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On nonlocal three-point boundary value problems of Duffing equation with mixed nonlinear forcing terms

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Abstract

In this paper, we investigate the existence and approximation of the solutions of a nonlinear nonlocal three-point boundary value problem involving the forced Duffing equation with mixed nonlinearities. Our main tool of the study is the generalized quasilinearization method due to Lakshmikantham. Some illustrative examples are also presented.

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1 Introduction

The Duffing equation plays an important role in the study of mechanical systems. There are multiple forms of the Duffing equation, ranging from dampening to forcing terms. This equation possesses the qualities of a simple harmonic oscillator, a nonlinear oscillator, and has indeed an ability to exhibit chaotic behavior. Chaos can be defined as disorder and confusion. In physics, chaos is defined as behavior so unpredictable as to appear random, allowing great sensitivity to small initial conditions. The chaotic behavior can emerge in a system as simple as the logistic map. In that case, the "route to chaos" is called period-doubling. In practice, one would like to understand the route to chaos in systems described by partial differential equations such as flow in a randomly stirred fluid. This is, however, very complicated and difficult to treat either analytically or numerically. The Duffing equation is found to be an appropriate candidate for describing chaos in dynamic systems. The advantage of a pseudo-chaotic equation like the Duffing equation is that it allows control of the amount of chaos it exhibits. Chaotic oscillators are important tools for creating and testing models that are more realistic. This is why the Duffing equation is of great interest. The use of the Duffing equation aids in the dynamic behavior of chaos and bifurcation, which studies how small changes in a function can cause a sudden change in behavior [1]. Another important application of the Duffing equation is in the field of the prediction of diseases. A careful measurement and analysis of a strongly chaotic voice has the potential to serve as an early warning system for more serious chaos and possible onset of disease. This chaos is with the help of the Duffing equation. In fact, the success at analyzing and predicting the onset of chaos in speech and its simulation by equations such as the Duffing equation has enhanced the hope that we might be able to predict the onset of arrhythmia and heart attacks someday [2].

The Duffing equation is a mathematical representation of the oscillator. Both the equation and oscillator are prone to many output waveforms. One of the simplest waveforms includes simple harmonic motion like a pendulum. Other waveforms are considerably more complex and can quickly be described as shear oscillatory chaos. The Duffing equation can be a forced or unforced damped chaotic harmonic oscillator. Exact solutions of second-order nonlinear differential equations like the forced Duffing equation are rarely possible due to the possible chaotic output. There do exist a number of powerful procedures for obtaining approximate solutions of nonlinear problems such as Galerkin's method, expansion methods, dynamic programming, iterative techniques, the method of upper and lower bounds, and Chapligin method to name a few. The monotone iterative technique coupled with the method of upper and lower solutions [3] manifests itself as an effective and flexible mechanism that offers theoretical as well as constructive existence results in a closed set, generated by the lower and upper solutions. In general, the convergence of the sequence of approximate solutions given by the monotone iterative technique is at most linear. To obtain a sequence of approximate solutions converging quadratically, we use the method of quasilinearization. The origin of the quasilinearization lies in the theory of dynamic programming [4,5]. Agarwal [6] discussed quasilinearization and approximate quasilinearization for multipoint boundary value problems. In fact, the quasilinearization technique is a variant of Newton's method. This method applies to semilinear equations with convex (concave) nonlinearities and generates a monotone scheme whose iterates converge quadratically to a solution of the problem at hand. The nineties brought new dimensions to this technique when Lakshmikantham [7,8] generalized the method of quasilinearization by relaxing the convexity assumption. This development was so significant that it attracted the attention of many researchers, and the method was extensively developed and applied to a wide range of initial and boundary value problems for different types of differential equations. A detailed description of the quasilinearization method and its applications can be found in the monograph [9] and the papers [10-26] and the references therein.

In this paper, we study a nonlinear nonlocal three-point boundary value problem of the forced Duffing equation with mixed nonlinearities given by

$$x''(t) + \lambda x'(t) = N(t, x(t)), \quad t \in J = [0, 1], \quad \lambda \in \mathbb{R} - \{0\}, \tag{1.1}$$

$$px(0) - qx'(0) = g_1(x(\sigma)), \qquad px(1) + qx'(1) = g_2(x(\sigma)), \quad 0 < \sigma < 1, \quad p, q > 0, \tag{1.2}$$

where $N(t, x) \in C[J \times \mathbb{R}, \mathbb{R}]$ is such that

$$N(t, x) = f(t, x) + k(t, x) + H(t, x), \tag{1.3}$$

and g_i: $\mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ are given continuous functions. The details of such a decomposition can be found in Section 1.5 of the text [9]. In (1.3), it is assumed that $f(t,x)$ is nonconvex, $k(t,x)$ is nonconcave, and $H(t,x)$ is a Lipschitz function:

$$H(t, x) - H(t, y) \geq -L(x - y), \;\; x \geq y, \;\; x, y \in \mathbb{R}, \;\; L > 0.$$

A quasilinearization technique due to Lakshmikantham [9] is applied to obtain an analytic approximation of the solution of the problem (1.1-1.2). In fact, we obtain sequences of upper and lower solutions converging monotonically and quadratically to a unique solution of the problem at hand. It is worth mentioning that the forced Duffing equation with mixed nonlinearities has not been studied so far.

