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Blow-up problems for a compressible reactive gas model

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Abstract

This paper investigates a compressible reactive gas model with homogeneous Dirichlet boundary conditions. Under the parameters and the initial data satisfying some conditions, we prove that the solutions have global blow-up, and the blow-up rate is uniform in all compact subsets of the domain. Moreover, the blow-up rates of $|u(t)|_\infty$ and $|v(t)|_\infty$ are precisely determined.

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1 Introduction and main results

In this paper, we investigate blow-up and the blow-up rate of nonnegative solutions for the following degenerate reaction-diffusion system with nonlocal sources:

$$\begin{cases} u_t = \nabla\cdot(u^m\nabla u) + au^{p_1}\|v\|^{p_2}_{B,\alpha_1}, & (x,t)\in B\times(0,T),\\ v_t = \nabla\cdot(v^n\nabla v) + bv^{q_1}\|u\|^{q_2}_{B,\alpha_2}, & (x,t)\in B\times(0,T),\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x\in B,\\ u(x,t) = v(x,t) = 0, & (x,t)\in\partial B\times(0,T), \end{cases} \tag{1.1}$$

where $B = B(0,R)\subset\mathbb{R}^N$ $(N\geq 1)$ is a ball centered at the origin with the radius $R\in\mathbb{R}^+$, $a,b>0$, exponents $p_2,q_2,\alpha_1,\alpha_2\geq 1$, $m,n,p_1,q_1>0$, and $T<\infty$ is the maximal existence time of a solution, $\|\cdot\|^\alpha_{B,\alpha} = \int_B|\cdot|^\alpha\,dx$.

The system (1.1) models such as heat propagations in a two-components combustible mixture gases [1]. This problem is worth studying because of the applications to heat and mass transport processes (see [2, 3]). In addition, there exist interesting interactions among the multi-nonlinearities described by these exponents in the problem (1.1).

In the past decades, many physical phenomena have been formulated into nonlocal mathematical models and studied by many authors. Here, we will recall some of those results concerning the first initial boundary value problem.

At first, the global solutions and blow-up problems for a single parabolic equation with nonlocal nonlinearity sources had been studied extensively, see [4–10] and references therein. As a typical example, in [4] Souplet considered the equation with spatial integral term

$$u_t = \Delta u + g\left(\int_\Omega f\big(u(x,t)\big)\,\mathrm{d}x\right) \tag{1.2}$$

and the equation with both local and nonlocal terms

$$u_t = \Delta u + \int_\Omega f\big(u(x,t)\big)\,\mathrm{d}x + h\big(u(x,t)\big). \tag{1.3}$$

These two equations are related to some ignition models for compressible reactive gases. The author introduced a method to investigate the profile of blow-up solutions of (1.2) and (1.3) and observed the asymptotic blow-up behaviors of the solutions. In addition, an important model in the theory of nuclear reactor dynamics can be described by the following equation with the space-time integral term:

$$u_t = \Delta u + f\left(\int_0^t \int_\Omega g\big(u(y,s)\big)\beta(y)\,\mathrm{d}y\,\mathrm{d}s\right). \tag{1.4}$$

The blow-up of its solutions was studied by Pao [5], Guo and Su [6].

In 2003, Li and Xie [7] considered the following problem:

$$v_\tau = \Delta v^m + a v^{p_1} \int_\Omega v^{q_1}\,\mathrm{d}x. \tag{1.5}$$

By introducing some transformations $u = v^m$, $t = m\tau$ (1.5) takes the form

$$u_t = u^p\left(\Delta u + a u^r \int_\Omega u^s\,\mathrm{d}x\right). \tag{1.6}$$

Then they proved that the solution of (1.6) blows up in finite time for large initial data and obtained the blow-up rate. Recently, Liu *et al.* in [8] investigated the blow-up rate of solutions to diffusion equation (1.6). Their approach was based on sub- and super-solution methods which were very different from those previously used in the study of the blow-up rate. They proved, by using the maximum principle, that the solutions have global blow-up, and the rate of blow-up is uniform in all compact subsets of the domain. Here the global blow-up means that there exists $0 < T < +\infty$ such that

$$\lim_{t\to T^-} \big|u(\cdot,t)\big| = \infty \quad \text{or} \quad \lim_{t\to T^-} \big|v(\cdot,t)\big| = \infty \qquad \text{for all } x \in \Omega.$$

Secondly, we should point out that in the case of $m = n = 0$, the system (1.1) becomes a semilinear system. To our knowledge, there do not seem to be any results in the literature on blow-up problems of these types. But other related works of the semilinear case have been deeply investigated by many authors, *e.g.*, see [11, 12], and the authors of this paper in [13] studied the system

$$u_t = \Delta u + a(x)u^{p_1}(x,t)v^{q_1}(0,t), \qquad v_t = \Delta v + b(x)v^{p_2}(x,t)u^{q_2}(0,t),$$

where the simultaneous and non-simultaneous blow-up criteria were obtained by using the fundamental solution of the heat equation. On the other hand, there are many known results concerning the global solutions and blow-up problems for the parabolic system with local nonlinearities, localized nonlinearities and nonlinear boundary conditions, see [14–17] and references therein. In particular, Ling and Wang in [18] considered the following degenerate parabolic system:

$$u_t = \Delta u^m + v^{p_1}\|u\|_\alpha^{p_2}, \qquad v_t = \Delta v^n + u^{q_1}\|v\|_\beta^{q_2}$$

in a bounded domain Ω, with the help of the super- and sub-solution methods, the critical exponent of the system was determined. Motivated by the above works, under the following conditions:

$$0 < m < p_1 < 1, \quad 0 < n < q_1 < 1, \quad p_2q_2 > (1-p_1)(1-q_1), \tag{1.7}$$

we consider a more general degenerate parabolic system (1.1) which includes the problems considered in [7, 8] and [17] as special cases. Employing the ideas in [7, 8], we describe the blow-up rate of the radially symmetric solutions to (1.1). Here we discuss the blow-up of radially symmetric solutions as well as derive their blow-up rate. Moreover, we get the accurate coefficient of the blow-up rate. For the related discussion on a radially symmetric solution, we refer the readers to [19] and references therein.

In this paper, we always assume that the initial data $(u_0, v_0) \in \mathbf{V}$ ($\mathbf{V}$ is defined by (1.8)) and satisfies the following (H1)-(H3) or (H4):

(H1) $u_0(x), v_0(x) \in C^{2+\alpha}(B) \cap C(\overline{B})$, $\alpha \in (0,1)$.

(H2) $u_0(x), v_0(x) > 0$ in B, $u_0(x) = v_0(x) = 0$, $\frac{\partial u_0}{\partial \nu}, \frac{\partial v_0}{\partial \nu} < 0$ on ∂B.

(H3) $u_0(x)$, $v_0(x)$ are radially symmetric, $u_0'(r), v_0'(r) \le 0$ for $r \in (0,R)$, $r = |x|$.

