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Positive solutions of third-order nonlocal boundary value problems at resonance

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Abstract

In this paper, we investigate the existence of positive solutions for a class of third-order nonlocal boundary value problems at resonance. Our results are based on the Leggett-Williams norm-type theorem, which is due to O'Regan and Zima. An example is also included to illustrate the main results.

MSC: 34B10; 34B15

Keywords: third-order; nonlocal; at resonance; positive solution

1 Introduction

This paper is devoted to the existence of positive solutions for the following third-order nonlocal boundary value problem (BVP for short):

$$\begin{cases} x'''(t) + f(t, x(t)) = 0, & 0 < t < 1, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), & x'(0) = x'(1) = 0, \end{cases} \quad (1.1)$$

where $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $\alpha_i \in R^+$ ($i = 1, 2, \dots, m-2$) and $\sum_{i=1}^{m-2} \alpha_i = 1$. The problem (1.1) happens to be at resonance in the sense that the associated linear homogeneous BVP

$$\begin{cases} -x'''(t) = 0, & 0 < t < 1, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), & x'(0) = x'(1) = 0, \end{cases} \quad (1.2)$$

has nontrivial solutions. Clearly, the resonant condition is $\sum_{i=1}^{m-2} \alpha_i = 1$. Third-order differential equations arise in a variety of different areas of applied mathematics and physics, *e.g.*, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity-driven flows and so on [1].

Recently, the existence of positive solutions for third-order two-point or multi-point BVPs has received considerable attention; we mention a few works: [2–11] and the references therein. However, all of the papers on third-order BVPs focused their attention on the positive solutions with non-resonance cases. It is well known that the problem of the existence of positive solutions to BVPs is very difficult when the resonant case is considered. Only few papers deal with the existence of positive solutions to BVPs at resonance, and just to second-order BVPs [12–15]. It is worth mentioning that Infante and Zima [13]

studied the existence of positive solutions for the second-order m -point BVP

$$\begin{cases} -x''(t) + f(t, x(t)) = 0, & 0 < t < 1, \\ x'(0) = 0, & x(1) = \sum_{i=1}^{m-2} \alpha_i x(\eta_i) \end{cases} \quad (1.3)$$

by means of the Leggett-Williams norm-type theorem due to O'Regan and Zima [16], where $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, $\alpha_i \in R^+$ ($i = 1, 2, \dots, m - 2$) and $\sum_{i=1}^{m-2} \alpha_i = 1$.

However, third-order or higher-order derivatives do not have the convexity; to the best of our knowledge, no results are available for the existence of positive solutions for third-order or higher-order BVPs at resonance. The main purpose of this paper is to fill the gap in this area. Motivated greatly by the above-mentioned excellent works, in this paper we will investigate the third-order nonlocal BVP (1.1) at resonance, where $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $\alpha_i \in R^+$ ($i = 1, 2, \dots, m - 2$) and $\sum_{i=1}^{m-2} \alpha_i = 1$. Some new existence results of at least one positive solution are established by applying the Leggett-Williams norm-type theorem due to O'Regan and Zima [16]. An example is also included to illustrate the main results.

2 Some definitions and a fixed point theorem

For the convenience of the reader, we present here the necessary definitions and a new fixed point theorem due to O'Regan and Zima.

Definition 2.1 Let X and Z be real Banach spaces. A linear operator $L : \text{dom } L \subset X \rightarrow Z$ is called a Fredholm operator if the following two conditions hold:

- (i) $\text{Ker } L$ has a finite dimension, and
- (ii) $\text{Im } L$ is closed and has a finite codimension.

Throughout the paper, we will assume that

1° L is a Fredholm operator of index zero, that is, $\text{Im } L$ is closed and $\dim \text{Ker } L = \text{codim } \text{Im } L < \infty$.