2 Preliminaries

As argued in [12], the solution $x(t)$ of the problem (1.1-1.2) can be written in terms of the Green's function as

$$x(t) = g_1(x(\sigma))\Big(\frac{(p-q\lambda)e^{-\lambda} - p\,e^{-\lambda t}}{p[(p-q\lambda)e^{-\lambda} - (p+q\lambda)]}\Big) + g_2(x(\sigma))\Big(\frac{p\,e^{-\lambda t} - (p+q\lambda)}{p\;[(p-\lambda q)\;e^{-\lambda} - (p+\lambda q)]}\Big) + \int_0^1 G(t,s)N(s,x(s))ds,$$

where

$$G(t,s) = \frac{p\;e^{\lambda s}}{\lambda[(p+q\lambda)\;e^{\lambda} - (p-q\lambda)]}\begin{cases}\left(e^{\lambda(1-s)} - \frac{(p-q\lambda)}{p}\right)\left(e^{-\lambda t} - \frac{(p+q\lambda)}{p}\right), & \text{if} \quad 0 \le t \le s \le 1,\\ \left(e^{\lambda(1-t)} - \frac{(p-q\lambda)}{p}\right)\left(e^{-\lambda s} - \frac{(p+q\lambda)}{p}\right), & \text{if} \quad 0 \le s \le t \le 1.\end{cases}$$

Observe that $G(t,s) < 0$ on $[0,1] \times [0,1]$.

Definition 2.1. We say that $\alpha \in C^2[J, \mathbb{R}]$ is a lower solution of the problem (1.1-1.2) if

$$\alpha''(t) + \lambda\alpha'(t) \ge N(t,\alpha), \;\; t \in J,$$

$$p\alpha(0) - q\alpha'(0) \le g_1(\alpha(\sigma)), \qquad\qquad p\alpha(1) + q\alpha'(1) \le g_2(\alpha(\sigma)),$$

and $\beta \in C^2[J, \mathbb{R}]$ will be an upper solution of the problem (1.1-1.2) if the inequalities are reversed in the definition of lower solution.

Now we state some basic results that play a pivotal role in the proof of the main result. We do not provide the proof as the method of proof is similar to the one described in the text [9].

Theorem 2.1. Let α and β be lower and upper solutions of (1.1-1.2), respectively. Assume that

(i) $f_x(t,x) + k_x(t,x) - L > 0$ for every $(t,x) \in J \times \mathbb{R}$.

(ii) g_1 and g_2 are continuous on $\mathbb{R}$ satisfying the one-sided Lipschitz condition:

$$g_i(x) - g_i(y) \le L_i(x-y), \quad 0 \le L_i < 1, \;\; i = 1, 2.$$

Then $\alpha(t) \le \beta(t)$, $t \in J$.

Theorem 2.2. Let α and β be lower and upper solutions of (1.1-1.2), respectively, such that $\alpha(t) \le \beta(t)$, $t \in J$. Then, there exists a solution $x(t)$ of (1.1-1.2) such that $\alpha(t) \le x(t) \le \beta(t)$, $t \in J$.

3 Main result

Theorem 3.1. Assume that

(A_1) $\alpha_0, \beta_0 \in C^2[J, \mathbb{R}]$ are lower and upper solutions of (1.1-1.2), respectively.

(A_2) $N \in C[J \times \mathbb{R}, \mathbb{R}]$ be such that

$$N(t,x) = f(t,x) + k(t,x) + H(t,x),$$

where $f_x(t, x)$, $k_x(t, x)$, $f_{xx}(t, x)$, $k_{xx}(t, x)$ exist and are continuous, and for continuous functions φ, χ,$(f_{xx}(t, x) + \varphi_{xx}(t, x)) \geq 0$, $(k_{xx}(t, x) + \chi_{xx}(t, x)) \leq 0$ with $\varphi_{xx} \geq 0$, $\chi_{xx} \leq 0$ for every $(t, x) \in S$, where $S = \{(t, x) \in J \times \mathbb{R}: \alpha_0(t) \leq x(t) \leq \beta_0(t)\}$. $H(t, x)$ satisfies the one-sided Lipschitz condition:

$$H(t,x) - H(t,y) \geq -L(x-y), \ \ x \geq y, \ \ x, y \in \mathbb{R},$$

where $L > 0$ is a Lipschitz constant and $f_x(t, x) + k_x(t, x) - L > 0$ for every $(t, x) \in S$.

(A_3) For $i = 1, 2, g_i, g_i^{'}, g_i^{''}$ are continuous on $\mathbb{R}$ satisfying $0 \leq g_i^{'} \leq 1$ and $(g_i^{''}(x) + \psi_i^{''}(x)) \leq 0$ with $\psi_i^{ii} \leq 0$ on $\mathbb{R}$ for some continuous functions $\psi_i(x)$.

Then, there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that converge in the space of continuous functions on J quadratically to a unique solution $x(t)$ of the problem (1.1-1.2).

