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Existence results for a class of nonlocal problems involving p-Laplacian

Yang Yang[1*] and Jihui Zhang[2]

* Correspondence: yynjnu@126.com
[1]School of Science, Jiangnan University, Wuxi, 214122, People's Republic of China
Full list of author information is available at the end of the article

Abstract

This paper is concerned with the existence of solutions to a class of p-Kirchhoff type equations with Neumann boundary data as follows:

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\right]^{p-1} \Delta_p u = f(x,u), & \text{in } \Omega; \\ \frac{\partial u}{\partial \upsilon} = 0, & \text{on } \partial\Omega. \end{cases}$$

By means of a direct variational approach, we establish conditions ensuring the existence and multiplicity of solutions for the problem.

Keywords: Nonlocal problems, Neumann problem, p-Kirchhoff's equation

1. Introduction

In this paper, we deal with the nonlocal p-Kirchhoff type of problem given by:

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\right]^{p-1} \Delta_p u = f(x,u), & \text{in } \Omega; \\ \frac{\partial u}{\partial \upsilon} = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^\mathbf{N}$, $1 < p < N$, v is the unit exterior vector on $\partial\Omega$, Δ_p is the p-Laplacian operator, that is, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, the function $M : \mathbf{R}^+ \to \mathbf{R}^+$ is a continuous function and there is a constant $m_0 > 0$, such that

(M_0) $\quad M(t) \geq m_0$ for all $t \geq 0$.

$f(x,t) : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a continuous function and satisfies the subcritical condition:

$$\left|f(x,t)\right| \leq C(|t|^{q-1} + 1), \quad \text{for some} \quad p < q < p^* = \begin{cases} \frac{Np}{N-p}, & N \geq 3; \\ +\infty, & N = 1, 2. \end{cases} \tag{1.2}$$

where C denotes a generic positive constant.

Problem (1.1) is called nonlocal because of the presence of the term M, which implies that the equation is no longer a pointwise identity. This provokes some mathematical difficulties which makes the study of such a problem particulary interesting. This problem has a physical motivation when $p = 2$. In this case, the operator $M(\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u$ appears in the Kirchhoff equation which arises in nonlinear vibrations, namely

$$\begin{cases} u_{tt} - M(\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = f(x,u), & \text{in } \Omega \times (0,T); \\ u = 0, & \text{on } \partial\Omega \times (0,T); \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x). \end{cases}$$

P-Kirchhoff problem began to attract the attention of several researchers mainly after the work of Lions [1], where a functional analysis approach was proposed to attack it. The reader may consult [2-8] and the references therein for similar problem in several cases.

This work is organized as follows, in Section 2, we present some preliminary results and in Section 3 we prove the main results.

2. Preliminaries

By a weak solution of (1.1), then we say that a function $u \varepsilon W^{1,p}(\Omega)$ such that

$$\left[M\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\right]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \mathrm{d}x = \int_\Omega f(x,u)\varphi \mathrm{d}x, \quad \text{for all} \quad \varphi \in W^{1,p}(\Omega)$$

So we work essentially in the space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u|^p + |u|^p)\mathrm{d}x\right)^{\frac{1}{p}},$$

and the space $W^{1,p}(\Omega)$ may be split in the following way. Let $W_c = \langle 1 \rangle$, that is, the subspace of $W^{1,p}(\Omega)$ spanned by the constant function 1, and $W_0 = \{z \in W^{1,p}(\Omega), \int_\Omega z = 0\}$, which is called the space of functions of $W^{1,p}(\Omega)$ with null mean in Ω. Thus

$$W^{1,p}(\Omega) = W_0 \oplus W_c.$$

As it is well known the Poincaré's inequality does not hold in the space $W^{1,p}(\Omega)$. However, it is true in W_0.