From Definition 2.1, it follows that there exist continuous projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ such that

$$\text{Im } P = \text{Ker } L; \quad \text{Ker } Q = \text{Im } L; \quad X = \text{Ker } L \oplus \text{Ker } P; \quad Z = \text{Im } L \oplus \text{Im } Q$$

and that the isomorphism

$$L|_{\text{dom } L \cap \text{Ker } P} : \text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$$

is invertible. We denote the inverse of $L|_{\text{dom } L \cap \text{Ker } P}$ by $K_P : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$. The generalized inverse of L denoted by $K_{P,Q} : Z \rightarrow \text{dom } L \cap \text{Ker } P$ is defined by $K_{P,Q} = K_P(I - Q)$. Moreover, since $\dim \text{Im } Q = \text{codim } \text{Im } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$. Consider a nonlinear operator $N : X \rightarrow Z$. It is known (see [17, 18]) that the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_{P,Q}Nx.$$

Definition 2.2 Let X be a real Banach space. A nonempty closed convex set P is said to be a cone provided that

- (1) $\lambda x \in P$ for all $x \in P$ and $\lambda \geq 0$, and
- (2) $x, -x \in P$ implies $x = \theta$.

Note that every cone $P \subset X$ induces a partial order \leq in X by defining $x \leq y$ if and only if $y - x \in P$. The following property is valid for every cone in a Banach space.

Lemma 2.3 ([19]) *Let P be a cone in X . Then for every $u \in P \setminus \{\theta\}$, there exists a positive number $\sigma(u)$ such that $\|x + u\| \geq \sigma(u)\|x\|$ for all $x \in P$.*

Let $\gamma : X \rightarrow P$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in P$. Set

$$\Psi = P + JQN + K_{P,Q}N$$

and

$$\Psi_\gamma = \Psi \circ \gamma.$$

Our main results are based on the following theorem due to O'Regan and Zima.

Theorem 2.4 ([16]) *Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that 1° is satisfied and if the following assumptions hold:*

- (H1) $QN : X \rightarrow Z$ is continuous and bounded, and $K_{P,Q} : X \rightarrow X$ is compact on every bounded subset of X ;
- (H2) $Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \text{dom}L$ and $\lambda \in (0, 1)$;
- (H3) γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C ;
- (H4) $d_B([I - (P + JQN)\gamma]|_{\text{Ker}L}, \text{Ker}L \cap \Omega_2, \theta) \neq 0$, where d_B stands for the Brouwer degree;
- (H5) there exists $u_0 \in C \setminus \{\theta\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi(x)\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ is such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for all $x \in C$;
- (H6) $(P + JQN)\gamma(\partial\Omega_2) \subset C$;
- (H7) $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$,

then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

For simplicity of notation, we set

$$h_i(s) = \begin{cases} \frac{2\xi_i s - \xi_i^2 s - s^2}{2}, & 0 \leq s \leq \xi_i, \\ \frac{\xi_i^2(1-s)}{2}, & \xi_i \leq s \leq 1, \end{cases}$$

where $i = 1, 2, \dots, m - 2$, and

$$G(t, s) = \begin{cases} \frac{(1-s)(s^2 - 2s + 3t^2)}{6} - \frac{(t-s)^2}{2} + \frac{4t^3 - 6t^2 + 23}{2 \sum_{i=1}^{m-2} \alpha_i (3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i h_i(s), & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)(s^2 - 2s + 3t^2)}{6} + \frac{4t^3 - 6t^2 + 23}{2 \sum_{i=1}^{m-2} \alpha_i (3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i h_i(s), & 0 \leq t \leq s \leq 1. \end{cases}$$

It is easy to check that $G(t, s) \geq 0$, $t, s \in [0, 1]$, and since $0 \leq h_i(s) \leq \frac{1}{2}s(1-s)$, we get

$$1 - \frac{12\delta}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i h_i(s) \geq 0, \quad s \in [0, 1]$$

for every $\delta \in (0, \frac{2\xi_1^2(3-2\xi_1)}{3}]$. We also let $\delta := \min\{\frac{2\xi_1^2(3-2\xi_1)}{3}, \frac{1}{\max_{t,s \in [0,1]} G(t,s)}, 1\}$.

We can now state our result on the existence of a positive solution for the BVP (1.1).