Proof. Let us define $F: J \times \mathbb{R} \to \mathbb{R}$ by $F(t, x) = f(t, x) + \varphi(t, x)$, $K: J \times \mathbb{R} \to \mathbb{R}$ by $K(t, x) = k(t, x) + \chi(t, x)$, $G_i: \mathbb{R} \to \mathbb{R}$ by $G_i(x) = g_i(x) + \psi_i(x)$, $i = 1, 2$. By the assumption (A_2) and the generalized mean value theorem, we get

$$f(t,x) \geq f(t,y) + F_x(t,y)(x-y) - \phi(t,x) + \phi(t,y). \tag{3.1}$$

$$k(t,x) \geq k(t,y) + K_x(t,x)(x-y) + \psi(t,y) - \psi(t,x), \tag{3.2}$$

Interchanging x and y, (3.1) and (3.2) take the form

$$f(t,x) \leq f(t,y) + F_x(t,x)(x-y) - \phi(t,x) + \phi(t,y), \tag{3.3}$$

$$k(t,x) \leq k(t,y) + K_x(t,y)(x-y) - \chi(t,x) + \chi(t,y). \tag{3.4}$$

By the assumption (A_3), we obtain

$$g_i(x) \geq g_i(y) + G_i'(x)(x-y) + \psi_i(y) - \psi_i(x), \quad i = 1, 2, \tag{3.5}$$

which, on interchanging x and y yields

$$g_i(x) \leq g_i(y) + G_i'(y)(x-y) + \psi_i(y) - \psi_i(x), \quad i = 1, 2. \tag{3.6}$$

We set

$$\begin{aligned} A(t,x;\alpha_0,\beta_0) &= f(t,\alpha_0) + k(t,\alpha_0) + H(t,x) \\ &\quad + [F_x(t,\beta_0) + K_x(t,\alpha_0) - \phi_x(t,\alpha_0) - \chi_x(t,\beta_0)](x-\alpha_0), \\ B(t,x;\alpha_0,\beta_0) &= f(t,\beta_0) + k(t,\beta_0) + H(t,x) \\ &\quad + [F_x(t,\beta_0) + K_x(t,\alpha_0) - \phi_x(t,\alpha_0) - \chi_x(t,\beta_0)](x-\beta_0), \end{aligned}$$

and for $i = 1{,}2$,

$$h_i(x(\sigma);\alpha_0,\beta_0) = g_i(\alpha_0(\sigma)) + G_i'(\beta_0(\sigma))(x(\sigma)-\alpha_0(\sigma)) + \psi_i(\alpha_0(\sigma)) - \psi_i(x(\sigma)),$$
$$\hat{h}_i(x(\sigma);\beta_0) = g_i(\beta_0(\sigma)) + G_i'(\beta_0(\sigma))(x(\sigma)-\beta_0(\sigma)) + \psi_i(\beta_0(\sigma)) - \psi_i(x(\sigma)).$$

Observe that

$$A(t,\alpha_0;\alpha_0,\beta_0) = N(t,\alpha_0), \qquad N(t,x) \le A(t,x;\alpha_0,\beta_0), \tag{3.7}$$

$$h_i(\alpha_0(\sigma);\alpha_0,\beta_0) = g_i(\alpha_0(\sigma)), \qquad g_i(x) \ge h_i(x(\sigma);\alpha_0,\beta_0),\ i = 1,2, \tag{3.8}$$

and

$$B(t,\beta_0;\alpha_0,\beta_0) = N(t,\beta_0), \qquad N(t,x) \ge B(t,x;\alpha_0,\beta_0), \tag{3.9}$$

$$\hat{h}_i(\beta_0(\sigma);\beta_0) = g_i(\beta_0(\sigma)), \qquad g_i(x) \le \hat{h}_i(x(\sigma);\beta_0),\ i = 1,2. \tag{3.10}$$

Now, we consider the problem

$$x''(t) + \lambda x'(t) = A(t,x;\alpha_0,\beta_0), \quad t \in J, \tag{3.11}$$

$$px(0) - qx'(0) = h_1(x(\sigma);\alpha_0,\beta_0), \qquad px(1) + qx'(1) = h_2(x(\sigma);\alpha_0,\beta_0). \tag{3.12}$$

Using (A_1), (3.7) and (3.8), we obtain

$$\begin{aligned} \alpha_0''(t) + \lambda\alpha_0'(t) &\ge N(t,\alpha_0(t)) = A(t,\alpha_0;\alpha_0,\beta_0),\\ p\alpha_0(0) - q\alpha_0'(0) &\le g_1(\alpha_0(\sigma)) = h_1(\alpha_0(\sigma);\alpha_0,\beta_0),\\ p\alpha_0(1) + q\alpha_0'(1) &\le g_2(\alpha_0(\sigma)) = h_2(\alpha_0(\sigma);\alpha_0,\beta_0), \end{aligned}$$

and

$$\begin{aligned} \beta_0''(t) + \lambda\beta_0' &\le N(t,\beta_0(t)) \le A(t,\beta_0;\beta_0,\beta_0),\\ p\beta_0(0) - q\beta_0'(0) &\ge g_1(\beta_0(\sigma)) \ge h_1(\beta_0(\sigma);\alpha_0,\beta_0),\\ p\beta_0(1) + q\beta_0'(1) &\ge g_2(\beta_0(\sigma)) \ge h_2(\beta_0(\sigma);\alpha_0,\beta_0), \end{aligned}$$

which imply that α_0 and β_0 are, respectively, lower and upper solutions of (3.11-3.12). Thus, by Theorems 2.1 and 2.2, there exists a solution α_1 for the problem (3.11-3.12) such that

$$\alpha_0(t) \le \alpha_1(t) \le \beta_0(t), \quad t \in J. \tag{3.13}$$

Next, consider the problem

$$x''(t) + \lambda x'(t) = B(t,x;\alpha_0,\beta_0), \quad t \in J, \tag{3.14}$$

$$px(0) - qx'(0) = \hat{h}_1(x(\sigma);\beta_0), \qquad px(1) + qx'(1) = \hat{h}_2(x(\sigma);\beta_0). \tag{3.15}$$