Denote the set of initial data, depending only on the radial variable in the spherical coordinate system of $\mathbb{R}^N$:

$$\mathbf{V} = \{(u_0(r), v_0(r)) | \Phi_1(r) > 0, \Phi_2(r) > 0\}, \tag{1.8}$$

where

$$\Phi_1(r) = m(u_0'(r))^2 u_0^{m-1}(r) + u_0^m(r)\left(u_0''(r) + \frac{N-1}{r}u_0'(r)\right) + au_0^{p_1}(r)\|v_0(r)\|_{B,\alpha_1}^{p_2},$$

$$\Phi_2(r) = n(v_0'(r))^2 v_0^{n-1}(r) + v_0^n(r)\left(v_0''(r) + \frac{N-1}{r}v_0'(r)\right) + bv_0^{q_1}(r)\|u_0(r)\|_{B,\alpha_2}^{q_2}.$$

It is noted that the set $\mathbf{V}$ is not empty. For example, for the simplest case $N = R = 1$ and $a = b = 1$, for any constant exponents m, n and p_i, q_i, α_i, $i = 1,2$, there exist positive constants a_1, a_2 such that $(u_0, v_0) \in \mathbf{V}$ with $u_0(r) = a_1/2 - a_1r^2/2$, $v_0(r) = a_2/2 - a_2r^2/2$, $r \in [0,1)$.

(H4) Let δ_0, k_1, k_2 be positive constants (will be given in Section 3), and there exists a constant $\delta > \delta_0$ such that

$$\begin{cases} \Delta u_{20} + a(1+m-p_1)\|v_{20}\|_{B,\mu_1}^{\sigma_1} - \delta u_{20}^{k_1+1-r_1} \ge 0, \\ \Delta v_{20} + b(1+n-q_1)\|u_{20}\|_{B,\mu_2}^{\sigma_2} - \delta v_{20}^{k_2+1-r_2} \ge 0, \end{cases} \tag{1.9}$$

here u_{20}, v_{20} and σ_i, μ_i, r_i are defined by (2.9) and (2.6).

Then, our main results read as follows in detail.

Theorem 1 *Assume that* $(u_0, v_0) \in \mathbf{V}$ *and satisfies (H1)-(H3). If* $\rho^2 > \frac{1}{a(1+m-p_1)}\frac{1}{b(1+n-q_1)}$, *then the positive solution* (u, v) *of (1.1) blows up in finite time, where* ρ *is defined by (2.12).*

Theorem 2 *Under the assumptions of Theorem 1, if* $\Delta u_0, \Delta v_0 \le 0$ *on* $\overline{B}$ *and* (u_0, v_0) *satisfies (H4), then the following statements hold uniformly on any compact subset of B:*

$$\lim_{t\to T}\frac{u^{1-p_1}(x,t)}{(1-p_1)\widetilde{G}_1(t)} = a, \qquad \lim_{t\to T}\frac{v^{1-q_1}(x,t)}{(1-q_1)\widetilde{G}_2(t)} = b, \tag{1.10}$$

where $\widetilde{G}_1(t)$ *and* $\widetilde{G}_2(t)$ *are defined by (3.1).*

Theorem 3 *Under the assumptions of Theorem 2, if* $(1-q_1)(1+m-p_1) < p_2(q_2-m)$ *and* $(1-p_1)(1+n-q_1) < q_2(p_2-n)$, *then*

$$\begin{aligned}
&\lim_{t\to T} u(x,t)(T-t)^{\frac{1}{k_1}} \\
&\quad = \frac{(1+m-p_1)^{\frac{1}{1-p_1}}|B|^{\theta_1}}{d^{\frac{1}{k_1}}}\left(\frac{\beta_2}{a(1+m-p_1)}\right)^{\frac{1-q_1}{d}}\left(\frac{\beta_1}{b(1+n-q_1)}\right)^{\frac{p_2}{d}},
\end{aligned} \tag{1.11}$$

$$\begin{aligned}
&\lim_{t\to T} v(x,t)(T-t)^{\frac{1}{k_2}} \\
&\quad = \frac{(1+n-q_1)^{\frac{1}{1-q_1}}|B|^{\theta_2}}{d^{\frac{1}{k_2}}}\left(\frac{\beta_1}{b(1+n-q_1)}\right)^{\frac{1-p_1}{d}}\left(\frac{\beta_2}{a(1+m-p_1)}\right)^{\frac{q_2}{d}},
\end{aligned} \tag{1.12}$$

uniformly on compact subsets of B, where

$$\theta_1 = -\left(\frac{\sigma_1}{\mu_1} + \frac{\sigma_1}{\beta_2}\left(\frac{\sigma_2}{\mu_2} - \frac{\sigma_1}{\mu_1}\right)\right)\frac{1}{k_1},$$

$$\theta_1 = -\left(\frac{\sigma_2}{\mu_2} + \frac{\sigma_2}{\beta_1}\left(\frac{\sigma_1}{\mu_1} - \frac{\sigma_2}{\mu_2}\right)\right)\frac{1}{k_2}.$$

This paper is organized as follows. The result pertaining to blow-up of a solution in finite time is presented in Section 2, while results regarding the blow-up rates are established in Section 3. Some discussions are given in Section 4.

2 Proof of Theorem 1

In this section, we will discuss the blow-up of the solution to (1.1) and prove Theorem 1. By a simple computation, we have

$$\nabla\cdot\left(u^m\nabla u\right) = mu^{m-1}|\nabla u|^2 + u^m\Delta u, \tag{2.1}$$

$$\Delta u^{1+m-p_1} = (1+m-p_1)(m-p_1)u^{m-p_1-1}|\nabla u|^2 + (1+m-p_1)u^{m-p_1}\Delta u. \tag{2.2}$$

Since $1+m-p_1 > 0$, $m < p_1$, from (2.2), we can derive the inequality

$$u^{m-p_1}\Delta u \ge \frac{1}{1+m-p_1}\Delta u^{1+m-p_1}. \tag{2.3}$$

Moreover, by (1.1), (2.1) and (2.3), we have

$$\begin{aligned}\frac{1}{1-p_1}\frac{\partial u^{1-p_1}}{\partial t} &= u^{-p_1}u_t = u^{-p_1}\left(mu^{m-1}|\nabla u|^2 + u^m\Delta u + au^{p_1}\|v\|_{B,\alpha_1}^{p_2}\right)\\ &\geq \frac{1}{1+m-p_1}\Delta u^{1+m-p_1} + a\|v\|_{B,\alpha_1}^{p_2}.\end{aligned}$$

Thus,

$$\frac{1+m-p_1}{1-p_1}\frac{\partial (u^{1+m-p_1})^{\frac{1-p_1}{1+m-p_1}}}{\partial t} \geq \Delta u^{1+m-p_1} + a(1+m-p_1)\|v\|_{B,\alpha_1}^{p_2}. \tag{2.4}$$

Similarly,

$$\frac{1+n-q_1}{1-q_1}\frac{\partial (v^{1+n-q_1})^{\frac{1-q_1}{1+n-q_1}}}{\partial t} \geq \Delta v^{1+n-q_1} + b(1+n-q_1)\|u\|_{B,\alpha_2}^{q_2}. \tag{2.5}$$

Denote $u_1 = u^{1+m-p_1}$, $v_1 = v^{1+n-q_1}$ and

$$\begin{cases} r_1 = \frac{m}{1+m-p_1}, & \sigma_1 = \frac{p_2}{1+n-q_1}, \quad \mu_1 = \frac{\alpha_1}{1+n-q_1},\\ r_2 = \frac{n}{1+n-q_1}, & \sigma_2 = \frac{q_2}{1+m-p_1}, \quad \mu_2 = \frac{\alpha_2}{1+m-p_1}.\end{cases} \tag{2.6}$$