Lemma 2.1 [8] (Poincaré-Wirtinger's inequality) *There exists a constant* $\eta > 0$ *such that* $\int_\Omega |z|^p \mathrm{d}x \leq \eta \int_\Omega |\nabla z|^p \mathrm{d}x$*for all* $z \in W_0$.

Let us also recall the following useful notion from nonlinear operator theory. If X is a Banach space and $A : X \to X^*$ is an operator, we say that A is of type (S_+), if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \rightharpoonup x$ weakly in X, and $\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0$. we have that $x_n \to x$ in X.

Let us consider the map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ corresponding to $-\Delta_p$ with Neumann boundary data, defined by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \mathrm{d}x, \quad \forall u, v \in W^{1,p}(\Omega). \tag{2.1}$$

We have the following result:

Lemma 2.2 [9,10]*The map* $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *defined by* (2.1) *is continuous and of type* (S_+).

In the next section, we need the following definition and the lemmas.

Definition 2.1. *Let E be a real Banach space, and D an open subset of E. Suppose that a functional* $J : D \to R$ *is Fréchet differentiable on D. If* $x_0 \in D$ *and the Fréchet derivative* $J'(x_0) = 0$, *then we call that* x_0 *is a critical point of the functional J and* $c = J(x_0)$ *is a critical value of J.*

Definition 2.2. *For $J \in C^1(E, \mathbf{R})$, we say J satisfies the Palais-Smale condition (denoted by (PS)) if any sequence $\{u_n\} \subset E$ for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.*

Lemma 2.3 [11]*Let X be a Banach space with a direct sum decomposition $X = X_1 \oplus X_2$, with $k = dimX_2 < \infty$, let J be a C^1 function on X, satisfying (PS) condition. Assume that, for some $r > 0$,*

$$J(u) \le 0 \, for \, u \in X_1, \quad \|u\| \le r;$$
$$J(u) \ge 0 \, for \, u \in X_2, \quad \|u\| \le r.$$

Assume also that J is bounded below and $\inf_X J < 0$. Then J has at least two nonzero critical points.

Lemma 2.4 [12]*Let $X = X_1 \oplus X_2$, where X is a real Banach space and $X_2 \neq \{0\}$, and is finite dimensional. Suppose $J \in C^1(X, R)$ satisfies (PS) and*

(i) there is a constant α and a bounded neighborhood D of 0 in X_2 such that $J|_{\partial D} \le \alpha$ and,

(ii) there is a constant $\beta > \alpha$ such that $J|_{X_1} \ge \beta$,

then J possesses a critical value $c \ge \beta$, moreover, c can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{D}} J(h(u)).$$

where $\Gamma = \{h \in C(\overline{D}, X) | h = id \text{ on } \partial D\}$.

Definition 2.3. *For $J \in C^1(E, \mathbf{R})$, we say J satisfies the Cerami condition (denoted by (C)) if any sequence $\{u_n\} \subset E$ for which $J(u_n)$ is bounded and $(1 \, ||u_n||) \, J'(u_n)|| \to 0$ as $n \to \infty$ possesses a convergent subsequence.*

Remark 2.1 If J satisfies the (C) condition, Lemma 2.4 still holds.

In the present paper, we give an existence theorem and a multiplicity theorem for problem (1.1). Our main results are the following two theorems.

Theorem 2.1 *If following hold:*

(F_0) $0 \le \lim\limits_{|u| \to 0} \frac{pF(x,u)}{|u|^p} < \frac{m_0^{p-1}}{\eta}$ *a.e.* $x \in \Omega$, *where* $F(x,u) = \int_0^u f(x,s)ds$, η ***appears in Lemma 2.1***;

(F_1) $\lim_{|u| \to \infty} \frac{pF(x,u)}{|u|^p} \le 0$ *a.e.* $x \in \Omega$;

(F_2) $\lim_{|u| \to \infty} \int_\Omega F(x,u) dx = -\infty$.

Then the problem (1.1) has least three distinct weak solutions in $W^{1,p}(\Omega)$.