Using (A_1), (3.9) and (3.10), we get

$$\begin{aligned} \alpha_0''(t) + \lambda\alpha_0'(t) &\ge N(t,\alpha_0(t)) \ge B(t,\alpha_0;\alpha_0,\beta_0),\\ p\alpha_0(0) - q\alpha_0'(0) &\le g_1(\alpha_0(\sigma)) \le \hat{h}_1(\alpha_0(\sigma);\beta_0),\\ p\alpha_0(1) + q\alpha_0'(1) &\le g_2(\alpha_0(\sigma)) \le \hat{h}_2(\alpha_0(\sigma);\beta_0), \end{aligned}$$

and

$$\begin{aligned}\beta_0''(t) + \lambda\beta_0' &\leq N(t, \beta_0(t)) = B(t, \beta_0; \alpha_0, \beta_0),\\ p\beta_0(0) - q\beta_0'(0) &\geq g_1(\beta_0(\sigma)) = \hat{h}_1(\beta_0(\sigma); \beta_0),\\ p\beta_0(1) + q\beta_0'(1) &\geq g_2(\beta_0(\sigma)) = \hat{h}_2(\beta_0(\sigma); \beta_0),\end{aligned}$$

which imply that α_0 and β_0 are, respectively, lower and upper solutions of (3.14-3.15). Again, by Theorems 2.1 and 2.2, there exists a solution β_1 of (3.14-3.15) satisfying

$$\alpha_0(t) \leq \beta_1(t) \leq \beta_0(t), \quad t \in J. \tag{3.16}$$

Now we show that $\alpha_1(t) \leq \beta_1(t)$. For that, we prove that $\alpha_1(t)$ is a lower solution and $\beta_1(t)$ is an upper solution of (1.1-1.2). Using the fact that $\alpha_1(t)$ is a solution of (3.11-3.12) satisfying $\alpha_0(t) \leq \alpha_1(t) \leq \beta_0(t)$ and (3.7-3.8), we obtain

$$\alpha_1''(t) + \lambda\alpha_1'(t) = A(t, \alpha_1; \alpha_0, \beta_0) \geq N(t, \alpha_1(t)),$$

$$p\alpha_1(0) - q\alpha_1'(0) = h_1(\alpha_1(\sigma); \alpha_0, \beta_0) \leq g_1(\alpha_1(\sigma)),$$

$$p\alpha_1(1) + q\alpha_1'(1) = h_2(\alpha_1(\sigma); \alpha_0, \beta_0) \leq g_2(\alpha_1(\sigma)).$$

By the above inequalities, it follows that α_1 is a lower solution of (1.1-1.2).

In view of the fact that $\beta_1(t)$ is a solution of (3.14-3.15) together with (3.9), we get

$$\beta_1''(t) + \lambda\beta_1'(t) = B(t, \beta_1; \alpha_0, \beta_0) \leq N(t, \beta_1(t)),$$

and by virtue of (3.10), we have

$$p\beta_1(0) - q\beta_1'(0) = \hat{h}_1(\beta_1(\sigma); \beta_0) \geq g_1(\beta_1(\sigma)),$$

$$p\beta_1(1) + q\beta_1'(1) = \hat{h}_2(\beta_1(\sigma); \beta_0) \geq g_2(\beta_1(\sigma)).$$

Thus, β_1 is an upper solution of (1.1-1.2). Hence, by Theorem 2.1, it follows that

$$\alpha_1(t) \leq \beta_1(t), \quad t \in J. \tag{3.17}$$

Combining (3.13, 3.16) and (3.17) yields

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t), \quad t \in J.$$

Now, by induction, we prove that

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \alpha_{n+1}(t) \leq \beta_{n+1}(t) \leq \beta_n(t) \leq \cdots . \leq \beta_1(t) \leq \beta_0(t).$$

For that, we consider the boundary value problems

$$x''(t) + \lambda x'(t) = A(t, x; \alpha_n, \beta_n), \quad t \in J, \tag{3.18}$$

$$px(0) - qx'(0) = h_1(x(\sigma); \alpha_n, \beta_n), \qquad px(1) + qx'(1) = h_2(x(\sigma); \alpha_n, \beta_n), \tag{3.19}$$

and

$$x''(t) + \lambda x'(t) = B(t, x; \alpha_n, \beta_n), \quad t \in J, \tag{3.20}$$

$$px(0) - qx'(0) = \hat{h}_1(x(\sigma); \beta_n), \qquad px(1) + qx'(1) = \hat{h}_2(x(\sigma); \beta_n). \tag{3.21}$$

Assume that for some $n > 1$, $\alpha_0(t) \le \alpha_n(t) \le \beta_n(t) \le \beta_0(t)$ and we will show that $\alpha_{n+1}(t) \le \beta_{n+1}(t)$.

