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Vanishing heat conductivity limit for the 2D Cahn-Hilliard-Boussinesq system

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Abstract

This article studies the vanishing heat conductivity limit for the 2D Cahn-Hilliard-boussinesq system in a bounded domain with non-slip boundary condition. The result has been proved globally in time.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, simply connected domain with smooth boundary $\partial\Omega$, and n is the unit outward normal vector to $\partial\Omega$. We consider the following Cahn-Hilliard-Boussinesq system in $\Omega \times (0, \infty)$ [1]:

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi - \Delta u = \mu\nabla\phi + \theta e_2, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t\theta + u \cdot \nabla\theta = \varepsilon\Delta\theta, \tag{1.3}$$

$$\partial_t\phi + u \cdot \nabla\phi = \Delta\mu, \tag{1.4}$$

$$-\Delta\phi + f'(\phi) = \mu, \tag{1.5}$$

$$u = 0, \theta = 0, \frac{\partial\phi}{\partial n} = \frac{\partial\mu}{\partial n} = 0 \quad on \quad \partial\Omega \times (0, \infty), \tag{1.6}$$

$$(u, \theta, \phi)(x, 0) = (u_0, \theta_0, \phi_0)(x), x \in \Omega, \tag{1.7}$$

where u, π, θ and φ denote unknown velocity field, pressure scalar, temperature of the fluid and the order parameter, respectively. ε >0 is the heat conductivity coefficient and e_2 : = $(0, 1)^t$. μ is a chemical potential and $f(\phi) := \frac{1}{4}(\phi^2 - 1)^2$ is the double well potential.

When $\varphi = 0$, (1.1), (1.2) and (1.3) is the well-known Boussinesq system. In [2] Zhou and Fan proved a regularity criterion $\omega \dot{=} curlu \in L^1(0, T; \dot{B}^0_{\infty,\infty})$ for the 3D Boussinesq system with partial viscosity. Later, in [3] Zhou and Fan studied the Cauchy problem of certain Boussinesq-α equations in n dimensions with $n = 2$ or 3. We establish regularity for the solution under $\nabla u \in L^1(0,T;\dot{B}^0_{\infty,\infty})$. Here $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space. Chae [4] studied the vanishing viscosity limit $\varepsilon \to 0$ when $\Omega = \mathbb{R}^2$. The aim of this article is to prove a similar result. We will prove that

Theorem 1.1. *Let* $(u_0,\theta_0) \in H_0^1 \cap H^2$, $\varphi_0 \in H^4$, $\operatorname{div} u_0 = 0$ *in* Ω *and* $\frac{\partial \phi_0}{\partial n} = \frac{\partial \mu_0}{\partial n} = 0$ *on* $\partial\Omega$. *Then, there exists a positive constant C independent of* ε *such that*

$$\begin{aligned} &\| u_\varepsilon \|_{L^\infty(0,T;H^2)} \le C, \| \theta_\varepsilon \|_{L^\infty(0,T;H^2)} \le C, \\ &\| \phi_\varepsilon \|_{L^\infty(0,T;H^4)} \le C, \| \partial_t(u_\epsilon,\theta_\varepsilon,\phi_\epsilon)\|_{L^2(0,T;L^2)} \le C, \end{aligned} \tag{1.8}$$

for any $T > 0$, *which implies*

$$(u_\varepsilon,\theta_\varepsilon,\phi_\varepsilon) \to (u,\theta,\phi) \quad \textit{strongly in} \;\; L^2(0,T;H^1) \;\; \textit{when} \;\; \varepsilon \to 0. \tag{1.9}$$

Here, (u, θ, φ) *is the solution of the problem (1.1)-(1.7) with* $\varepsilon = 0$.

2 Proof of Theorem 1.1

Since (1.9) follows easily from (1.8) by the Aubin-Lions compactness principle, we only need to prove the a priori estimates (1.8). From now on, we will drop the subscript ε and throughout this section C will be a constant independent of ε.

First, by the maximum principle, it follows from (1.2), (1.3), and (1.6) that

$$\| \theta \|_{L^\infty(0,T;L^\infty)} \le \| \theta_0 \|_{L^\infty} \le C. \tag{2.1}$$