Then $0 < r_1, r_2 < 1$, $\sigma_1, \sigma_2, \mu_1, \mu_2 > 1$ and u_1, v_1 satisfy

$$\begin{cases} u_{1t} \geq u_1^{r_1}(\Delta u_1 + a(1+m-p_1)\|v_1\|_{B,\mu_1}^{\sigma_1}),\\ v_{1t} \geq v_1^{r_2}(\Delta v_1 + b(1+n-q_1)\|u_1\|_{B,\mu_2}^{\sigma_2}).\end{cases} \tag{2.7}$$

Consider now the following problem:

$$\begin{cases} u_{2t} = u_2^{r_1}(\Delta u_2 + a(1+m-p_1)\|v_2\|_{B,\mu_1}^{\sigma_1}), & (x,t) \in B \times (0,T),\\ v_{2t} = v_2^{r_2}(\Delta v_2 + a(1+n-q_1)\|u_2\|_{B,\mu_2}^{\sigma_2}), & (x,t) \in B \times (0,T),\\ u_2(x,t) = v_2(x,t) = 0, & (x,t) \in \partial B \times (0,T),\\ u_2(x,0) = u_{20}(x), \quad v_2(x,0) = v_{20}(x), & x \in B,\end{cases} \tag{2.8}$$

where

$$\begin{cases} \|v_2\|_{B,\mu_1}^{\sigma_1} = \|v\|_{B,\alpha_1}^{p_2}, & \|u_2\|_{B,\mu_2}^{\sigma_2} = \|u\|_{B,\alpha_2}^{q_2},\\ u_{20}(x) = (u_0(x))^{1+m-p_1}, & v_{20}(x) = (v_0(x))^{1+n-q_1}.\end{cases} \tag{2.9}$$

Since $u_0(x)$, $v_0(x)$ satisfy (H1)-(H2), then (2.8) has a unique classical solution (u_2, v_2) (see [20]). In the meantime, by the comparison principle, we observe

$$u_2(x,t) \leq u^{1+m-p_1}(x,t), \qquad v_2(x,t) \leq v^{1+n-q_1}(x,t), \quad (x,t) \in B \times (0,T). \tag{2.10}$$

Let G be a bounded domain of $\mathbb{R}^N$. Consider the problem

$$\begin{cases} \omega_t = d\omega^{r_0}(\Delta\omega + a_0 \int_G \omega \,\mathrm{d}x), & x \in G, t > 0, \\ \omega(x,t) = c, & x \in \partial G, t > 0, \\ \omega(x,0) = c, & x \in G, \end{cases} \tag{2.11}$$

where $0 < r_0 < 1$ and $d, a_0, c > 0$ are some constants. By the standard method (see [3]), we can show that (2.11) has a unique classical solution $\omega(x,t)$ and $\omega(x,t) \geq c$. Denote by $\varphi_0(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\varphi_0(x) = 1, \quad x \in G; \qquad \varphi_0(x) = 0, \quad x \in \partial G.$$

Set $\rho_0 = \int_G \varphi_0(x)\,\mathrm{d}x$, then we have:

Lemma 1 *If $\rho_0 > 1/a_0$, then the positive solution $\omega(x,t)$ of (2.11) blows up in finite time.*

Proof Set $H(t) = \int_G \omega^{1-r_0}\varphi_0 \,\mathrm{d}x$, then

$$\begin{aligned} \frac{1}{1-r_0}H'(t) &= d\left(\int_G \Delta\omega\varphi_0 \,\mathrm{d}x + a_0 \int_G \omega \,\mathrm{d}x \int_G \varphi_0 \,\mathrm{d}x\right) \\ &\geq d(a_0\rho_0 - 1)\int_G \omega \,\mathrm{d}x \geq d(a_0\rho_0 - 1)\left(\int_G \omega\varphi_0 \,\mathrm{d}x\right)/M, \end{aligned}$$

where $M = \max_{x\in\bar{G}} \varphi_0(x)$. Let $z = \omega^{1-r_0}$, then

$$\int_G z_t(x,t)\varphi_0(x)\,\mathrm{d}x \geq d(1-r_0)(a_0\rho_0 - 1)\left(\int_G z^{1/(1-r_0)}\varphi_0 \,\mathrm{d}x\right)/M.$$

Since $1/(1-r_0) > 1$, from Jensen's inequality, it follows that

$$\int_G z_t(x,t)\varphi_0(x)\,\mathrm{d}x \geq d(1-r_0)(a_0\rho_0 - 1)(\rho_0)^{-r_0/(1-r_0)}\left(\int_G z\varphi_0 \,\mathrm{d}x\right)^{1/(1-r_0)}/M.$$

That is $H'(t) \geq C_0 H^{1/(1-r_0)}(t)$. In view of $H(0) > 0$, it follows that there exists $T < \infty$ such that $\lim_{t\to T} H(t) = +\infty$, and hence $\omega(x,t)$ blows up in finite time. □

Let $\varphi(x)$ be the unique positive solution of the following linear elliptic problem:

$$-\Delta\varphi(x) = 1, \quad x \in B, \qquad \varphi(x) = 0, \quad x \in \partial B$$

and

$$\rho = \min\{\rho_{01} = \|\varphi\|_{B,\mu_1}^{\sigma_1}, \rho_{02} = \|\varphi\|_{B,\mu_2}^{\sigma_2}\}. \tag{2.12}$$

Lemma 2 *If $\rho^2 > \frac{1}{a(1+m-p_1)}\frac{1}{b(1+n-q_1)}$, then for the solution (u_2, v_2) of (2.8), there exists a sufficiently small constant $\varepsilon > 0$ such that*

$$\varepsilon\varphi(x) \leq u_2(x,t), \qquad \varepsilon\varphi(x) \leq v_2(x,t)$$

for all $(x,t) \in \overline{B} \times [0,T)$.

Proof From (H1) and (H2) we see that there exists a sufficiently small constant $\varepsilon > 0$ such that

$$\varepsilon\varphi(x) \le u_{20}(x), \qquad \varepsilon\varphi(x) \le v_{20}(x), \quad x \in \overline{B} \tag{2.13}$$

and

$$a(1+m-p_1)\rho > \varepsilon^{1-\sigma_1} \ge \varepsilon^{\sigma_2-1} > \bigl(b(1+n-q_1)\rho\bigr)^{-1}. \tag{2.14}$$

Let $s_1(x,t) = \varepsilon\varphi(x)$, $s_2(x,t) = \varepsilon\varphi(x)$, then we have by (2.14)

$$\begin{cases} s_{1t} - s_1^{r_1}(\Delta s_1 + a(1+m-p_1)\|s_2\|_{B,\mu_1}^{\sigma_1}) \le s_1^{r_1}(\varepsilon - a(1+m-p_1)\varepsilon^{\sigma_1}\rho) \le 0, \\ s_{2t} - s_2^{r_2}(\Delta s_2 + b(1+n-q_1)\|s_1\|_{B,\mu_2}^{\sigma_2}) \le 0, \quad x \in B, 0 < t < T, \\ s_1(x,t) = s_2(x,t) = 0, \quad x \in \partial B, 0 < t < T. \end{cases} \tag{2.15}$$

Thus it follows from (2.13) and (2.15) that (s_1, s_2) is a sub-solution of (2.8). Hence, $(\varepsilon\varphi, \varepsilon\varphi) \le (u_2, v_2)$ by the comparison principle. □

Lemma 3 *The solution (u_2, v_2) of (2.8) blows up in finite time if $\rho^2 > \frac{1}{a(1+m-p_1)}\frac{1}{b(1+n-q_1)}$ and u_0, v_0 satisfy (H1)-(H3).*