Using (3.7), we have

$$\alpha_n''(t) + \lambda\alpha_n'(t) = A(t, \alpha_n; \alpha_{n-1}, \beta_{n-1}) \ge N(t, \alpha_n) = A(t, \alpha_n; \alpha_n, \beta_n).$$

By (3.8), we obtain

$$h_i(\alpha_n(\sigma); \alpha_{n-1}, \beta_{n-1}) \le g_i(\alpha_n(\sigma)) = h_i(\alpha_n(\sigma); \alpha_n, \beta_n),$$

which yields

$$p\alpha_n(0) - q\alpha_n'(0) \le h_1(\alpha_n(\sigma); \alpha_n, \beta_n), \qquad p\alpha_n(1) + q\alpha_n'(1) \le h_2(\alpha_n(\sigma); \alpha_n, \beta_n).$$

Thus, α_n is a lower solution of (3.18-3.19). In a similar manner, we find that β_n is an upper solution of (3.18-3.19). Thus, by Theorems 2.1 and 2.2, there exists a solution $\alpha_{n+1}(t)$ of (3.18-3.19) such that $\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_n(t)$, $t \in J$. Similarly, it can be proved that $\alpha_n(t) \le \beta_{n+1}(t) \le \beta_n(t)$, $t \in J$, where $\beta_{n+1}(t)$ is a solution of (3.20-3.21) and $\alpha_n(t)$, $\beta_n(t)$ are lower and upper solutions of (3.20-3.21), respectively. Next, we show that $\alpha_{n+1}(t) \le \beta_{n+1}(t)$.

For that, we have to show that $\alpha_{n+1}(t)$ and $\beta_{n+1}(t)$ are lower and upper solutions of (1.1-1.2), respectively. Using (3.7, 3.8) together with the fact that $\alpha_{n+1}(t)$ is a solution of (3.18-3.19), we get

$$\alpha_{n+1}''(t) + \lambda\alpha_{n+1}'(t) = A(t, \alpha_{n+1}; \alpha_n, \beta_n) \ge N(t, \alpha_{n+1}),$$

$$p\alpha_{n+1}(0) - q\alpha_{n+1}'(0) = h_i(\alpha_{n+1}(\sigma); \alpha_n, \beta_n) \le g_1(\alpha_{n+1}(\sigma)),$$

$$p\alpha_{n+1}(1) + q\alpha_{n+1}'(1) = h_i(\alpha_{n+1}(\sigma); \alpha_n, \beta_n) \le g_2(\alpha_{n+1}(\sigma)),$$

which implies that α_{n+1} is a lower solution of (1.1-1.2). Employing a similar procedure, it can be proved that β_{n+1} is an upper solution of (1.1-1.2). Hence, by Theorem 2.1, it follows that $\alpha_{n+1}(t) \le \beta_{n+1}(t)$. Therefore, by induction, we have

$$\alpha_0(t) \le \alpha_1(t) \le \cdots \le \alpha_n(t) \le \alpha_{n+1}(t) \le \beta_{n+1}(t) \le \beta_n(t) \le \cdots \le \beta_1(t) \le \beta_0(t), \ \forall n \in \mathbb{N}.$$

Since [0,1] is compact and the monotone convergence is pointwise, it follows that $\{\alpha_n\}$ and $\{\beta_n\}$ are uniformly convergent with

$$\lim_{n\to\infty} \alpha_n(t) = x(t), \qquad \lim_{n\to\infty} \beta_n(t) = y(t),$$

such that $\alpha_0(t) \le x(t) \le y(t) \le \beta_0(t)$, where

$$\begin{aligned}\alpha_n(t) &= h_1(\alpha_n(\sigma); \alpha_{n-1}, \beta_{n-1}) \frac{(p - q\lambda)e^{-\lambda} - p\,e^{-\lambda t}}{p[(p - q\lambda)e^{-\lambda} - (p + q\lambda)]} \\ &\quad + h_2(\alpha_n(\sigma); \alpha_{n-1}, \beta_{n-1}) \frac{(p + q\lambda) - p\,e^{-\lambda t}}{p[(p + \lambda q) - (p - \lambda q)e^{-\lambda}]} \\ &\quad + \int_0^1 G(t, s) A(s, \alpha_n(s); \alpha_{n-1}, \beta_{n-1}) ds,\end{aligned}$$

and

$$\beta_n(t) = \hat{h}_1(\beta_n(\sigma); \beta_{n-1}) \frac{(p - q\lambda)e^{-\lambda} - p\, e^{-\lambda t}}{p[(p - q\lambda)e^{-\lambda} - (p + q\lambda)]} + \hat{h}_2(\beta_n(\sigma); \beta_{n-1}) \frac{(p + q\lambda) - p\, e^{-\lambda t}}{p[(p + \lambda q) - (p - \lambda q)e^{-\lambda}]} + \int_0^1 G(t, s)B(s, \beta_n(s); \beta_{n-1}, \beta_{n-1})ds.$$

By the uniqueness of the solution (which follows by the hypotheses of Theorem 2.1), we conclude that $x(t) = y(t)$. This proves that the problem (1.1-1.2) has a unique solution $x(t)$ given by

$$x(t) = g_1(x(\sigma)) \frac{(p - q\lambda)e^{-\lambda} - pe^{-\lambda t}}{p[(p - q\lambda)e^{-\lambda} - (p + q\lambda)]} + g_2(x(\sigma)) \frac{(p + q\lambda) - pe^{-\lambda t}}{p[(p + \lambda q) - (p - \lambda q)e^{-\lambda}]} + \int_0^1 G(t, s)N(s, x(s))\mathrm{d}s.$$