Theorem 3.1 *Assume that $f : [0, 1] \times [0, +\infty) \rightarrow R$ is continuous and*

- (1) *there exists a constant $M \in (0, \infty)$ such that $f(t, x) > -\delta x$ for all $(t, x) \in [0, 1] \times [0, M]$;*
- (2) *there exist $r \in (0, M)$, $t_0 \in [0, 1]$, $a \in (0, 1)$, $b \in (0, 1)$ and continuous functions $g : [0, 1] \rightarrow [0, \infty)$, $h : (0, r] \rightarrow [0, \infty)$ such that $f(t, x) \geq g(t)h(x)$ for $(t, x) \in [0, 1] \times (0, r]$, $\frac{h(x)}{x^a}$ is non-increasing on $(0, r]$ with*

$$\frac{h(r)}{(r)^a} \int_0^1 G(t_0, s)g(s) ds \geq \frac{1-b}{b^a}.$$

- (3) $f_\infty = \liminf_{x \rightarrow +\infty} \min_{t \in [0,1]} \frac{f(t,x)}{x} < 0$.

Then the resonant BVP (1.1) has at least one positive solution on $[0, 1]$.

Proof Consider the Banach spaces $X = Z = C[0, 1]$ with $\|x\| = \max_{t \in [0,1]} |x(t)|$.

Let $L : \text{dom } L \subset X \rightarrow Z$ and $N : X \rightarrow Z$ with

$$\text{dom } L = \left\{ x \in X : x''' \in C[0, 1], x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), x'(0) = x'(1) = 0 \right\}$$

be given by $(Lx)(t) = -x'''(t)$ and $(Nx)(t) = f(t, x(t))$ for $t \in [0, 1]$. Then

$$\text{Ker } L = \{x \in \text{dom } L : x(t) \equiv c \text{ on } [0, 1]\}$$

and

$$\text{Im } L = \left\{ y \in Z : \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s)y(s) ds = 0 \right\}.$$

Clearly, $\dim \text{Ker } L = 1$ and $\text{Im } L$ is closed. It follows from $Z_1 = Z \setminus \text{Im } L$ that

$$Z_1 = \left\{ y_1 \in Z : y_1 = \frac{12}{2 \sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s)y(s) ds, y \in Z \right\}.$$

In fact, for each $y \in Z$, we have

$$\begin{aligned} & \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s)(y(s) - y_1(s)) ds \\ &= \left(1 - \frac{12}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s) ds \right) \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s)y(s) ds = 0, \end{aligned}$$

which shows that $y - y_1 \in \text{Im } L$, which together with $Z_1 \cap \text{Im } L = \{\theta\}$ implies that $Z = Z_1 \oplus \text{Im } L$. Note that $\dim Z_1 = 1$ and thus $\text{codim Im } L = 1$. Therefore, L is a Fredholm operator of index zero.

Next, define the projections $P : X \rightarrow X$ by

$$Px = \int_0^1 x(s) ds$$

and $Q : Z \rightarrow Z$ by

$$Qy = \frac{12}{\sum_{i=1}^{m-2} \alpha_i (3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s)y(s) ds.$$

Clearly, $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$ and $\text{Ker } L = \{x \in X : \int_0^1 x(s) ds = 0\}$. Note that for $y \in \text{Im } L$, the inverse K_P of L_P is given by

$$(K_P y)(t) = \int_0^1 k(t,s)y(s) ds,$$

where

$$k(t,s) = \begin{cases} \frac{(1-s)(s^2-2s+3t^2)}{6} - \frac{(t-s)^2}{2}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)(s^2-2s+3t^2)}{6}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Considering that f can be extended continuously to $[0,1] \times R$, it is easy to check that $QN : X \rightarrow Z$ is continuous and bounded, and $K_{P,QN} : X \rightarrow X$ is compact on every bounded subset of X , which ensures that (H1) of Theorem 2.4 is fulfilled.

Define the cone of nonnegative functions

$$C = \{x \in X : x(t) \geq 0, \text{ on } [0,1]\}.$$

Let

$$\Omega_1 = \{x \in X : b\|x\| < |x(t)| < r, t \in [0,1]\}$$

and

$$\Omega_2 = \{x \in X : \|x\| < M\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\overline{\Omega_1} = \{x \in X : b\|x\| \leq |x(t)| \leq r, t \in [0,1]\} \subset \Omega_2.$$

Moreover, $C \cap (\overline{\Omega_2} \setminus \Omega_1) \neq \theta$. Let $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega_2}$ into bounded subsets of C , which means that (H3) of Theorem 2.4 holds.

