Haghaieghi and Afrouzi *Boundary Value Problems* 2011, **2011**:52
http://www.boundaryvalueproblems.com/content/2011/1/52

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Sub-super solutions for (*p*-*q*) Laplacian systems

Somayeh Haghaieghi[1*] and Ghasem Alizadeh Afrouzi[2]

* Correspondence:
Haghaieghi_ch86@yahoo.com
[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
Full list of author information is available at the end of the article

Abstract

In this work, we consider the system:

$$\begin{cases} -\Delta_p u = \lambda[g(x)a(u) + f(v)] \text{ in } \Omega \\ -\Delta_q v = \lambda[g(x)b(v) + h(u)] \text{ in } \Omega \\ \quad u = v = 0 \quad\quad\quad\quad\quad \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded region in R^N with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian operator defined by $\Delta_p u$ = div $(|\nabla u|^{p-2}\nabla u)$, p, $q > 1$ and g (x) is a C^1 sign-changing the weight function, that maybe negative near the boundary. f, h, a, b are C^1 non-decreasing functions satisfying $a(0) \geq 0$, $b(0) \geq 0$. Using the method of sub-super solutions, we prove the existence of weak solution.

1 Content

In this paper, we study the existence of positive weak solution for the following system:

$$\begin{cases} -\Delta_p u = \lambda[g(x)a(u) + f(v)] \text{ in } \Omega \\ -\Delta_q v = \lambda[g(x)b(v) + h(u)] \text{ in } \Omega \\ \quad u = v = 0 \quad\quad\quad\quad\quad \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded region in R^N with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian operator defined by $\Delta_p u = div(|\nabla u|^{p-2}\,\nabla u)$, p, $q > 1$ and $g(x)$ is a C^1 sign-changing the weight function, that maybe negative near the boundary. f, h, a, b are C^1 non-decreasing functions satisfying $a(0) \geq 0$, $b(0) \geq 0$.

This paper is motivated by results in [1-5]. We shall show the system (1) with sign-changing weight functions has at least one solution.

2 Preliminaries

In this article, we use the following hypotheses:

(Al) $\lim \dfrac{f\left(M(h(s))^{\frac{1}{q-1}}\right)}{s^{p-1}} = 0$ as $s \to \infty$, $\forall M > 0$

(A2) $\lim f(s) = \lim h(s) = \infty$ as $s \to \infty$.

(A3) $\lim \frac{a(s)}{s^{p-1}} = lim \frac{b(s)}{s^{q-1}} = 0$ as $s \to \infty$.

Let λ_p, λ_q be the first eigenvalue of $-\Delta_p$, $-\Delta_q$ with Dirichlet boundary conditions and ϕ_p, ϕ_q be the corresponding positive eigenfunctions with $||\phi_p||_\infty = ||\phi_q||_\infty = 1$.

Let m, δ, γ, μ_p, $\mu_q > 0$ be such that

$$\begin{cases} |\nabla\varphi_p|^p - \lambda_p\varphi_p \geq m & \text{in } \overline{\Omega}_\delta \\ \varphi_p \geq \mu_p & \text{on } \Omega - \Omega_\delta \end{cases} \tag{2}$$

and

$$\begin{cases} |\nabla\varphi_q|^q - \lambda_q\varphi_q \geq m & \text{in } \overline{\Omega}_\delta \\ \varphi_p \geq \mu_p & \text{on } \Omega - \Omega_\delta. \end{cases} \tag{3}$$

$$\overline{\Omega}_\delta = \{x \in \Omega;\ d(x,\ \partial\Omega) \leq \delta\}.$$

We assume that the weight function $g(x)$ take negative values in Ω_δ, but it requires to be strictly positive in Ω-Ω_δ. To be precise, we assume that there exist positive constants β and η such that $g(x) \geq -\beta$ on $\overline{\Omega}_\delta$ and $g(x) \geq \eta$ on Ω-Ω_δ. Let $s_0 \geq 0$ such that $\eta a(s) + f(s) > 0$, $\eta b(s) + h(s) > 0$ for $s > s_0$ and

$$f_0 = \max\{0,\ -f(0)\},\ h_0 = \max\{0,\ -h(0)\}.$$

For γ such that $\gamma^{r-1}\ t > s_0$; $t = \min\ \{\alpha_p, \alpha_q\}$, $r = \min\{p, q\}$ we define

$$A = \max\left[\frac{\gamma\lambda_p}{\eta a\left(\gamma^{\frac{1}{p-1}}\alpha_p\right)+f\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)}, \frac{\gamma\lambda_q}{\eta b\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)+h\left(\gamma^{\frac{1}{p-1}}\alpha_p\right)}\right]$$

$$B = \min\left[\frac{m\gamma}{\beta a\left(\gamma^{\frac{1}{p-1}}\right)+f_0}, \frac{m\gamma}{\beta b\left(\gamma^{\frac{1}{q-1}}\right)+h_0}\right]$$

where $\alpha_p = \frac{p-1}{p}\mu_p{}^{\frac{p}{p-1}}$ and $\alpha_q = \frac{q-1}{q}\mu_q{}^{\frac{q}{q-1}}$.