Proof In view of $\rho^2 > \frac{1}{a(1+m-p_1)}\frac{1}{b(1+n-q_1)}$, we can choose a smooth sub-ball $B_1 \subset\subset B$ such that

$$\rho_1^2 > \frac{1}{a(1+m-p_1)}\frac{1}{b(1+n-q_1)},$$

where $\rho_1 = \min\{\rho_{11} = \|\varphi_1\|_{B_1,\mu_1}^{\sigma_1}, \rho_{12} = \|\varphi_1\|_{B_1,\mu_2}^{\sigma_2}\}$ and $\varphi_1(x) > 0$ satisfies

$$-\Delta\varphi_1(x) = 1, \quad x \in B_1; \qquad \varphi_1(x) = 0, \quad x \in \partial B_1.$$

On the other hand, there exists a sufficiently small $\varepsilon_0 > 0$ such that

$$\int_{B_1} \varphi_1(x)\,\mathrm{d}x \ge \varepsilon_0\|\varphi_1\|_{B_1,\mu_1}^{\sigma_1}, \qquad \int_{B_1} \varphi_1(x)\,\mathrm{d}x \ge \varepsilon_0\|\varphi_1\|_{B_1,\mu_2}^{\sigma_2}. \tag{2.16}$$

Let $\eta = \varepsilon \min_{\overline{B}_1} \varphi$, here ε is determined by Lemma 2. Then $\eta > 0$ and

$$u_2(x,t) \ge \eta, \qquad v_2(x,t) \ge \eta, \quad (x,t) \in \overline{B}_1 \times (0,T)$$

by Lemma 2. Therefore, (u_2, v_2) in $B_1 \times (0,T)$ satisfies

$$\begin{cases} u_{2t} = u_2^{r_1}(\Delta u_2 + a(1+m-p_1)\|v_2\|_{B,\mu_1}^{\sigma_1}) \\ \qquad \ge u_2^{r_1}(\Delta u_2 + a(1+m-p_1)\|v_2\|_{B_1,\mu_1}^{\sigma_1}), \\ v_{2t} = v_2^{r_2}(\Delta v_2 + b(1+n-q_1)\|u_2\|_{B,\mu_2}^{\sigma_2}) \ge v_2^{r_2}(\Delta v_2 + b(1+n-q_1)\|u_2\|_{B_1,\mu_2}^{\sigma_2}), \\ u_2(x,t) \ge \eta, \qquad v_2(x,t) \ge \eta, \quad (x,t) \in \partial B_1 \times (0,T), \\ u_2(x,0) = u_{20}(x) \ge \eta, \qquad v_2(x,0) = v_{20}(x) \ge \eta, \quad x \in B_1. \end{cases} \tag{2.17}$$

Now, consider the following system:

$$\begin{cases} u_{3t} = u_3^{r_1}(\Delta u_3 + a(1+m-p_1)\|v_3\|_{B_1,\mu_1}^{\sigma_1}), & x \in B_1, t > 0, \\ v_{3t} = v_3^{r_2}(\Delta v_3 + b(1+n-q_1)\|u_3\|_{B_1,\mu_2}^{\sigma_2}), & x \in B_1, t > 0, \\ u_3(x,t) = v_3(x,t) = \eta, & x \in \partial B_1, t > 0, \\ u_3(x,0) = v_3(x,0) = \eta, & x \in B_1. \end{cases} \tag{2.18}$$

Similarly, we can show that there exists a nonnegative classical solution (u_3, v_3) of (2.18) for $(x,t) \in B_1 \times (0, T')$, where T' denotes the maximal existence time. The standard comparison principle for a parabolic system implies that $T' \geq T$ and

$$u_2(x,t) \geq u_3(x,t), \qquad v_2(x,t) \geq v_3(x,t), \quad (x,t) \in \overline{B}_1 \times (0,T). \tag{2.19}$$

Therefore, it suffices to show that (u_3, v_3) blows up in finite time, because if so, its upper bound (u_2, v_2) does exist up to a finite time T.

Since the initial data (η, η) is a sub-solution of (2.18), the standard super-solution and sub-solution methods assert that $u_{3t} \geq 0$, $v_{3t} \geq 0$, which implies that

$$\Delta u_3 + a(1+m-p_1)\|v_3\|_{B_1,\mu_1}^{\sigma_1} \geq 0, \qquad \Delta v_3 + b(1+n-q_1)\|u_3\|_{B_1,\mu_2}^{\sigma_2} \geq 0.$$

Hence $u_3, v_3 \geq \eta$ for $(x,t) \in \overline{B}_1 \times [0, T')$. Thus, (u_3, v_3) satisfies

$$\begin{cases} u_{3t} \geq \eta^{r_1 - r} u_3^r(\Delta u_3 + a(1+m-p_1)\|v_3\|_{B_1,\mu_1}^{\sigma_1}), & (x,t) \in B_1 \times (0,T'), \\ v_{3t} \geq \eta^{r_2 - r} v_3^r(\Delta v_3 + b(1+n-q_1)\|u_3\|_{B_1,\mu_2}^{\sigma_2}), & (x,t) \in B_1 \times (0,T') \end{cases} \tag{2.20}$$

with the corresponding initial and boundary conditions and $0 < r < \min\{r_1, r_2\}$.

Since $\rho_1^2 > \frac{1}{a(1+m-p_1)} \frac{1}{b(1+n-q_1)}$, there exist positive constants l_1, l_2 with $l_1, l_2 > 1$, and l such that

$$\begin{cases} a(1+m-p_1)\rho_1 > \frac{l_1}{l_2} > \frac{1}{b(1+n-q_1)\rho_1}, \\ \rho_1 > \frac{1}{l} > \frac{l_1 |B_1|^{\frac{\mu_1 - 1}{\mu_1}}}{\varepsilon_0 a(1+m-p_1) l_2^{\sigma_1}}, \qquad \rho_1 > \frac{1}{l} > \frac{l_2 |B_1|^{\frac{\mu_2 - 1}{\mu_2}}}{\varepsilon_0 b(1+n-q_1) l_1^{\sigma_2}}. \end{cases} \tag{2.21}$$

Let

$$\omega_1(x,t) = l_1 \omega(x,t), \qquad \omega_2(x,t) = l_2 \omega(x,t),$$

where $\omega(x,t)$ is a unique positive solution of (2.11) with

$$d = \min\{\eta^{r_1 - r}, \eta^{r_2 - r}\}, \qquad r_0 = r, \qquad a_0 = \frac{l}{\varepsilon_0}, \qquad c = \min\left\{\frac{1}{l_1}, \frac{1}{l_2}\right\}\eta, \qquad G = B_1.$$

From (2.21) and Lemma 1, we know that $\omega(x,t)$ blows up in finite time $T_0 < \infty$. Moreover, $\omega_t \geq 0$, that is, $\Delta\omega + a_0 \int_{B_1} \omega \, \mathrm{d}x \geq 0$, since the initial data is a sub-solution of (2.11). In addition, from $\sigma_1, \sigma_2 > 1$ and Hölder's inequality, we have

$$\begin{cases} \int_{B_1} \omega \, \mathrm{d}x \leq |B_1|^{\frac{\mu_1 - 1}{\mu_1}} (\int_{B_1} \omega^{\mu_1} \, \mathrm{d}x)^{\frac{1}{\mu_1}} \leq |B_1|^{\frac{\mu_1 - 1}{\mu_1}} \|\omega\|_{B_1,\mu_1}^{\sigma_1}, \\ \int_{B_1} \omega \, \mathrm{d}x \leq |B_1|^{\frac{\mu_2 - 1}{\mu_2}} (\int_{B_1} \omega^{\mu_2} \, \mathrm{d}x)^{\frac{1}{\mu_2}} \leq |B_1|^{\frac{\mu_2 - 1}{\mu_2}} \|\omega\|_{B_1,\mu_2}^{\sigma_2}. \end{cases} \tag{2.22}$$