Let $0 < r' = \min\{r_1, r_2\}$, where r_1 and r_2 will be defined in the following proof.

Suppose that there exist $x_0 \in C \cap \partial\Omega_2 \cap \text{dom}L$ and $\lambda_0 \in (0, 1)$ such that $Lx_0 = \lambda_0 Nx_0$. Then

$$x_0'''(t) + \lambda_0 f(t, x_0(t)) = 0.$$

Let $x_0(t_1) = \|x_0\| = \max_{t \in [0, 1]} |x_0(t)| = M$. Now, we verify that $t_1 \neq 0$ and $t_1 \neq 1$.

First, we show $t_1 \neq 0$. Suppose, on the contrary, that $x_0(t)$ achieves maximum value M only at $t = 0$. Then $x_0(0) = \sum_{i=1}^{m-2} \alpha_i x_0(\xi_i)$ in combination with $\sum_{i=1}^{m-2} \alpha_i = 1$ yields that $\max_{1 \leq i \leq m-2} \{x_0(\xi_i)\} \geq M$, which is a contradiction.

Next, we show $t_1 \neq 1$. It follows from $x_0'(0) = x_0'(1) = 0$ that there is a constant $\eta \in (0, 1)$ such that $x_0''(\eta) = 0$, and thus $x_0''(t) = -\lambda_0 \int_{\eta}^t f(s, x_0(s)) ds$. By the condition (3) we have, for $x \geq r'$ ($M \geq r' > r > 0$), there exists $\sigma \in (0, \delta)$ such that

$$f(t, x) \leq -\sigma x, \quad \forall t \in [0, 1]. \tag{3.1}$$

Suppose, on the contrary, that $x_0(1) = M$. The step is divided into two cases:

Case 1. Assume that $x_0(t) > 0$ on $[\eta, 1]$. Let $r_1 = \min_{t \in [\eta, 1]} x_0(t)$. Then (3.1) yields

$$x_0''(1) = -\lambda_0 \int_{\eta}^1 f(s, x_0(s)) ds \geq \lambda_0 \sigma \int_{\eta}^1 x_0(s) ds > 0,$$

which implies $x_0'(t)$ is increasing close to 1. This together with $x_0'(1) = 0$ induces $x_0'(t) < 0$ (t close to 1), that is, $x_0(t)$ is decreasing close to 1, which contradicts $x_0(1) = M$.

Case 2. Assume that $x_0(t)$ has zero points on $[\eta, 1]$; we may choose η' nearest to 1 with $x_0(\eta') = 0$. Then there is a constant $\eta \in (\eta', 1)$ such that $x_0''(\eta) = 0$. Similar to the above arguments, we easily get a contradiction too.

Hence, we can choose $t_1 \in (0, 1)$ so that $x_0(t_1) = M$. This gives $x_0'(t_1) = 0$ and $x_0''(t_1) \leq 0$. By $x_0 \in \text{dom}L$, we know $x_0(t) \geq 0$ and $x_0'(0) = 0$. Similarly, we also divide the part of the proof into two cases.

Case 1. If $x_0(t) > 0$ on $[0, t_1]$, then there is a constant $\eta \in (0, t_1)$ such that $x_0''(\eta) = 0$. Thus we have

$$x_0''(t) = -\lambda_0 \int_{\eta}^t f(s, x_0(s)) ds.$$

Let $r_2 = \min_{t \in [\eta, t_1]} x_0(t)$. Then it follows from (3.1) that

$$0 \geq x_0''(t_1) = -\lambda_0 \int_{\eta}^{t_1} f(s, x_0(s)) ds \geq \lambda_0 \sigma \int_{\eta}^{t_1} x_0(s) ds > 0,$$

which is a contradiction.