We use the following lemma to prove our main results.

Lemma 1.1 [6]. *Suppose there exist sub and supersolutions* (ψ_1, ψ_2) *and* (z_1, z_2) *respectively of (1) such that* $(\psi_1, \psi_2) \leq (z_1, z_2)$. *then (1) has a solution* (u, v) *such that* $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

3 Main result

Theorem 3.1*Suppose that (A1)-(A3) hold, then for every* $\lambda \in [A, B]$, *system (1) has at least one positive solution.*

Proof of Theorem 3.1 We shall verify that (ψ_1, ψ_2) is a sub solution of (1.1) where

$$\psi_1 = \gamma^{\frac{1}{p-1}}\frac{p-1}{p}\varphi_p{}^{\frac{p}{p-1}}$$

$$\psi_2 = \gamma^{\frac{1}{q-1}}\frac{q-1}{q}\varphi_q{}^{\frac{q}{q-1}}.$$

Let $W \in {H_0}^1(\Omega)$ with $w \geq 0$. Then

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\nabla w \mathrm{d}x = \gamma\int_\Omega(\lambda_p{\varphi_p}^p - |\nabla\varphi_p|^p)w\mathrm{d}x \tag{4}$$

Now, on $\overline{\Omega}_\delta$ by (2),(3) we have

$$\gamma(\lambda_p{\varphi_p}^p - |\nabla\varphi_p|^p) \leq -m\gamma$$

Since $\lambda \leq B$ then

$$\lambda \leq \frac{m\gamma}{\beta a(\gamma^{\frac{1}{p-1}}) + f_0}.$$

thus

$$\begin{aligned}\gamma(\lambda_p{\varphi_p}^p - |\nabla\varphi_p|^p) &\leq -m\gamma\\ &\leq \lambda\left(-\beta a(\gamma^{\frac{1}{p-1}}) - f_0\right)\\ &\leq \lambda\left(g(x)a(\gamma^{\frac{1}{p-1}}) - f_0\right)\ \lambda\ \left(g(x)\,a\left(\tfrac{p-1}{p}\gamma^{\frac{1}{p-1}}{\varphi_p}^{\frac{1}{p-1}}\right)\right.\\ &\left.+f\left(\tfrac{q-1}{q}\gamma^{\frac{1}{q-1}}{\varphi_q}^{\frac{1}{q-1}}\right)\right)\end{aligned}$$

then by (4)

$$\begin{aligned}\int_{\overline{\Omega}_\delta}|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla w\mathrm{d}x \quad &\leq \int_{\overline{\Omega}_\delta}\lambda\left(g(x)\,a\left(\tfrac{p-1}{p}\gamma^{\frac{1}{p-1}}{\varphi_p}^{\frac{p}{p-1}}\right)\right.\\ &\left.+f\left(\tfrac{q-1}{q}\gamma^{\frac{1}{q-1}}{\varphi_q}^{\frac{q}{q-1}}\right)\right)w\mathrm{d}x\end{aligned}$$

A similar argument shows that

$$\begin{aligned}\int_{\overline{\Omega}_\delta}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla w\mathrm{d}x \quad &\leq \int_{\overline{\Omega}_\delta}\lambda\left(g(x)\,b\left(\tfrac{q-1}{q}\gamma^{\frac{1}{q-1}}{\varphi_q}^{\frac{1}{q-1}}\right)\right.\\ &\left.+h\left(\tfrac{p-1}{p}\gamma^{\frac{1}{p-1}}{\varphi_p}^{\frac{1}{p-1}}\right)\right)w\mathrm{d}x\end{aligned}$$

Next, on $\Omega - \overline{\Omega}_\delta$. Since $\lambda \geq A$, then

$$\lambda \geq \frac{\gamma\lambda_p}{\eta a\left(\gamma^{\frac{1}{p-1}}\alpha_p\right) + f\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)}$$

so we have

$$\begin{aligned}\gamma(\lambda_p{\varphi_p}^p - |\nabla\varphi_p|^p) \ &\leq \gamma\lambda_p\\ &\leq \lambda\left[\eta a\left(\gamma^{\frac{1}{p-1}}\alpha_p\right) + f\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)\right]\\ &\leq \lambda[g(x)a(\psi_1) + f(\psi_2)],\ \Omega - \overline{\Omega_\delta}\end{aligned}$$

Then by (4) on we have

$$-\Delta_p \psi_1 \le \lambda[g(x)a(\psi_1) + f(\psi_2)] \quad \text{on } \Omega - \overline{\Omega_\delta}$$