Theorem 2.2 *If the following hold:*

(M_1) *The function M that appears in the classical Kirchhoff equation satisfies $\widehat{M}(t) \le (M(t))^{p-1} t$ for all $t \ge 0$, where $\widehat{M}(t) = \int_0^t [M(s)]^{p-1} ds$;*

(F_3) $f(x,u)u > 0$ *for all* $u \neq 0$;

(F_4) $\lim_{|u| \to \infty} \frac{pF(x,u)}{|u|^p} = 0$ *a.e.* $x \in \Omega$;

(F_5) $\lim_{|u| \to \infty} (f(x,u)u - pF(x,u)) = -\infty$.

Then the problem (1.1) has at least one weak solution in $W^{1,p}(\Omega)$.

Remark 2.2 We exhibit now two examples of nonlinearities that fulfill all of our hypotheses

$$f(x,u) = \frac{m_0^{p-1}}{2\eta} |u|^{p-2} u - |u|^{q-2} u,$$

hypotheses (F_0), (F_1), (F_2) and (1.2) are clearly satisfied.

$$f(x,u) = \arctan u + \frac{u}{1+u^2},$$

hypotheses (F_3), (F_4) and (F_5) and (1.2) are clearly satisfied.

3. Proofs of the theorems

Let us start by considering the functional $J : W^{1,p}(\Omega) \to \mathbf{R}$ given by

$$J(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right) - \int_\Omega F(x,u)\mathrm{d}x.$$

Proof of Theorem 2.1 By (F_0), we know that $f(x, 0) = 0$, and hence $u(x) = 0$ is a solution of (1.1).

To complete the proof we prove the following lemmas.

Lemma 3.1 *Any bounded (PS) sequence of J has a strongly convergent subsequence.*

Proof: Let $\{u_n\}$ be a bounded (PS) sequence of J. Passing to a subsequence if necessary, there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$. From the subcritical growth of f and the Sobolev embedding, we see that

$$\int_\Omega f(x,u_n)\,(u_n - u)\mathrm{d}x \to 0.$$

and since $J'(u_n)(u_n - u) \to 0$, we conclude that

$$\left[M\left(\int_\Omega |\nabla u_n|^p \mathrm{d}x\right)\right]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x \to 0.$$

In view of condition (M_0), we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x \to 0.$$

Using Lemma 2.2, we have $u_n \to u$ as $n \to \infty$. □

Lemma 3.2 *If condition* (M_0), (F_1) *and* (F_2) *hold, then* $\lim_{||u||\to\infty} J(u) = +\infty$.

Proof: If there are a sequence $\{u_n\}$ and a constant C such that $||u_n|| \to \infty$ as $n \to \infty$, and $J(u_n) \le C$ $(n = 1, 2 \cdots)$, let $v_n = \frac{u_n}{\|u_n\|}$, then there exist $v_0 \in W^{1,p}(\Omega)$ and a subsequence of $\{v_n\}$, we still note by $\{v_n\}$, such that $v_n \rightharpoonup v_0$ in $W^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$.

For any $\varepsilon > 0$, by (F_1), there is a $H > 0$ such that $F(x,u) \le \frac{\varepsilon}{p}|u|^p$ for all $|u| \ge H$ and a. e. $x \in \Omega$, then there exists a constant $C > 0$ such that $F(x,u) \le \frac{\varepsilon}{p}|u|^p + C$ for all $u \in R$, and a.e. $x \in \Omega$, Consequently

$$\begin{aligned}
\frac{C}{||u_n||^p} \ge \frac{J(u_n)}{||u_n||^p} &= \frac{1}{||u_n||^p}\left(\frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u_n|^p \mathrm{d}x\right) - \int_\Omega F(x,u_n)\mathrm{d}x\right) \\
&\ge \frac{1}{p}m_0^{p-1}\int_\Omega |\nabla v_n|^p \mathrm{d}x - \frac{\varepsilon}{p}\int_\Omega |v_n|^p \mathrm{d}x - \frac{C|\Omega|}{||u_n||^p} \\
&= \frac{1}{p}m_0^{p-1} - \left(\frac{1}{p}m_0^{p-1} + \frac{\varepsilon}{p}\right)\int_\Omega |v_n|^p \mathrm{d}x - \frac{C|\Omega|}{||u_n||^p}.
\end{aligned}$$