Case 2. If $x_0(t)$ has zero points on $[0, t_1]$, we may choose η' nearest to t_1 with $x_0(\eta') = 0$. Then there is a constant $\eta \in (\eta', t_1)$ such that $x_0''(\eta) = 0$. Thus we have

$$x_0''(t) = -\lambda_0 \int_{\eta}^t f(s, x_0(s)) ds.$$

Let $r_2 = \min_{t \in [\eta, t_1]} x_0(t)$. Then it follows from (3.1) that

$$0 \geq x_0''(t_1) = -\lambda_0 \int_{\eta}^{t_1} f(s, x_0(s)) ds \geq \lambda_0 \sigma \int_{\eta}^{t_1} x_0(s) ds > 0,$$

which is a contradiction. Therefore, (H2) of Theorem 2.4 holds.

Consider $x \in \text{Ker } L \cap \Omega_2$. Then $x(t) \equiv c$ on $[0, 1]$. Similar to [13], we define

$$H(x, \lambda) = x - \lambda|x| - \frac{12\lambda}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s) f(s, |x|) ds,$$

$$x \in \text{Ker } L \cap \Omega_2 \text{ and } \lambda \in [0, 1].$$

Suppose $H(x, \lambda) = 0$. Then in view of (1), we obtain

$$\begin{aligned} c &= \lambda c + \frac{12\lambda}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s) f(s, |c|) ds \\ &\geq \lambda c - \frac{12\lambda}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s) \delta |c| ds \geq \lambda |c| (1 - \delta) \geq 0. \end{aligned}$$

Hence, $H(x, \lambda) = 0$ implies $c \geq 0$. Furthermore, if $H(M, \lambda) = 0$, then we have

$$0 \leq M(1 - \lambda) = \frac{12\lambda}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s) f(s, M) ds,$$

contradicting (3.1). Thus $H(x, \lambda) = 0$ for $x \in \partial\Omega_2$ and $\lambda \in [0, 1]$. Therefore,

$$d_B(H(x, 0), \text{Ker } L \cap \Omega_2, \theta) = d_B(H(x, 1), \text{Ker } L \cap \Omega_2, \theta).$$

However,

$$d_B(H(x, 0), \text{Ker } L \cap \Omega_2, \theta) = d_B(I, \text{Ker } L \cap \Omega_2, \theta).$$

This gives

$$d_B([I - (P + JQN)\gamma]|_{\text{Ker } L}, \text{Ker } L \cap \Omega_2, \theta) = d_B(H(x, 1), \text{Ker } L \cap \Omega_2, \theta) \neq 0,$$

which shows that (H4) of Theorem 2.4 holds.

Let $x \in \overline{\Omega_2} \setminus \Omega_1$ and $t \in [0, 1]$. Then

$$\begin{aligned} (\Psi_\gamma x)(t) &= \int_0^1 |x(s)| ds + \frac{12}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s) f(s, |x(s)|) ds \\ &\quad + \int_0^1 k(t, s) \left[f(s, |x(s)|) - \frac{12}{\sum_{i=1}^{m-2} \alpha_i(3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(\tau) f(\tau, |x(\tau)|) d\tau \right] ds \\ &= \int_0^1 |x(s)| ds + \int_0^1 G(t, s) f(s, |x(s)|) ds. \end{aligned}$$

From (1), we know that

$$\begin{aligned} (\Psi_\gamma)(t) &\geq \int_0^1 |x(s)| ds - \delta \int_0^1 G(t,s) |x(s)| ds \\ &= \int_0^1 (1 - \delta G(t,s)) |x(s)| ds \geq 0. \end{aligned}$$

Hence $\Psi_\gamma(\overline{\Omega_2} \setminus \Omega_1) \subset C$. Moreover, for $x \in \partial\Omega_2$, we have

$$\begin{aligned} (P + JQN)\gamma x &= \int_0^1 |x(s)| ds + \frac{12}{\sum_{i=1}^{m-2} \alpha_i (3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i \int_0^1 h_i(s) f(s, |x(s)|) ds \\ &\geq \int_0^1 \left[1 - \frac{12\delta}{2 \sum_{i=1}^{m-2} \alpha_i (3\xi_i^2 - 2\xi_i^3)} \sum_{i=1}^{m-2} \alpha_i h_i(s) \right] |x(s)| ds \geq 0, \end{aligned}$$

which shows that $(P + JQN)\gamma(\partial\Omega_2) \subset P$. These ensure that (H6), (H7) of Theorem 2.4 hold. It remains to show that (H5) is satisfied.