Thus, a series of computations yields

$$\begin{cases} \omega_{1t} - \eta^{r_1-r}\omega_1^r(\Delta\omega_1 + a(1+m-p_1)\|\omega_2\|_{B_1,\mu_1}^{\sigma_1}) \\ \quad = l_1 d\omega^r(\Delta\omega + \frac{l}{\varepsilon_0}\int_{B_1}\omega\,\mathrm{d}x) - l_1\eta^{r_1-r}(l_1\omega)^r(\Delta\omega + \frac{a(1+m-p_1)l_2^{\sigma_1}}{l_1}\|\omega\|_{B_1,\mu_1}^{\sigma_1}) \\ \quad \le l_1 d\omega^r(\Delta\omega + \frac{l}{\varepsilon_0}|B_1|^{\frac{\mu_1-1}{\mu_1}}\|\omega\|_{B_1,\mu_1}^{\sigma_1}) \\ \qquad - l_1\eta^{r_1-r}(l_1\omega)^r(\Delta\omega + \frac{a(1+m-p_1)l_2^{\sigma_1}}{l_1}\|\omega\|_{B_1,\mu_1}^{\sigma_1}) \le 0, \\ \omega_{2t} - \eta^{r_2-r}\omega_2^r(\Delta\omega_2 + b(1+n-q_1)\|\omega_1\|_{B_1,\mu_2}^{\sigma_2}) \le 0, \quad x\in B_1, 0<t<T_0, \\ \omega_1(x,t) = l_1 c \le \eta, \qquad \omega_2(x,t) = l_2 c \le \eta, \quad x\in\partial B_1, 0\le t<T_0, \\ \omega_1(x,0) = l_1 c \le \eta, \qquad \omega_2(x,0) = l_2 c \le \eta, \quad x\in B_1. \end{cases} \tag{2.23}$$

It follows from (2.20), (2.23) and the comparison principle that $(\omega_1, \omega_2) \le (u_3, v_3)$. Hence (u_3, v_3) blows up in finite time, and so does the solution (u_2, v_2) of (2.8) from (2.19). The proof now is completed. □

Considering Lemma 3 and (2.10), we directly obtain the results of Theorem 1.

3 Proofs of Theorems 2 and 3

In this section, we assume that the solution (u, v) of (1.1) blows up in finite time T and will prove Theorems 2 and 3. We use c or C to denote the generic constant depending only on the structural data of the problem, and it may be different even in the same formula.

For the problem (1.1), denote

$$\begin{aligned} &\tilde{g}_1(t) = \|v\|_{B,\alpha_1}^{p_2}, \qquad \tilde{g}_2(t) = \|u\|_{B,\alpha_2}^{q_2}, \\ &\widetilde{G}_1(t) = \int_0^t \tilde{g}_1(s)\,\mathrm{d}s, \qquad \widetilde{G}_2(t) = \int_0^t \tilde{g}_2(s)\,\mathrm{d}s. \end{aligned} \tag{3.1}$$

Then we have

Lemma 4 *Suppose that u_0, v_0 satisfy (H1)-(H3), then we have*

$$\lim_{t\to T}\widetilde{G}_i(t) = \lim_{t\to T}\sup\tilde{g}_i(t) = \infty, \quad i = 1, 2.$$

Proof According to the hypotheses, we know that $u(0,t) \ge u(x,t)$, $v(0,t) \ge v(x,t)$, $(x,t) \in B\times(0,T)$. Let

$$\widetilde{U}(t) = \max_{x\in\overline{B}} u(x,t) = u(0,t), \qquad \widetilde{V}(t) = \max_{x\in\overline{B}} v(x,t) = v(0,t). \tag{3.2}$$

Then, $\widetilde{U}(t)$, $\widetilde{V}(t)$ are Lipschitz continuous (see [21]) and $\nabla\widetilde{U} = \nabla u(0,t) = 0$, $\nabla\widetilde{V} = \nabla v(0,t) = 0$. Since (u_0, v_0) is radially symmetric and non-increasing in $r = |x|$, (u, v) is also a radially symmetric and non-increasing function, *i.e.*, $u_r(r,t), v_r(r,t) \le 0$ with $r = |x|$. Thus, $u(x,t)$ and $v(x,t)$ always reach their maxima at $x = 0$, which means that $\Delta u(0,t), \Delta v(0,t) \le 0$ for any $0 < t < T$, *i.e.*, $\Delta\widetilde{U}, \Delta\widetilde{V} \le 0$ for any $0 < t < T$. Therefore, it follows from (2.1) and (1.1) that

$$\widetilde{U}'(t) \le a\widetilde{U}^{p_1}(t)\tilde{g}_1(t), \qquad \widetilde{V}'(t) \le b\widetilde{V}^{q_1}(t)\tilde{g}_2(t). \tag{3.3}$$

Integrating (3.3) over $(0,t)$, we obtain

$$\begin{cases} \frac{1}{1-p_1}\widetilde{U}^{1-p_1}(t) \le a\widetilde{G}_1(t) + \frac{1}{1-p_1}\widetilde{U}^{1-p_1}(0), \\ \frac{1}{1-q_1}\widetilde{V}^{1-q_1}(t) \le b\widetilde{G}_2(t) + \frac{1}{1-q_1}\widetilde{V}^{1-q_1}(0). \end{cases} \tag{3.4}$$

From $\lim_{t\to T}\widetilde{U}(t) = \lim_{t\to T}\widetilde{V}(t) = \infty$ and $0 < \widetilde{U}(0), \widetilde{V}(0) < \infty$, we get

$$\lim_{t\to T}\widetilde{G}_i(t) = \lim_{t\to T}\sup\tilde{g}_i(t) = \infty, \quad i = 1,2. \qquad \square$$

Next, we first give some auxiliary lemmas about the solutions of (2.8), which will be used in the proofs of theorems. Similar to (3.2), we let

$$U(t) = \max_{x\in\overline{B}} u_2(x,t), \qquad V(t) = \max_{x\in\overline{B}} v_2(x,t). \tag{3.5}$$

By (2.8), we see that $U(t)$ and $V(t)$ satisfy

$$\begin{aligned} &U_t \le a(1+m-p_1)|B|^{\frac{\sigma_1}{\mu_1}}U^{r_1}V^{\sigma_1}, \\ &V_t \le b(1+n-q_1)|B|^{\frac{\sigma_2}{\mu_2}}V^{r_2}U^{\sigma_2}, \quad \text{a.e. } t\in(0,T). \end{aligned} \tag{3.6}$$

Let $\beta_1 = 1 - r_1 + \sigma_2$, $\beta_2 = 1 - r_2 + \sigma_1$, then $\beta_1, \beta_2 > 0$. By Young's inequality, we have

$$\begin{aligned} (U^{\beta_1} + V^{\beta_2})_t &\le \bigl(\beta_1 a(1+m-p_1)|B|^{\frac{\sigma_1}{\mu_1}} + \beta_2 b(1+n-q_1)|B|^{\frac{\sigma_2}{\mu_2}}\bigr)U^{\beta_1\frac{\sigma_2}{\beta_1}}V^{\beta_2\frac{\sigma_1}{\beta_2}} \\ &\le C\bigl(U^{\beta_1} + V^{\beta_2}\bigr)^{\frac{\sigma_2}{\beta_1}+\frac{\sigma_1}{\beta_2}}. \end{aligned}$$

Integrating the above inequality over (t,T), we obtain

$$U^{\beta_1} + V^{\beta_2} \ge C(T-t)^{-\frac{\beta_1\beta_2}{d}}, \tag{3.7}$$

where $d = \sigma_1\sigma_2 - (1-r_1)(1-r_2) > 0$ by (1.7).