In order to prove that each of the sequences $\{\alpha_n\}$, $\{\beta_n\}$ converges quadratically, we set $z_n(t) = \beta_n(t) - x(t)$ and $r_n(t) = x(t) - \alpha_n(t)$, and note that $z_n \geq 0$, $r_n \geq 0$. We will only prove the quadratic convergence of the sequence $\{r_n\}$ as that of $\{z_n\}$ is similar. By the mean value theorem, we find that

$$\begin{aligned}
& r''_{n+1}(t) + \lambda r'_{n+1}(t) \\
&= x''(t) - \alpha''_{n+1}(t) + \lambda[x'(t) - \alpha'_{n+1}(t)] \\
&= [x''(t) + \lambda x'(t)] - [\alpha''_{n+1}(t) + \lambda\alpha'_{n+1}(t))] \\
&= N(t, x) - A(t, \alpha_{n+1}, \alpha_n, \beta_n) \\
&= F(t, x) + K(t, x) + H(t, x) - \phi(t, x) - \chi(t, x) - F(t, \alpha_n) \\
&- K(t, \alpha_n) - H(t, \alpha_{n+1}) + \phi(t, \alpha_n) + \chi(t, \alpha_n) \\
&- [F_x(t, \beta_n) + K_x(t, \alpha_n) - \phi_x(t, \alpha_n) - \chi_x(t, \beta_n)](\alpha_{n+1} - \alpha_n) \\
&= F(t, x) + K(t, x) + H(t, x) - \phi(t, x) - \chi(t, x) - F(t, \alpha_n) \\
&- K(t, \alpha_n) - H(t, \alpha_{n+1}) + \phi(t, \alpha_n) + \chi(t, \alpha_n) \\
&- [F_x(t, \alpha_n) + K_x(t, \alpha_n) - \phi_x(t, \alpha_n) - \chi_x(t, \beta_n)](r_n - r_{n+1}) \\
&\geq F_x(t, \xi_1)r_n + K_x(t, \xi_2)r_n - Lr_{n+1} - \phi_x(t, \xi_3)r_n - \chi_x(t, \xi_4)r_n \\
&- [F_x(t, \beta_n) + K_x(t, \alpha_n) - \phi_x(t, \alpha_n) - \chi_x(t, \beta_n)](r_n - r_{n+1}) \\
&\geq [F_x(t, \alpha_n) - F_x(t, \beta_n)]r_n + [K_x(t, x) - K_x(t, \alpha_n)]r_n \\
&- [\phi_x(t, x) - \phi_x(t, \alpha_n)]r_n + [\chi_x(t, \beta_n) - \chi_x(t, \alpha_n)]r_n \\
&+ [-L + F_x(t, \beta_n) + K_x(t, \alpha_n) - \phi_x(t, \alpha_n) - \chi_x(t, \beta_n)]r_{n+1} \\
&\geq [-F_{xx}(t, \zeta_5) + \chi_{xx}(t, \zeta_8)]r_n(\beta_n - \alpha_n) + K_{xx}(t, \zeta_6)r_n^2 - \phi_{xx}(t, \zeta_7)r_n^2 \\
&+ [-L + F_x(t, \alpha_n) + K_x(t, \alpha_n) - \phi_x(t, \alpha_n) - \chi_x(t, \alpha_n)]r_{n+1} \\
&\geq [-F_{xx}(t, \zeta_5) + \chi_{xx}(t, \zeta_8)]r_n(z_n + r_n) + K_{xx}(t, \zeta_6)r_n^2 - \phi_{xx}(t, \zeta_7)r_n^2 \\
&\geq [F_{xx}(t, \zeta_5) - \chi_{xx}(t, \zeta_8)] \left(\frac{-3}{2}r_n^2 - \frac{1}{2}z_n^2\right) + [K_{xx}(t, \zeta_6) - \phi_{xx}(t, \zeta_7)]r_n^2 \\
&\geq \left[\frac{-3}{2}F_{xx}(t, \zeta_5) + \frac{3}{2}\chi_{xx}(t, \zeta_8) + K_{xx}(t, \zeta_6) - \phi_{xx}(t, \zeta_7)\right] r_n^2 \\
&+ \left[\frac{-1}{2}F_{xx}(t, \zeta_5) + \frac{1}{2}\chi_{xx}(t, \zeta_8)\right] z_n^2 \\
&\geq -\left[\frac{3}{2}C_1 + \frac{3}{2}C_4 + C_2 + C_3\right] \|r_n\|^2 - \frac{1}{2}[C_1 + C_4]\|z_n\|^2 \\
&\geq -\left(M_1\|r_n\|^2 + M_2\|z_n\|^2\right),
\end{aligned}$$

where $\alpha_n \le \zeta_5, \ \zeta_8 \le \beta_n, \ \alpha_n \le \zeta_6, \ \zeta_7 \le x$, and $|F_{xx}| \le C_1, \ |K_{xx}| \le C_2, \ |\phi_{xx}| \le C_3, \ |\chi_{xx}| \le C_4, \ M_1 = \frac{3}{2}C_1 + \frac{3}{2}C_4 + C_2 + C_3$ and $M_2 = \frac{1}{2}(C_1 + C_2)$.

