for some positive constants K and μ .

Similar to those remarks as in [25], we have the following remark.

Remark. Note that there is a wide class of relaxation functions satisfying (A3). More precisely, if $\zeta(t) = e^{\alpha t}$, $\alpha > 0$, then $\eta(t) = \alpha$, this gives the exponential decay estimate $E(t) \leq c_1 e^{-c_2 \alpha t}$, for some positive constants c_1 and c_2 . Similarly, if $\zeta(t) = (1 + t)^\alpha$, $\alpha > 0$, then we obtain the polynomial decay estimate $E(t) \leq c_3 (1 + t)^{-\mu}$, for some positive constants c_3 and μ .

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Competing interests

The author declares that they have no competing interests.

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