Lemma 5 *Suppose that u_0, v_0 satisfy (H1)-(H3) and the solution (u_2, v_2) of (2.8) blows up in finite time T. Then, we have*

$$\lim_{t\to T}\sup g_i(t) = \lim_{t\to T} G_i(t) = \infty, \quad i = 1,2,$$

where

$$\begin{aligned} &g_1(t) = \|v_2\|_{B,\mu_1}^{\sigma_1}, \qquad g_2(t) = \|u_2\|_{B,\mu_2}^{\sigma_2}, \\ &G_1(t) = \int_0^t g_1(s)\,\mathrm{d}s, \qquad G_2(t) = \int_0^t g_2(s)\,\mathrm{d}s. \end{aligned} \tag{3.8}$$

Proof Let $U(t), V(t)$ be as (3.5), then from (2.8), we have

$$\begin{aligned} &U'(t) \le a(1+m-p_1)U^{r_1}(t)g_1(t), \\ &V'(t) \le b(1+n-q_1)V^{r_2}(t)g_2(t), \quad \text{a.e. } t\in[0,T). \end{aligned} \tag{3.9}$$

Integrating (3.9) over $(0,t)$, we obtain

$$\begin{cases} \frac{1}{1-r_1}U^{1-r_1}(t) \leq a(1+m-p_1)G_1(t) + \frac{1}{1-r_1}U^{1-r_1}(0), \\ \frac{1}{1-r_2}V^{1-r_2}(t) \leq b(1+n-q_1)G_2(t) + \frac{1}{1-r_2}V^{1-r_2}(0). \end{cases} \tag{3.10}$$

Similar to the proofs of $\widetilde{G}_i(t)$ and $\tilde{g}_i(t)$, we have

$$\lim_{t\to T} G_i(t) = \lim_{t\to T}\sup g_i(t) = \infty, \quad i = 1,2.$$

□

Lemma 6 *Suppose that u_0, v_0 satisfy (H1)-(H4). Then, we have*

$$u_{2t} - \delta u_2^{k_1+1} \geq 0, \qquad v_{2t} - \delta v_2^{k_2+1} \geq 0, \quad (x,t) \in B \times (0,T), \tag{3.11}$$

here $k_1 = d/\beta_2$, $k_2 = d/\beta_1$.

Proof Set $J_1(x,t) = u_{2t} - \delta u_2^{k_1+1}$, $J_2(x,t) = u_{2t} - \delta u_2^{k_2+1}$. Then,

$$\lim_{x\to\partial B} J_1(x,t) \geq 0, \qquad \lim_{x\to\partial B} J_2(x,t) \geq 0; \quad J_1(x,0) \geq 0, J_2(x,0) \geq 0, x \in B.$$

A series of computations yields $J_{1t} = u_{2tt} - \delta(k_1+1)u_2^{k_1}u_{2t}$ and

$$\begin{aligned} u_{2tt} &= r_1 u_2^{-1}\big(J_1^2 + 2\delta u_2^{k_1+1}J_1 + \delta^2 u_2^{2k_1+2}\big) + u_2^{r_1}\Delta J_1 + \delta(k_1+1)k_1 u_2^{k_1-1+r_1}|\nabla u_2|^2 \\ &\quad + \delta(k_1+1)u_2^{k_1+r_1}\Delta u_2 + a(1+m-p_1)\sigma_1 u_2^{r_1}\|v_2\|_{\mu_1}^{\sigma_1-\mu_1}\int_B v_2^{\mu_1-1}\big(J_2 + \delta v_2^{k_2+1}\big)\,\mathrm{d}x \\ &= u_2^{r_1}\Delta J_1 + \big(2r_1\delta u_2^{k_1} + \delta(k_1+1)u_2^{k_1}\big)J_1 + r_1 u_2^{-1}J_1^2 + \delta(k_1+1)k_1 u_2^{k_1-1+r_1}|\nabla u_2|^2 \\ &\quad + \big(r_1\delta^2 + (k_1+1)\delta^2\big)u_2^{2k_1+1} - (k_1+1)a(1+m-p_1)\delta u_2^{k_1+r_1}\|v_2\|_{\mu_1}^{\sigma_1} \\ &\quad + a(1+m-p_1)\sigma_1 u_2^{r_1}\|v_1\|_{\mu_1}^{\sigma_1-\mu_1}\int_B v_2^{\mu_1-1}J_2\,\mathrm{d}x \\ &\quad + a(1+m-p_1)\sigma_1\delta u_2^{r_1}\|v_2\|_{\mu_1}^{\sigma_1-\mu_1}\|v_2\|_{\mu_1+k_2}^{\mu_1+k_2}. \end{aligned}$$

From the condition (1.7), it is easy to calculate that $k_1 + 1 > r_1$. Then, it entails

$$\begin{aligned} &J_{1t} - u_2^{r_1}\Delta J_1 - 2r_1\delta u_2^{k_1}J_1 - a(1+m-p_1)\sigma_1 u_2^{r_1}\|v_2\|_{\mu_1}^{\sigma_1-\mu_1}\int_B v_2^{\mu_1-1}J_2\,\mathrm{d}x \\ &\quad\geq r_1\delta^2 u_2^{2k_1+1} + a(1+m-p_1)\sigma_1\delta u_2^{r_1}\|v_2\|_{\mu_1}^{\sigma_1-\mu_1}\|v_2\|_{\mu_1+k_2}^{\mu_1+k_2} \\ &\qquad - (k_1+1)a(1+m-p_1)\delta u_2^{k_1+r_1}\|v_2\|_{\mu_1}^{\sigma_1} \\ &\quad= a(1+m-p_1)(k_1+1)\delta u_2^{r_1}\bigg(\frac{r_1\delta u_2^{2k_1+1-r_1}}{a(1+m-p_1)(k_1+1)} \\ &\qquad + \frac{\sigma_1}{k_1+1}\|v_2\|_{\mu_1}^{\sigma_1-\mu_1}\|v_2\|_{\mu_1+k_2}^{\mu_1+k_2} - u_2^{k_1}\|v_2\|_{\mu_1}^{\sigma_1}\bigg). \end{aligned} \tag{3.12}$$

By the Hölder inequality, for any $0 < \theta < 1$, it follows that

$$\|v_2\|_{\mu_1}^{\sigma_1} = \|v_2\|_{\mu_1}^{(\sigma_1-\mu_1)\theta}\|v_2\|_{\mu_1}^{\sigma_1-(\sigma_1-\mu_1)\theta} \leq \|v_2\|_{\mu_1}^{(\sigma_1-\mu_1)\theta}\|v_2\|_{\mu_1+k_2}^{\sigma_1-(\sigma_1-\mu_1)\theta}|B|^{\frac{k_2[\sigma_1-(\sigma_1-\mu_1)\theta]}{\mu_1(\mu_1+k_2)}}.$$