Now we define

$$N_1(t) = \frac{(p - q\lambda)e^{-\lambda} - p\,e^{-\lambda t}}{p[(p - q\lambda)e^{-\lambda} - (p + q\lambda)]}, \quad N_2(t) = \frac{(p + q\lambda) - p\,e^{-\lambda t}}{p[(p + \lambda q) - (p - \lambda q)e^{-\lambda}]}$$

and obtain

$$\begin{aligned}
&r_{n+1}(t) = x(t) - \alpha_{n+1}(t)\\
&= N_1(t)[g_1(x(\sigma)) - h_1(\alpha_{n+1}(\sigma); \alpha_n, \beta_n)] + N_2(t)[g_2(x(\sigma)) - h_2(\alpha_{n+1}(\sigma); \alpha_n, \beta_n)]\\
&+ \int_0^1 G(t,s)[[N(s, x(s)) - A(s, \alpha_{n+1}(s); \alpha_n, \beta_n)]ds\\
&= N_1(t)[g_1(x(\sigma)) - h_1(\alpha_{n+1}(\sigma); \alpha_n, \beta_n)] + N_2(t)[g_2(x(\sigma)) - h_2(\alpha_{n+1}(\sigma); \alpha_n, \beta_n)]\\
&+ \int_0^1 G(t,s)[r''_{n+1}(s) + \lambda r'_{n+1}(s)]ds\\
&\le N_1(t)[g_1(x(\sigma)) - g_1(\alpha_n(\sigma)) - G'_1(\beta_n(\sigma))(\alpha_{n+1}(\sigma) - \alpha_n(\sigma))\\
&- \psi_1(\alpha_n(\sigma)) + \psi_1(\alpha_{n+1}(\sigma))] + N_2(t)[g_2(x(\sigma)) - g_2(\alpha_n(\sigma))\\
&- G'_2(\beta_n(\sigma))(\alpha_{n+1}(\sigma) - \alpha_n(\sigma)) - \psi_2(\alpha_n(\sigma)) + \psi_2(\alpha_{n+1}(\sigma))]\\
&+ (M_1\|r_n\|^2 + M_2\|z_n\|^2)\int_0^1 |G(t,s)|ds\\
&\le N_1(t)[g'_1(\gamma_1)r_n - G'_1(\beta_n(\sigma))(r_n - r_{n+1}) + \psi'_1(\gamma_2)(r_n - r_{n+1})]\\
&+ N_2(t)[g'_2(\delta_1)r_n - G'_2(\beta_n(\sigma))(r_n - r_{n+1}) + \psi'_2(\delta_2)(r_n - r_{n+1})]\\
&+ M_0(M_1\|r_n\|^2 + M_2\|z_n\|^2).\\
&\le N_1(t)[G'_1(\gamma_1)r_n - \psi'_1(\gamma_1)r_n - G'_1(\beta_n(\sigma))r_n + G'_1(\beta_n(\sigma))r_{n+1})\\
&+ \psi'_1(\gamma_2)r_n - \psi'_1(\gamma_2)r_{n+1})] + N_2(t)[G'_2(\delta_1)r_n - \psi'_2(\delta_1)r_n\\
&- G'_2(\beta_n(\sigma))r_n + G'_2(\beta_n(\sigma))r_{n+1}) + \psi'_2(\delta_2)r_n - \psi'_2(\delta_2)r_{n+1})]\\
&+ M_0(M_1\|r_n\|^2 + M_2\|z_n\|^2).\\
&\le N_1(t)[(G'_1(\alpha_n(\sigma)) - G'_1(\beta_n(\sigma)))r_n - (\psi'_1(x(\sigma))r_n - \psi'_1(u_n(\sigma))r_n)\\
&+ (G'_1(\beta_n(\sigma)) - \psi'_1(\alpha_n(\sigma))r_{n+1}] + N_2(t)[G'_2(\alpha_n(\sigma))r_n - G'_2(\beta_n(\sigma))r_n\\
&- \psi'_2(x(\sigma))r_n + \psi'_2(\alpha_n(\sigma))r_n + (G'_2(\beta_n(\sigma)) - \psi'_2(\alpha_n(\sigma)))r_{n+1})]\\
&+ M_0(M_1\|r_n\|^2 + M_2\|z_n\|^2)\\
&\le N_1(t)[(-G''_1(\rho_1)r_n(z_n + r_n) - \psi''_1(\rho_2)r_n^2 + g'_1(\alpha_n(\sigma))r_{n+1}]\\
&+ N_2(t)[-G''_2(\sigma_1)r_n(z_n + r_n) - \psi''_2(\sigma_2)r_n^2 + g'_2(\alpha_n(\sigma))r_{n+1})]\\
&+ M_0(M_1\|r_n\|^2 + M_2\|z_n\|^2)\\
&\le N_1(t)[(-G''_1(\rho_1)\left(\frac{3}{2}r_n^2 + \frac{1}{2}z_n^2\right) - \psi''_1(\rho_2)r_n^2 + r_{n+1})] + N_2(t)\left[-G''_2(\sigma_1)\left(\frac{3}{2}r_n^2 + \frac{1}{2}z_n^2\right)\right.\\
&\left. - \psi''_2(\sigma_2)r_n^2 + r_{n+1})\right] + M_0\left(M_1\|r_n\|^2 + M_2\|z_n\|^2\right).
\end{aligned}$$