It implies $\int_\Omega |v_0|^p \mathrm{d}x \ge 1$. On the other hand, by the weak lower semi-continuity of the norm, one has

$$||v_0|| \leq \liminf_{n\to\infty} ||v_n|| = 1.$$

Hence $\int_\Omega |\nabla v_0|^p \mathrm{d}x = 0$, so $|v_0(x)| = constant \neq 0$ a.e. $x \in \Omega$. By (F_2), $\lim_{|u_n|\to\infty} \int_\Omega F(x, u_n)\mathrm{d}x \to -\infty$. Hence

$$\begin{aligned} C \geq J(u_n) &= \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u_n|^p \mathrm{d}x\right) - \int_\Omega F(x, u_n)\mathrm{d}x \\ &\geq -\int_\Omega F(x, u_n)\mathrm{d}x \to +\infty \text{ as } n \to \infty. \end{aligned}$$

This is a contradiction. Hence J is coercive on $W^{1,p}(\Omega)$, bounded from below, and satisfies the (PS) condition. □

By Lemma 3.1 and 3.2, we know that J is coercive on $W^{1,p}(\Omega)$, bounded from below, and satisfies the (PS) condition. From condition (F_0), we know, there exist $r > 0$, $\varepsilon > 0$ such that

$$0 \leq F(x, u) \leq \left(\frac{m_0^{p-1}}{p\eta} - \varepsilon\right)|u|^p, \quad \text{for} |u| \leq r.$$

If $u \in W_c$, for $||u|| \leq \rho_1$, then $|u| \leq r$, we have

$$\begin{aligned} J(u) &= \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right) - \int_\Omega F(x, u)\mathrm{d}x \\ &= -\int_\Omega F(x, u)\mathrm{d}x \leq 0. \end{aligned}$$

If $u \in W_0$, then from condition (F_0) and (1.2), we have

$$F(x, u) \leq \left(\frac{m_0^{p-1}}{p\eta} - \varepsilon\right)|u|^p + C|u|^q, \quad \text{for} \quad u \in R, \quad q \in (p, p^*).$$

Noting that

$$\int_\Omega |u|^p \mathrm{d}x \leq \eta \int_\Omega |\nabla u|^p \mathrm{d}x, \quad u \in W_0,$$

we can obtain

$$\begin{aligned} J(u) &= \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right) - \int_\Omega F(x, u)\mathrm{d}x \\ &\geq \frac{1}{p}m_0^{p-1}\int_\Omega |\nabla u|^p \mathrm{d}x - \frac{m_0^{p-1}}{p\eta}\int_\Omega |u|^p \mathrm{d}x + \varepsilon\int_\Omega |u|^p \mathrm{d}x - C\int_\Omega |u|^q \mathrm{d}x \\ &\geq C\varepsilon||u||^p - CC_1||u||^q. \end{aligned}$$

Choose $||u|| = \rho_2$ small enough, such that $J(u) \geq 0$ for $||u|| \leq \rho_2$ and $u \in W_0$.
Now choose $\rho = \min\{\rho_1, \rho_2\}$, then, we have

$$J(u) \leq 0 \;\; \text{for } u \in W_c, \quad ||u|| \leq \rho;$$

$$J(u) \leq 0 \;\; \text{for } u \in W_0, \quad ||u|| \leq \rho.$$

If $\inf\{J(u), u \in W^{1,p}(\Omega)\} = 0$, then all $u \in W_c$ with $||u|| \le \rho$ are minimum of J, which implies that J has infinite critical points. If $\inf\{J(u), u \in W^{1,p}(\Omega)\} < 0$ then by Lemma 2.3, J has at least two nontrivial critical points. Hence problem (1.1) has at least two nontrivial solutions in $W^{1,p}(\Omega)$, Therefore, problem (1.1) has at least three distinct solutions in $W^{1,p}(\Omega)$. □

Proof of Theorem 2.2. We divide the proof into several lemmas.