Furthermore, by Young's inequality, for any $\varepsilon > 0$ and $l_1, l_2 > 1$ satisfying $1/l_1 + 1/l_2 = 1$, the following inequality holds:

$$\begin{aligned} u_2^{k_1}\|v_2\|_{\mu_1}^{\sigma_1} &\leq u_2^{k_1}\|v_2\|_{\mu_1}^{(\sigma_1-\mu_1)\theta}\|v_2\|_{\mu_1+k_2}^{\sigma_1-(\sigma_1-\mu_1)\theta}|B|^{\frac{k_2[\sigma_1-(\sigma_1-\mu_1)\theta]}{\mu_1(\mu_1+k_2)}} \\ &\leq |B|^{\frac{k_2[\sigma_1-(\sigma_1-\mu_1)\theta]}{\mu_1(\mu_1+k_2)}}\left(\frac{(\varepsilon u_2^{k_1})^{l_1}}{l_1} + \frac{1}{l_2}\left(\frac{1}{\varepsilon}\|v_2\|_{\mu_1}^{(\sigma_1-\mu_1)\theta}\|v_2\|_{\mu_1+k_2}^{\sigma_1-(\sigma_1-\mu_1)\theta}\right)^{l_2}\right). \end{aligned} \tag{3.13}$$

Now, we take

$$l_1 = \frac{k_2+\sigma_1}{k_2}, \qquad l_2 = \frac{k_2+\sigma_1}{\sigma_1}, \qquad \theta = \frac{\sigma_1}{k_2+\sigma_1}, \quad \varepsilon = \left(\frac{k_1+1}{k_2+\sigma_1}|B|^{\frac{k_2\sigma_1}{\mu_1(k_2+\sigma_1)}}\right)^{\frac{\sigma_1}{k_2+\sigma_1}}.$$

Therefore, by (3.12) and (3.13), it follows that

$$\begin{aligned} &J_{1t} - u_2^{r_1}\Delta J_1 - 2r_1\delta u_2^{k_1}J_1 - a(1+m-p_1)\sigma_1 u_2^{r_1}\|v_2\|_{\mu_1}^{\sigma_1-\mu_1}\int_B v_2^{\mu_1-1}J_2\,\mathrm{d}x \\ &\quad\geq r_1\delta(\delta-\delta_1)u_2^{2k_1+2} \geq 0, \end{aligned}$$

where

$$\delta_1 = \frac{a(1+m-p_1)k_2}{r_1}|B|^{\frac{\sigma_1}{\mu_1}}\left(\frac{k_1+1}{k_2+\sigma_1}\right)^{\frac{\sigma_1}{k_2}+1}.$$

We can determine a number δ_2 in the similar way. Let $\delta_0 = \max\{\delta_1, \delta_2\}$, similar to the above, one has

$$J_{2t} - v_2^{r_2}\Delta J_2 - 2r_2\delta v_2^{k_2}J_2 - b(1+n-q_1)\sigma_2 v_2^{r_2}\|u_2\|_{\mu_2}^{\sigma_2-\mu_2}\int_B u_2^{\mu_2-1}J_1\,\mathrm{d}x \geq 0.$$

By the comparison principle of Lemma 1 in [20], we have $J_1, J_2 \geq 0$. This completes the proof. □

Lemma 7 *Suppose that u_0, v_0 satisfy (H1)-(H4), then there exist positive constants c and C such that*

$$\begin{cases} c \leq \max_{x\in\overline{B}} u_2(x,t)(T-t)^{1/k_1} \leq C, \\ c \leq \max_{x\in\overline{B}} v_2(x,t)(T-t)^{1/k_2} \leq C. \end{cases} \tag{3.14}$$

Proof It follows from (3.11) that

$$U_t \geq \delta U^{k_1+1}, \qquad V_t \geq \delta V^{k_2+1}, \quad t \in (0,T). \tag{3.15}$$

Combining with (3.6), we can obtain

$$U^{k_1+1-r_1} \leq \frac{a(1+m-p_1)}{\delta}|B|^{\frac{\sigma_1}{\mu_1}}V^{\sigma_1}, \qquad V^{k_2+1-r_2} \leq \frac{b(1+n-q_1)}{\delta}|B|^{\frac{\sigma_2}{\mu_2}}U^{\sigma_2}. \tag{3.16}$$

The direct computation yields $k_1 + 1 - r_1 = \sigma_1\beta_1/\beta_2$, $k_2 + 1 - r_1 = \sigma_2\beta_2/\beta_1$. It follows from (3.16) that

$$U^{\beta_1} \le \left(\frac{a(1+m-p_1)}{\delta}\right)^{\frac{\beta_2}{\sigma_1}} |B|^{\frac{\beta_2}{\mu_1}} V^{\beta_2}, \qquad V^{\beta_2} \le \left(\frac{b(1+n-q_1)}{\delta}\right)^{\frac{\beta_1}{\sigma_2}} |B|^{\frac{\beta_1}{\mu_2}} U^{\beta_1}. \tag{3.17}$$

Therefore, combining (3.17) with (3.7) gives

$$c \le U(t)(T-t)^{1/k_1}, \qquad c \le V(t)(T-t)^{1/k_2}.$$

Integrating (3.15) from t to T, we end the proof. □

Lemma 8 *Suppose that u_0, v_0 satisfy (H1)-(H4) and $\Delta u_0, \Delta v_0 \le 0$, then*

$$\lim_{t\to T} \frac{u_2^{1-r_1}(x,t)}{(1-r_1)G_1(t)} = a(1+m-p_1), \qquad \lim_{t\to T} \frac{v_2^{1-r_2}(x,t)}{(1-r_2)G_2(t)} = b(1+n-q_1) \tag{3.18}$$

uniformly on compact subsets of B.

Proof Here we consider the first eigenvalue problem

$$-\Delta\phi(x) = \lambda_1\phi(x), \quad x \in B; \qquad \phi(x) = 0, \quad x \in \partial B.$$

Normalize $\phi(x)$ as $\phi(x) > 0$ in B and $\int_B \phi(x)\,\mathrm{d}x = 1$. Define

$$z(x,t) = a(1+m-p_1)G_1(t) - \frac{1}{1-r_1}u_2^{1-r_1}(x,t), \quad \gamma(t) = \int_B z(y,t)\phi(y)\,\mathrm{d}y.$$

A series of computations yields

$$\begin{aligned}
\gamma'(t) &= \int_B \bigl(a(1+m-p_1)g_1(t) - u_2^{-r_1}u_{2t}\bigr)\phi(y)\,\mathrm{d}y \\
&= -\int_B \Delta u_2(y,t)\phi(y)\,\mathrm{d}y = \lambda_1 \int_B u_2(y,t)\phi(y)\,\mathrm{d}y \\
&= \lambda_1(1-r_1)^{\frac{1}{1-r_1}} \int_B \bigl(a(1+m-p_1)G_1(t) - z(y,t)\bigr)^{\frac{1}{1-r_1}}\phi(y)\,\mathrm{d}y \\
&\le \lambda_1(1-r_1)^{\frac{1}{1-r_1}} \int_B \bigl(a(1+m-p_1)G_1(t) + z^-(y,t)\bigr)^{\frac{1}{1-r_1}}\phi(y)\,\mathrm{d}y \\
&\le C\left(G_1^{\frac{1}{1-r_1}}(t) + \int_B \bigl(z^-(y,t)\bigr)^{\frac{1}{1-r_1}}\phi(y)\,\mathrm{d}y\right),
\end{aligned}$$

where $z^- = \max\{-z, 0\}$. By (3.10), we know that $\inf_B z(x,t) \ge -C$, which means $z^-(x,t) \le C$. Then

$$\gamma'(t) \le CG_1^{\frac{1}{1-r_1}}(t) + C. \tag{3.19}$$

Integrating (3.19) from 0 to t yields

$$\gamma(t) \le C\left(1 + \int_0^t G_1^{\frac{1}{1-r_1}}(s)\,\mathrm{d}s\right).$$