A similar argument shows that

$$-\Delta_q \psi_2 \le \lambda[g(x)b(\psi_2) + h(\psi_1)]$$

We suppose that κ_p and κ_q be solutions of

$$\begin{cases} -\Delta_p \kappa_p = 1 & \text{in } \Omega \\ \kappa_p = 0 & \text{on } \partial\Omega \end{cases}$$
$$\begin{cases} -\Delta_q \kappa_q = 1 & \text{in } \Omega \\ \kappa_q = 0 & \text{on } \partial\Omega \end{cases}$$

respectively, and $\mu'_p = ||\kappa_p||_\kappa$, $||\kappa_q||_\kappa = \mu'_q$.
Let

$$(z_1,\ z_2) = \left(\frac{c}{\mu'_p} \lambda^{\frac{1}{p-1}} \kappa_p,\ \left[2h\left(c\lambda^{\frac{1}{q-1}} \right) \right]^{\frac{1}{q-1}} \lambda^{\frac{1}{q-1}} \kappa_q \right).$$

Let $W \in H_0{}^1(\Omega)$ with $w \ge 0$.
For sufficient C large

$$\frac{\mu'_p{}^{p-1} \left[||g||_\infty a\ \left(C\lambda^{\frac{1}{p-1}} \right) + f\left(\left(2h(C\lambda^{\frac{1}{p-1}}) \right)^{\frac{1}{q-1}} \lambda^{\frac{1}{q-1}} \mu'_q \right) \right]}{C^{p-1}} \le 1$$

then

$$\begin{aligned}
\int |\nabla z_1|^{p-2} \nabla z_1 \nabla w \mathrm{d}x &= \lambda \left(\frac{C}{\mu'_p} \right)^{p-1} \int\ w\mathrm{d}x \\
&\ge \lambda \int\ \left[||g||_\infty a\,(C\lambda^{\frac{1}{p-1}}) + f\ \left((2h\,(C\lambda^{\frac{1}{p-1}}))^{\frac{1}{q-1}} \lambda^{\frac{1}{q-1}} \mu'_q \right) \right] \mathrm{d}x \\
&\ge \lambda \int\ \left[g(x)\,a\,(C\lambda^{\frac{1}{p-1}} \tfrac{\kappa_p}{\mu'_p}) + f\ \left((2h\,(C\lambda^{\frac{1}{p-1}}))^{\frac{1}{q-1}} \lambda^{\frac{1}{q-1}} \kappa_q \right) \right] \mathrm{d}x \\
&= \int\ [g(x)\,a\,(z_1) + f(z_2)]\,w\mathrm{d}x
\end{aligned}$$

Similarly, choosing C large so that

$$\frac{||g||_\infty \left(b\left(2h\ \left(C\lambda^{\frac{1}{p-1}} \right) \right)^{\frac{1}{q-1}} \lambda^{\frac{1}{q-1}} \mu'_q \right)}{h\left(C\lambda^{\frac{1}{p-1}} \right)} \le 1$$

then

$$\int |\nabla z_2|^{q-2}\nabla z_2 \nabla w \mathrm{d}x = 2\lambda h\left(C\lambda^{\frac{1}{p-1}}\right)\int w\mathrm{d}x$$
$$\geq \lambda \int \left[||g||_\infty b(z_2) + h(z_1)\right] w\mathrm{d}x.$$

Hence by Lemma (1.1), there exist a positive solution (u, v) of (1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$.

Author details
[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran [2]Department of Mathematics, Faculty of Mathematical Sciences University of Mazandaran, Babolsar, Iran

Authors' contributions
SH has presented the main purpose of the article and has used GAA contribution due to reaching to conclusions. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 13 August 2011 Accepted: 2 December 2011 Published: 2 December 2011

References

1. Ali, J, Shivaji, R: Existence results for classes of Laplacian system with sign-changing weight. Appl Math Anal. **20**, 558–562 (2007)
2. Rasouli, SH, Halimi, Z, Mashhadban, Z: A remark on the existence of positive weak solution for a class of (p, q)-Laplacian nonlinear system with sign-changing weight. Nonlinear Anal. **73**, 385–389 (2010). doi:10.1016/j.na.2010.03.027
3. Ali, J, Shivaji, R: Positive solutions for a class of (p)-Laplacian systems with multiple parameters. J Math Anal Appl. **335**, 1013–1019 (2007). doi:10.1016/j.jmaa.2007.01.067
4. Hai, DD, Shivaji, R: An existence results on positive solutions for class of semilinear elliptic systems. Proc Roy Soc Edinb A. **134**, 137–141 (2004). doi:10.1017/S0308210500003115
5. Hai, DD, Shivaji, R: An Existence results on positive solutions for class of p-Laplacian systems. Nonlinear Anal. **56**, 1007–1010 (2004). doi:10.1016/j.na.2003.10.024
6. Canada, A, Drabek, P, Azorero, PL, Peral, I: Existence and multiplicity results for some nonlinear elliptic equations. A survey Rend Mat Appl. **20**, 167–198 (2000)

doi:10.1186/1687-2770-2011-52
Cite this article as: Haghaieghi and Afrouzi: **Sub-super solutions for (*p*-*q*) Laplacian systems.** *Boundary Value Problems* 2011 **2011**:52.