Lemma 3.3 *If condition* (F_3) *and* (F_5) *hold, then* $J|_{W_c}$ *is anticoercive. (i.e. we have that* $J(u) \to -\infty$, *as* $|u| \to \infty$, $u \in R$.*)*

Proof: By virtue of hypothesis (F_5), for any given $L > 0$, we can find $R_1 = R_1(L) > 0$ such that

$$F(x,u) \ge \frac{1}{p}L + \frac{1}{p}f(x,u)u, \quad \text{for } a.e. x \in \Omega, \quad |u| > R_1.$$

Thus, using hypothesis (F_3), we have

$$F(x,u) \ge \frac{1}{p}L - C, \text{ for a.e.} x \in \Omega \, u \in \mathbf{R}$$

So

$$\int_\Omega F(x,u)\mathrm{d}x \ge \frac{1}{p}L|\Omega| - C|\Omega|.$$

Since $L > 0$ is arbitrary, it follows that

$$\int_\Omega F(x,u)\mathrm{d}x \to \infty, \quad \text{as} \quad |u| \to \infty,$$

and so

$$J(u)|_{W_C} = -\int_\Omega F(x,u)\mathrm{d}x \to -\infty, \quad \text{as} \quad |u| \to \infty.$$

This proves that $J|_{W_c}$ is anticoercive. □

Lemma 3.4 *If hypothesis* (F_4) *holds, then* $J|_{W_0} \ge -\infty$.

Proof: For a given $0 < \varepsilon < m_0^{p-1}$, we can find $C_\varepsilon > 0$ such that $F(x,u) \le \frac{\varepsilon}{p\eta}|u|^p + C_\varepsilon$ for a.e. $x \in \Omega$ all $u \in \mathbf{R}$. Then

$$\begin{aligned} J(u)|_{u\in W_0} &= \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right) - \int_\Omega F(x,u)\mathrm{d}x \\ &\ge \frac{1}{p}m_0^{p-1}\int_\Omega |\nabla u|^p\mathrm{d}x - \frac{m_0^{p-1}}{p\eta}\int_\Omega |u|^p\mathrm{d}x - C|\Omega| \\ &\ge -C|\Omega|. \end{aligned}$$

then $J|_{W_0} \ge -\infty$. □

Lemma 3.5 *If condition* (F_4) (F_5) *hold, then J satisfies the (C) condition.*

Proof: Let $\{u_n\}_{n\ge 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|J(u_n)| \le M_1, \quad \forall n \ge 1. \tag{3.1}$$

with some $M_1 > 0$ and

$$(1 + ||u_n||)J'(u_n) \to 0, \quad \text{in} \quad W^{1,p}(\Omega)^* \quad \text{as} \quad n \to \infty. \tag{3.2}$$

We claim that the sequence $\{u_n\}$ is bounded. We argue by contradiction. Suppose that $||u|| \to +\infty$, as $n \to \infty$, we set $v_n = \frac{u_n}{\|u_n\|}$, $\forall n \geq 1$. Then $||v_n|| = 1$ for all $n \geq 1$ and so, passing to a subsequence if necessary, we may assume that

$$v_n \rightharpoonup v \ \text{ in } \ W^{1,p}(\Omega);$$

$$v_n \to v \ \text{ in } \ L^p(\Omega).$$

from (3.2), we have $\forall h \in W^{1,p}(\Omega)$

$$\left| \left[M\left(\int_\Omega |\nabla u_n|^p \mathrm{d}x \right) \right]^{p-1} \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla h \mathrm{d}x - \int_\Omega \frac{f(x,u_n)h}{\|u_n\|^{p-1}} \mathrm{d}x \right| \leq \frac{\varepsilon_n}{1+\|u_n\|} \frac{\|h\|}{\|u_n\|^{p-1}} \tag{3.3}$$

with $\varepsilon_n \downarrow 0$.