That is

$$\int_B \left|z(y,t)\right|\phi(y)\,\mathrm{d}y \le C\left(1+\int_0^t G_1^{\frac{1}{1-r_1}}(s)\,\mathrm{d}s\right). \tag{3.20}$$

Denote $B_\varrho = \{y \in B : \varrho \le |y| < R\}$. Since $-\Delta z \le 0$, using Lemma 4.5 in [4], we obtain

$$\sup_{B_\varrho} z(x,t) \le \frac{C}{\varrho^{N+1}}\left(1+\int_0^t G_1^{\frac{1}{1-r_1}}(s)\,\mathrm{d}s\right). \tag{3.21}$$

It follows from (3.21) and (3.10) that

$$-\frac{C}{G_1(t)} \le a(1+m-p_1) - \frac{u_2^{1-r_1}}{(1-r_1)G_1(t)} \le \frac{C(1+\int_0^t G_1^{\frac{1}{1-r_1}}(s)\,\mathrm{d}s)}{G_1(t)} \tag{3.22}$$

for any $x \in B_\varrho$. On the other hand, we know from (3.10), (3.14) and $\Delta u_0, \Delta v_0 \le 0$ that

$$c \le (T-t)^{\frac{(1-r_1)(1-r_2+\sigma_1)}{d}} G_1(t) \le C.$$

Therefore,

$$\begin{cases} c \le (T-t)^{\frac{1-r_2+\sigma_1}{d}} G_1^{\frac{1}{1-r_1}}(t) \le C, \\ c \le (T-t)^{\frac{(1-r_1)(1-r_2+\sigma_1)}{d}+1} G_1'(t) \le C. \end{cases}$$

Noting that

$$\frac{1-r_2+\sigma_1}{d} < \frac{(1-r_1)(1-r_2+\sigma_1)}{d}+1 \quad\Longleftrightarrow\quad 1-r_2 < \sigma_1(\sigma_2-r_1).$$

Then

$$\lim_{t\to T} \frac{\int_0^t G_1^{\frac{1}{1-r_1}}(s)\,\mathrm{d}s}{G_1(t)} = \lim_{t\to T} \frac{G_1^{\frac{1}{1-r_1}}(t)}{G_1'(t)} = 0.$$

Thus

$$\lim_{t\to T} \frac{u_2^{1-r_1}(x,t)}{(1-r_1)G_1(t)} = a(1+m-p_1).$$

Similarly,

$$\lim_{t\to T} \frac{v_2^{1-r_2}(x,t)}{(1-r_2)G_2(t)} = b(1+n-q_1).$$

□

Proof of Theorem 2 According to $u_2 \le u^{1+m-p_1}$, it follows from (2.9), (3.1), (3.8) and (3.18) that

$$\liminf_{t\to T} \frac{u^{1-p_1}(x,t)}{(1-p_1)\widetilde{G}_1(t)} \ge \lim_{t\to T} \frac{u_2^{1-r_1}(x,t)}{(1-r_1)G_1(t)} \frac{1}{1+m-p_1} = a. \tag{3.23}$$

On the other hand, from (3.4), we estimate

$$\limsup_{t\to T} \frac{\widetilde{U}^{1-p_1}(t)}{(1-p_1)\widetilde{G}_1(t)} \le a. \tag{3.24}$$

Combining (3.23) with (3.24), we obtain

$$\lim_{t\to T} \frac{u^{1-p_1}(x,t)}{(1-p_1)\widetilde{G}_1(t)} = a.$$

Similarly,

$$\lim_{t\to T} \frac{v^{1-q_1}(x,t)}{(1-q_1)\widetilde{G}_2(t)} = b.$$

This completes the proof of the theorem. □

Proof of Theorem 3 By Theorem 2, we have that, as $t \to T$,

$$\begin{cases} \widetilde{G}_1'(t) = \|v\|_{B,\alpha_1}^{p_2} \sim |B|^{\frac{p_2}{\alpha_1}} (b(1-q_1))^{\frac{p_2}{1-q_1}} \widetilde{G}_2^{\frac{p_2}{1-q_1}}(t), \\ \widetilde{G}_2'(t) = \|u\|_{B,\alpha_2}^{q_2} \sim |B|^{\frac{q_2}{\alpha_2}} (a(1-p_1))^{\frac{q_2}{1-p_1}} \widetilde{G}_1^{\frac{q_2}{1-p_1}}(t), \end{cases}$$

where the notation $u \sim v$ means that $\lim_{t\to T} u(t)/v(t) = 1$. Hence, we obtain

$$\frac{\mathrm{d}\widetilde{G}_1}{\mathrm{d}\widetilde{G}_2} \sim |B|^{\frac{p_2}{\alpha_1}-\frac{q_2}{\alpha_2}} \bigl(a(1-p_1)\bigr)^{-\frac{q_2}{1-p_1}} \bigl(b(1-q_1)\bigr)^{\frac{p_2}{1-q_1}} \widetilde{G}_1^{-\frac{q_2}{1-p_1}} \widetilde{G}_2^{\frac{p_2}{1-q_1}}.$$

A series of computations yields

$$\begin{cases} \widetilde{G}_1(t) \sim \frac{|B|^{\theta_1(1-p_1)} d^{-\frac{\beta_2(1-p_1)}{d}}}{a(1-p_1)(1+m-p_1)} (\frac{\beta_2}{a(1+m-p_1)})^{\frac{(1-p_1)(1-q_1)}{d}} (\frac{\beta_1}{b(1+n-q_1)})^{\frac{p_2(1-p_1)}{d}} (T-t)^{-\frac{\beta_2(1-p_1)}{d}}, \\ \widetilde{G}_2(t) \sim \frac{|B|^{\theta_2(1-q_1)} d^{-\frac{\beta_1(1-q_1)}{d}}}{b(1-q_1)(1+n-q_1)} (\frac{\beta_1}{b(1+n-q_1)})^{\frac{(1-p_1)(1-q_1)}{d}} (\frac{\beta_2}{a(1+m-p_1)})^{\frac{q_2(1-q_1)}{d}} (T-t)^{-\frac{\beta_1(1-q_1)}{d}}. \end{cases}$$

Combining with Lemma 7, we obtain the results of Theorem 3 immediately. □

4 Discussions

The results in this paper show the interactions among the multi-nonlinearities in the reaction-diffusion system (1.1). Roughly speaking, either large exponents m, n, large coupling exponents p_2, q_2 or large constants a, b benefit from the occurrence of the finite blow-up. For example, to make a finite blow-up to the problem (1.1), for fixed m, p_1, p_2, α_1 and n, q_1, q_2, α_2, constants a and b should be properly large such that the following inequality

$$\rho^2 > \frac{1}{a(1+m-p_1)} \frac{1}{b(1+n-q_1)}$$

holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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