Testing (1.3) by θ, using (1.2) and (1.6), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \theta^2 \mathrm{d}x + \varepsilon \int |\nabla\theta|^2 \mathrm{d}x = 0,$$

whence

$$\sqrt{\varepsilon} \, \| \theta \|_{L^2(0,T;H^1)} \le C. \tag{2.2}$$

Testing (1.1) and (1.4) by u and μ, respectively, using (1.2), (1.6), (2.1), and summing up the result, we find that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}u^2 + \frac{1}{2}|\nabla\phi|^2 + f(\phi)\mathrm{d}x + \int |\nabla u|^2 + |\nabla\mu|^2 \mathrm{d}x \\ &= \int \theta e_2 u \mathrm{d}x \le \| \theta \|_{L^2} \| u \|_{L^2} \le C \| u \|_{L^2}, \end{aligned}$$

which gives

$$\| \phi \|_{L^\infty(0,T;H^1)} \le C, \tag{2.3}$$

$$\| u \|_{L^\infty(0,T;L^2)} + \| u \|_{L^2(0,T;H^1)} \le C, \tag{2.4}$$

$$\| \nabla\mu \|_{L^2(0,T;L^2)} \le C. \tag{2.5}$$

Testing (1.4) by φ, using (1.2), (1.5) and (1.6), we infer that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \phi^2 \mathrm{d}x + \int |\Delta\phi|^2 \mathrm{d}x = \int (\phi^3 - \phi)\Delta\phi \mathrm{d}x \\ &= -3\int \phi^2 |\nabla\phi|^2 \mathrm{d}x - \int \phi\Delta\phi \mathrm{d}x \le -\int \phi\Delta\phi \mathrm{d}x \\ &\le \frac{1}{2}\int |\Delta\phi|^2 \mathrm{d}x + \frac{1}{2}\int \phi^2 \mathrm{d}x, \end{aligned}$$

which leads to

$$\|\phi\|_{L^2(0,T;H^2)} \le C. \tag{2.6}$$

We will use the following Gagliardo-Nirenberg inequality:

$$\|\phi\|_{L^\infty}^2 \le C\|\phi\|_{L^6}\|\phi\|_{H^2}. \tag{2.7}$$

It follows from (2.6), (2.7), (2.5), (2.3) and (1.5) that

$$\begin{aligned}
&\int_0^T\int|\nabla\Delta\phi|^2\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int|\nabla(f'(\phi)-\mu)|^2\mathrm{d}x\mathrm{d}t\\
&\le C\int_0^T\int|\nabla\mu|^2\mathrm{d}x\mathrm{d}t+C\int_0^T\int|\nabla(\phi^3-\phi)|^2\mathrm{d}x\mathrm{d}t\\
&\le C+C\int_0^T\int\phi^4|\nabla\phi|^2\mathrm{d}x\mathrm{d}t\\
&\le C+C\|\nabla\phi\|_{L^\infty(0,T;L^2)}^2\int_0^T\|\phi\|_{L^\infty}^4\mathrm{d}t\\
&\le C+C\int_0^T\|\phi\|_{L^6}^2\|\phi\|_{H^2}^2\,\mathrm{d}t\\
&\le C+C\|\phi\|_{L^\infty(0,T;H^1)}^2\int_0^T\|\phi\|_{H^2}^2\mathrm{d}t\le C,
\end{aligned} \tag{2.8}$$

which yields

$$\|\phi\|_{L^2(0,T;H^3)} \le C, \tag{2.9}$$

$$\|\phi\|_{L^4(0,T;L^\infty)} \le C, \tag{2.10}$$

$$\|\nabla\phi\|_{L^2(0,T;L^\infty)} \le C. \tag{2.11}$$

Testing (1.4) by $\Delta^2\varphi$, using (1.5), (2.4), (2.3), (2.10) and (2.11), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\int|\Delta\phi|^2\mathrm{d}x+\int|\Delta^2\phi|^2\mathrm{d}x\\
&=-\int u\cdot\nabla\phi\cdot\Delta^2\phi\mathrm{d}x+\int\Delta(\phi^3-\phi)\cdot\Delta^2\phi\mathrm{d}x\\
&\le\|u\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\Delta^2\phi\|_{L^2}+\|\Delta(\phi^3-\phi)\|_{L^2}\|\Delta^2\phi\|_{L^2}\\
&\le C\|\nabla\phi\|_{L^\infty}\|\Delta^2\phi\|_{L^2}\\
&\quad+C(\|\phi\|_{L^\infty}^2\|\Delta\phi\|_{L^2}+\|\phi\|_{L^\infty}\|\nabla\phi\|_{L^\infty}\|\nabla\phi\|_{L^2}+\|\Delta\phi\|_{L^2})\|\Delta^2\phi\|_{L^2}\\
&\le C\|\nabla\phi\|_{L^\infty}\|\Delta^2\phi\|_{L^2}\\
&\quad+C(\|\phi\|_{L^\infty}^2\|\Delta\phi\|_{L^2}+\|\phi\|_{H^2}\|\nabla\phi\|_{L^\infty}+\|\Delta\phi\|_{L^2})\|\Delta^2\phi\|_{L^2}\\
&\le\frac{1}{2}\|\Delta^2\phi\|_{L^2}^2+C\|\nabla\phi\|_{L^\infty}^2+C\|\phi\|_{L^\infty}^4\|\Delta\phi\|_{L^2}^2\\
&\quad+C\|\nabla\phi\|_{L^\infty}^2\|\phi\|_{H^2}^2+C\|\Delta\phi\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|\phi\|_{L^\infty(0,T;H^2)}+\|\phi\|_{L^2(0,T;H^4)}\le C. \tag{2.12}$$