In (3.3), we choose $h = v_n - v \in W^{1,p}(\Omega)$, note that by virtue of hypothesis (F_4), we have

$$\frac{f(x,u_n)}{||u_n||^{p-1}} \rightharpoonup 0 \quad \text{in} \quad L^{p'}(\Omega),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

So we have

$$\left[M\left(\int_\Omega |\nabla u_n|^p \mathrm{d}x \right) \right]^{p-1} \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \mathrm{d}x \to 0.$$

Since $M(t) > m_0$ for all $t \geq 0$, so we have

$$\int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \mathrm{d}x \to 0.$$

Hence, using the (S_+) property, we have $v_n \to v$ in $W^{1,p}(\Omega)$ with $||v|| = 1$, then $v \neq 0$. Now passing to the limit as $n \to \infty$ in (3.3), we obtain

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla h \mathrm{d}x \to 0, \forall h \in W^{1,p}(\Omega),$$

then $v = \xi \in R$. Then $|u_n(x)| \to +\infty$ as $n \to +\infty$. Using hypothesis (F_5), we have $f(x, u_n(x))u_n(x) - pF(x, u_n(x)) \to -\infty$ for a.e $x \in \Omega$.

Hence by virtue of Fatou's Lemma, we have

$$\int_\Omega f(x,u_n)u_n - pF(x,u_n) \mathrm{d}x \to -\infty, \quad \text{as} \quad n \to +\infty. \tag{3.4}$$

From (3.1), we have

$$\widehat{M}\left(\int_\Omega |\nabla u_n|^p \right) \mathrm{d}x - p \int_\Omega F(x,u_n) \mathrm{d}x \geq -pM_1, \quad \forall n \geq 1. \tag{3.5}$$

From (3.2), we have

$$\left| \left[M\left(\int_\Omega |\nabla u_n|^p \mathrm{d}x \right) \right]^{p-1} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla h \mathrm{d}x - \int_\Omega f(x,u_n) h \mathrm{d}x \right| \leq \frac{\varepsilon_n ||h||}{1+||u_n||} \forall h \in W^{1,p}(\Omega).$$

With $\varepsilon_n \downarrow 0$. So choosing $h = u_n \in W^{1,p}(\Omega)$, we obtain

$$-\left[M\left(\int_\Omega |\nabla u_n|^p \mathrm{d}x\right)\right]^{p-1} \int_\Omega |\nabla u_n|^p \mathrm{d}x + \int_\Omega f(x, u_n) u_n \mathrm{d}x \geq -\varepsilon_n. \tag{3.6}$$

Adding (3.5) and (3.6), noting that $\widehat{M}(t) \leq (M(t))^{p-1} t$ for all $t \geq 0$, we obtain

$$\int_\Omega (f(x, u_n) u_n - pF(x, u_n)) \mathrm{d}x \geq -M_2, \quad \forall n \geq 1, \tag{3.7}$$

comparing (3.4) and (3.7), we reach a contradiction. So $\{u_n\}$in bounded in $W^{1,p}(\Omega)$. Similar with the proof of Lemma 3.1, we know that J satisfied the (C) condition. □

Sum up the above fact, from Lemma 2.4 and Remark 2.1, Theorem 2.2 follows from the Lemma 3.3 to 3.5.

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Author details
[1]School of Science, Jiangnan University, Wuxi, 214122, People's Republic of China [2]Institute of Mathematics, School of Mathematics Science, Nanjing Normal University, Nanjing, 210097, People's Republic of China

Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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