Taking $u_0(t) \equiv 1$ on $[0, 1]$ and $\sigma(u_0) = 1$, we confirm that

$$u_0 \in C \setminus \{\theta\}, \quad C(u_0) = \{x \in C : x(t) > 0 \text{ on } [0, 1]\}.$$

Let $x \in C(u_0) \cap \partial\Omega_1$. Then we have $x(t) > 0$, $t \in [0, 1]$, $0 < \|x\| < r$ and $x(t) \geq b\|x\|$, $t \in [0, 1]$. Therefore, in view of (2), for all $x \in C(u_0) \cap \partial\Omega_1$, we obtain

$$\begin{aligned} (\Psi x)(t_0) &= \int_0^1 x(s) ds + \int_0^1 G(t_0, s) f(s, x(s)) ds \\ &\geq b\|x\| + \int_0^1 G(t_0, s) g(s) h(x(s)) ds \\ &= b\|x\| + \int_0^1 G(t_0, s) g(s) \frac{h(x(s))}{x^a(s)} x^a(s) ds \\ &\geq b\|x\| + \frac{h(r)}{(r)^a} \int_0^1 G(t_0, s) g(s) x^a(s) ds \\ &\geq b\|x\| + \frac{h(r)}{(r)^a} \int_0^1 G(t_0, s) g(s) b^a \|x\|^a ds \\ &\geq b\|x\| + (1 - b)\|x\| = \|x\|. \end{aligned}$$

That is, $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$, which shows that (H5) of Theorem 2.4 holds.

Summing up, all the hypotheses of Theorem 2.4 are satisfied. Therefore, the equation $Lx = Nx$ has a solution $x \in C \cap (\overline{\Omega_2} \setminus \Omega_1)$. And so, the resonant BVP (1.1) has at least one positive solution on $[0, 1]$. \square

4 An example

Consider the BVP

$$\begin{cases} x'''(t) + f(t, x(t)) = 0, & 0 < t < 1, \\ x(0) = x(\frac{1}{2}), & x'(0) = x'(1) = 0. \end{cases} \tag{4.1}$$

Here $\alpha_1 = 1$, $\xi_1 = \frac{1}{2}$, $g(t) = -\frac{1}{4}(t^2 - t - 1)$, $h(x) = \sqrt{x+1}$ and

$$f(t, x) = \begin{cases} -\frac{1}{4}(t^2 - t - 1)(x^2 - 8x + 15)\sqrt{x+1}, & t \in [0, 1], x \in [0, 2], \\ -\frac{1}{4}(t^2 - t - 1)(2 - \frac{1}{2}\sqrt{x-1})\sqrt{x+1}, & t \in [0, 1], x \in (2, +\infty). \end{cases}$$

By a simple computation, we get

$$\delta = \frac{1}{3}, \quad \frac{1}{4} \leq g(t) \leq \frac{5}{16}, \quad t \in [0, 1], \quad x^2 - 8x + 15 \geq -x, \quad x \in [0, 3].$$

We may choose $M = 3$, $r = \frac{5}{2}$, $t_0 = 0$, $a = 1$, $b = \frac{13}{14}$. It is easy to check

- (1) $f(t, x) > -\frac{1}{3}x$ for all $(t, x) \in [0, 1] \times [0, 3]$;
- (2) $f(t, x) \geq g(t)h(x)$ for $(t, x) \in [0, 1] \times (0, \frac{5}{2}]$, $\frac{\sqrt{x+1}}{x}$ is non-increasing on $(0, \frac{5}{2}]$ with

$$\frac{h(r)}{(r)^a} \int_0^1 G(0, s)g(s) ds \geq \frac{7\sqrt{14}}{320} \geq \frac{1}{13} = \frac{1-b}{b^a};$$

- (3) $f_\infty = \liminf_{x \rightarrow +\infty} \min_{t \in [0, 1]} \frac{f(t, x)}{x} = -\frac{1}{8}$.

Thus, all the conditions of Theorem 3.1 are satisfied. Then the resonant problem (4.1) has at least one positive solution on $[0, 1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

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