Testing (1.1) by $-\Delta u+\nabla\pi$, using (1.2), (1.6), (2.12), (2.1) and (2.4), we reach

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\mathrm{d}x+\int(-\Delta u+\nabla\pi)^2\mathrm{d}x\\
=&\int(\mu\nabla\phi+\theta e_2-u\cdot\nabla u)(-\Delta u+\nabla\pi)\mathrm{d}x\\
\le&(\|\mu\|_{L^2}\|\nabla\phi\|_{L^\infty}+\|\theta\|_{L^2}+\|u\|_{L^4}\|\nabla u\|_{L^4})\|-\Delta u+\nabla\pi\|_{L^2}\\
\le&C(\|\nabla\phi\|_{L^\infty}+1+\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\cdot\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2})\|-\Delta u+\nabla\pi\|_{L^2}\\
\le&C\|\nabla\phi\|_{L^\infty}^2+C+C\|\nabla u\|_{L^2}^4+\frac{1}{2}\|-\Delta u+\nabla\pi\|_{L^2}^2,
\end{aligned}$$

which yields

$$\|u\|_{L^\infty(0,T;H^1)}+\|u\|_{L^2(0,T;H^2)}\le C. \tag{2.13}$$

Here, we have used the Gagliardo-Nirenberg inequalities:

$$\begin{aligned}
&\|u\|_{L^4}^2\le C\|u\|_{L^2}\|\nabla u\|_{L^2},\\
&\|\nabla u\|_{L^4}^2\le C\|\nabla u\|_{L^2}\|u\|_{H^2},
\end{aligned}$$

and the H^2-theory of the Stokes system:

$$\|u\|_{H^2}+\|\pi\|_{H^1}\le C\|-\Delta u+\nabla\pi\|_{L^2}. \tag{2.14}$$

Similarly to (2.13), we have

$$\|\partial_t u\|_{L^2(0,T;L^2)}\le C. \tag{2.15}$$

(1.1), (1.2), (1.6) and (1.7) can be rewritten as

$$\begin{cases}
\partial_t u-\Delta u+\nabla\pi=g:=\mu\nabla\phi+\theta e_2-u\cdot\nabla u, & in\ \ \Omega\times(0,\ \infty),\\
u=0,\ \ on\ \ \partial\Omega\times(0,\ \infty),\\
u(x,0)\ =\ u_0(x).
\end{cases}$$

Using (2.12), (2.1), (2.13), and the regularity theory of Stokes system, we have

$$\begin{aligned}
&\|\partial_t u\|_{L^2(0,T;L^p)}+\|u\|_{L^2(0,T;W^{2,p})}\le C\|g\|_{L^2(0,T;L^p)}\\
\le&C\|\mu\|_{L^2(0,T;L^\infty)}\|\nabla\phi\|_{L^\infty(0,T;L^p)}+C\|\theta\|_{L^\infty(0,T;L^\infty)}\\
&+C\|u\|_{L^\infty(0,T;L^{2p})}\|\nabla u\|_{L^2(0,T;L^{2p})}\le C,
\end{aligned} \tag{2.16}$$

for any $2<p<\infty$.
(2.16) gives

$$\|\nabla u\|_{L^2(0,T;L^\infty)}\le C. \tag{2.17}$$

It follows from (1.3) and (1.6) that

$$\Delta\theta=0\ \ on\ \ \partial\Omega\times(0,\infty). \tag{2.18}$$

Applying Δ to (1.3), testing by $\Delta\theta$, using (1.2), (1.6), (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Delta\theta|^2\mathrm{d}x+\varepsilon\int |\nabla\Delta\theta|^2\mathrm{d}x\\
&=-\int(\Delta(u\cdot\nabla\theta)-u\nabla\Delta\theta)\Delta\theta\mathrm{d}x\\
&\le C(\|\Delta u\|_{L^4}\|\nabla\theta\|_{L^4}+\|\nabla u\|_{L^\infty}\|\Delta\theta\|_{L^2})\|\Delta\theta\|_{L^2}\\
&\le C(\|\Delta u\|_{L^4}+\|\nabla u\|_{L^\infty})\|\Delta\theta\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|\theta\|_{L^\infty(0,T;H^2)}+\sqrt{\varepsilon}\|\theta\|_{L^2(0,T;H^3)}\le C. \tag{2.19}$$