where $\alpha_n \le \gamma_1, \delta_1, \rho_1, \sigma_1 \le x$, $\alpha_n \le \gamma_2 \le x$, and $\alpha_n \le \delta_2, \rho_2, \sigma_2 \le \alpha_{n+1}$. Letting $|G''_i| < D_i$, $|\psi''_i| < E_i, \max_{t\in[0,1]}|N_i| = \overline{N}_i (i = 1, 2)$ and M_0 as an upper bound on $M\int_0^1 G(t,s)\mathrm{d}s$, we obtain

$$\| r_{n+1}(t) \| \le \frac{\|r_n\|^2 W_1 + \|z_n\|^2)W_2}{(1 - \eta)},$$

where $\eta = (\overline{N}_1 + \overline{N}_2) < 1, W_1 = \left[\frac{3}{2}\overline{N}_1 D_1 + \overline{N}_1 E_1 + \frac{3}{2}\overline{N}_2 D_2 \overline{N}_2 E_2 + M_0 M_1\right],$ and $W_2 = \left[+\frac{1}{2}\overline{N}_1 D_1 + \frac{1}{2}\overline{N}_2 D_2 + M_0 M_2\right]$. This completes the proof.

4 Examples

Example 4.1. Consider the problem

$$x''(t) + x'(t) = 2x - t\cos(\pi x/2), \tag{4.1}$$

$$3x(0) - 2x'(0) = \frac{1}{3}x(1/2) + 1, \qquad 3x(1) + 2x'(1) = \frac{1}{2}x(1/2) + 2. \tag{4.2}$$

Here $f(t, x) = 2x - t\cos(\pi x/2)$, $k(t, x) \equiv 0$, $H(t,x) \equiv 0, g_1\left(x\left(\frac{1}{2}\right)\right) = \frac{1}{3}x(1/2) + 1, g_2\left(x\left(\frac{1}{2}\right)\right) = \frac{1}{2}x(1/2) + 2$. Let $\alpha_0 = 0$ and $\beta_0 = 1$ be lower and upper solutions of (4.1-4.2), respectively. We note that $f_x(t,x) = 2 - \frac{\pi}{2}t\sin(\pi x/2) > 0, g_1'\left(x\left(\frac{1}{2}\right)\right) = 1/3, g_2'\left(x\left(\frac{1}{2}\right)\right) = 1/2$. Further, we choose $\phi(t,x) = 3x^2, \psi_i(x) = -\hat{M}_i(x+1)^2, \hat{M}_i > 0,\ i = 1, 2$. We note that $f_{xx}(t,x) + \phi_{xx}(t,x) = -\frac{\pi^2}{4}t\cos(\pi x/2) + 6 \geq 0, g_i''(x) + \psi_i''(x) \leq 0$. Thus, all the conditions of Theorem (3.1) are satisfied. Hence, the conclusion of Theorem 3.1 applies to the problem (4.1-4.2).

Example 4.2. Consider the nonlinear boundary value problem given by

$$x''(t) + \lambda x'(t) = 7x + \sin(\pi xt/2) - t\cos(\pi x/2) + \frac{1}{2}|x|, \quad t \in [0, 1], \tag{4.3}$$

$$3x(0) - 2x'(0) = \frac{x(t)}{4} + 1, \qquad 3x(1) + 2x'(1) = \frac{x(t)}{2} + 2, \tag{4.4}$$

where $f(t, x) = 7x + \sin(\pi xt/2)$, $k(t, x) = -t\cos(\pi x/2)$, $H(t,x) = \frac{1}{2}|x|,\ L = \frac{1}{2},\ g_1(x) = x(t)/4 + 1,\ g_2(x) = x(t)/2 + 2$. Let $\alpha_0 = 0$ and $\beta_0 = 1$ be lower and upper solutions of (4.1-4.2), respectively. Observe that

$$f_x(t,x) + k_x(t,x) - \frac{1}{2} = 7 + \frac{t\pi}{2}\cos\frac{\pi}{2}xt + \frac{t\pi}{2}\sin\frac{\pi}{2}x - \frac{1}{2} > 0,$$

and $0 \leq g_i'(x) \leq 1$. For positive constants M_1, M_2, N_1, N_2, we choose

$$\phi(t,x) = M_1\frac{\pi^2}{4}t^2(1+x)^2,\ \chi(t,x) = -M_2\pi^2x^2,\ \psi_i(x) = -N_i(x+2)^2,$$

such that $f_{xx}(t, x) + \varphi_{xx}(t, x) = \pi^2t^2[2M_1 - \cos(\pi tx/2)]/4 \geq 0$, $k_{xx} + \chi_{xx} = -\pi^2[8M_2 - t\cos(\pi x/2)]/4 \leq 0$. Clearly, $g_i''(x) + \psi_i''(x) \leq 0$. Thus, all the conditions of Theorem 3.1 are satisfied. Hence, the conclusion of theorem (3.1) applies to the problem (4.3-4.4).

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Authors' contributions

Both authors, AA and MHA, contributed to each part of this work equally and read and approved the final version of the manuscript.

Competing interests

The authors declare that they have no competing interests.

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