It follows from (1.3), (1.6), (2.19) and (2.13) that

$$\|\partial_t\theta\|_{L^\infty(0,T;L^2)}\le C. \tag{2.20}$$

Taking ∂_t to (1.4) and (1.5), testing by $\partial_t\varphi$, using (1.2), (1.6), (2.12), and (2.15), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t\phi|^2\mathrm{d}x+\int |\Delta\partial_t\phi|^2\mathrm{d}x\\
&=-\int\partial_t u\cdot\nabla\phi\cdot\partial_t\phi\mathrm{d}x+\int\Delta(3\phi^2\partial_t\phi-\partial_t\phi)\cdot\partial_t\phi\mathrm{d}x\\
&=-\int\partial_t u\cdot\nabla\phi\cdot\partial_t\phi\mathrm{d}x+\int(3\phi^2\partial_t\phi-\partial_t\phi)\Delta\partial_t\phi\mathrm{d}x\\
&\le\|\partial_t u\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\partial_t\phi\|_{L^2}+(\|3\phi\|_{L^\infty}^2+1)\|\partial_t\phi\|_{L^2}\|\Delta\partial_t\phi\|_{L^2}\\
&\le\|\partial_t u\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\partial_t\phi\|_{L^2}+\frac{1}{2}\|\Delta\partial_t\phi\|_{L^2}^2+C\|\partial_t\phi\|_{L^2}^2,
\end{aligned}$$

which gives

$$\|\partial_t\phi\|_{L^\infty(0,T;L^2)}+\|\partial_t\phi\|_{L^2(0,T;H^2)}\le C. \tag{2.21}$$

By the regularity theory of elliptic equation, it follows from (1.4), (1.5), (1.6), (2.21), (2.13) and (2.12) that

$$\begin{aligned}
\|\phi\|_{L^\infty(0,T;H^4)}&\le C\|\Delta\phi\|_{L^\infty(0,T;H^2)}\le C\|\mu-f'(\phi)\|_{L^\infty(0,T;H^2)}\\
&\le C\|\mu\|_{L^\infty(0,T;H^2)}+C\|f'(\phi)\|_{L^\infty(0,T;H^2)}\\
&\le C\|\Delta\mu\|_{L^\infty(0,T;L^2)}+C\|f'(\phi)\|_{L^\infty(0,T;H^2)}\\
&\le C\|\partial_t\phi+u\cdot\nabla\phi\|_{L^\infty(0,T;L^2)}+C\|f'(\phi)\|_{L^\infty(0,T;H^2)}\\
&\le C\|\partial_t\phi\|_{L^\infty(0,T;L^2)}+C\|u\|_{L^\infty(0,T;L^4)}\|\nabla\phi\|_{L^\infty(0,T;L^4)}\\
&\quad+C\|f'(\phi)\|_{L^\infty(0,T;H^2)}\le C.
\end{aligned} \tag{2.22}$$

Taking ∂_t to (1.1), testing by $\partial_t u$, using (1.2), (1.6), (2.17), (2.22), (2.21) and (1.5), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_t u|^2\mathrm{d}x+\int |\nabla\partial_t u|^2\mathrm{d}x\\
&=-\int\partial_t u\cdot\nabla u\cdot\partial_t u\mathrm{d}x+\int(\partial_t\mu\cdot\nabla\phi+\mu\cdot\nabla\partial_t\phi+\partial_t\theta e_2)\partial_t u\mathrm{d}x\\
&\le\|\nabla u\|_{L^\infty}\|\partial_t u\|_{L^2}^2+(\|\partial_t u\|_{L^2}\|\nabla\phi\|_{L^\infty}+\|\mu\|_{L^\infty}\|\nabla\partial_t\phi\|_{L^2}+\|\partial_t\theta\|_{L^2})\|\partial_t u\|_{L^2}\\
&\le\|\nabla u\|_{L^\infty}\|\partial_t u\|_{L^2}^2+C(\|\Delta\partial_t\phi\|_{L^2}+\|\partial_t(\phi^3-\phi)\|_{L^2}+\|\nabla\partial_t\phi\|_{L^2}+1)\|\partial_t u\|_{L^2},
\end{aligned}$$

which implies

$$\| \partial_t u\|_{L^\infty(0,T;L^2)} + \| \partial_t u\|_{L^2(0,T;H^1)} \le C. \tag{2.23}$$

Using (2.23), (2.22), (2.1), (2.13), (1.1), (1.2), (1.6) and the H^2-theory of the Stokes system, we arrive at

$$\| u\|_{L^\infty(0,T;H^2)} \le C.$$

This completes the proof.

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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