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Existence of multiple positive solutions for fourth-order boundary value problems in Banach spaces

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Abstract

This paper deals with the positive solutions of a fourth-order boundary value problem in Banach spaces. By using the fixed-point theorem of strict-set-contractions, some sufficient conditions for the existence of at least one or two positive solutions to a fourth-order boundary value problem in Banach spaces are obtained. An example illustrating the main results is given.

MSC: 34B15

Keywords: u_0 -positive operator; boundary value problem; positive solution; fixed-point theorem; measure of noncompactness

1 Introduction

In this paper, we consider the existence of multiple positive solutions for the fourth-order ordinary differential equation boundary value problem in a Banach space E

$$\begin{cases} x^{(4)}(t) = f(t, x(t), -x''(t)), & t \in (0, 1), \\ x(0) = x(1) = x''(0) = x''(1) = \theta, \end{cases} \quad (1.1)$$

where $f : I \times E \times E \rightarrow E$ is continuous, $I = [0, 1]$, θ is the zero element of E . This problem models deformations of an elastic beam in equilibrium state, whose two ends are simply supported. Owing to its importance in physics, the existence of this problem in a scalar space has been studied by many authors using Schauder's fixed-point theorem and the Leray-Schauder degree theory (see [1–5] and references therein). On the other hand, the theory of ordinary differential equations (ODE) in abstract spaces has become an important branch of mathematics in last thirty years because of its application in partial differential equations and ODEs in appropriately infinite dimensional spaces (see, for example, [6–8]). For an abstract space, it is here worth mentioning that Guo and Lakshmikantham [9] discussed the multiple solutions of two-point boundary value problems of ordinary differential equations in a Banach space. Recently, Liu [10] obtained the sufficient condition for multiple positive solutions to fourth-order singular boundary value problems in an abstract space. In [11], by using the fixed-point index theory in a cone for a strict-set-contraction operator, the authors have studied the existence of multiple positive solutions for the singular boundary value problems with an integral boundary condition.

However, the above works in a Banach space were carried out under the assumption that the second-order derivative x'' is not involved explicitly in the nonlinear term f . This is because the presence of second-order derivatives in the nonlinear function f will make the study extremely difficult. As a result, the goal of this paper is to fill up the gap, that is, to investigate the existence of solutions for fourth-order boundary value problems of (1.1) in which the nonlinear function f contains second-order derivatives, *i.e.*, f depends on x'' .

The main features of this paper are as follows. First, we discuss the existence results in an abstract space E , not $E = \mathbb{R}$. Secondly, we will consider the nonlinear term which is more extensive than the nonlinear term of [10, 11]. Finally, the technique for dealing with fourth-order BVP is completely different from [10, 11]. Hence, we improve and generalize the results of [10, 11] to some degree, and so, it is interesting and important to study the existence of positive solutions of BVP (1.1). The arguments are based upon the u_0 -positive operator and the fixed-point theorem in a cone for a strict-set-contraction operator.

The paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, various conditions on the existence of positive solutions to BVP (1.1) are discussed. In Section 4, we give an example to demonstrate our result.

2 Preliminaries

Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a cone P of E , *i.e.*, $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. We consider problem (1.1) in $C[I, E]$. Evidently, $C[I, E]$ is a Banach space with norm $\|x\|_C = \max_{t \in I} \|x(t)\|$ and $Q = \{x \in C[I, E] : x(t) \geq \theta, \text{ for } t \in I\}$ is a cone of the Banach space $C[I, E]$. In the following, $x \in C^2[I, E] \cap C^4[(0, 1), E]$ is called a solution of problem (1.1) if it satisfies (1.1). x is a positive solution of (1.1) if, in addition, x is nonnegative and nontrivial, *i.e.*, $x \in C[I, P]$ and $x(t) \not\equiv \theta$ for $t \in I$.

For a bounded set V in a Banach space, we denote by $\alpha(V)$ the Kuratowski measure of noncompactness (see [6–8] for further understanding). In this paper, we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness of a bounded set in E and in $C[I, E]$.

Lemma 2.1 [6] *Let $D \subset E$ and D be a bounded set, f be uniformly continuous and bounded from $I \times D$ into E , then*

$$\alpha(f(I \times V)) = \max_{t \in I} \alpha(f(t, V)), \quad \forall V \subset D.$$

The key tool in our approach is the following fixed-point theorem of strict-set-contractions:

Theorem 2.1 [8] *Let P be a cone of a real Banach space E and $P_{r,R} = \{u \in P | r \leq \|u\| \leq R\}$ with $0 < r < R$. Suppose that $A : P_{r,R} \rightarrow P$ is a strict set contraction such that one of the following two conditions is satisfied:*

- (i) $Au \not\leq u$ for $u \in P$, $\|u\| = r$ and $Au \not\geq u$ for $u \in P$, $\|u\| = R$;
- (ii) $Au \not\geq u$ for $u \in P$, $\|u\| = r$ and $Au \not\leq u$ for $u \in P$, $\|u\| = R$;

then the operator A has at least one fixed point $u \in P_{r,R}$ such that $r < \|u\| < R$.

The following concept is due to Krasnosel'skii [12, 13], with a slightly more general definition in [12].

Definition 2.1 We say that a bounded linear operator $T : C[I, \mathbb{R}] \rightarrow C[I, \mathbb{R}]$ is u_0 -positive on a cone $K = \{v \in C[I, \mathbb{R}] : v(t) \geq 0, \forall t \in [0, 1]\}$ if there exists $u_0 \in K \setminus \{\theta\}$ such that for every $u \in K \setminus \{\theta\}$, there are positive constants $k_1(u), k_2(u)$ such that

$$k_1(u)u_0 \leq Tu \leq k_2(u)u_0.$$

Lemma 2.2 Let T be u_0 -positive on a cone K . If T is completely continuous, then $r(L)$, the spectral radius of T , is the unique positive eigenvalue of T with its eigenfunction in K . Moreover, if $\lambda_1 Tu_0 = u_0$ holds with $\lambda_1 = (r(T))^{-1}$, then for an arbitrary non-zero $u \in K$ ($u \neq ku_0$) the elements u and $\lambda_1 Tu$ are incomparable

$$\lambda_1 Tu \not\leq u, \quad \lambda_1 Tu \not\geq u.$$

In the following, the closed balls in spaces E and $C[I, E]$ are denoted, respectively, by $\Omega_l = \{x \in E : \|x\| \leq l\}$ ($l > 0$) and $B_l = \{x \in C[I, E] : \|x\|_C \leq l\}$ ($l > 0$).

For convenience, let us list the following assumptions:

(H₁) $f \in C[I \times P \times P, P]$, f is bounded and uniformly continuous in t on $I \times (P \cap \Omega_r)^2$ for any $r > 0$, and there exist two nonnegative constants l_1, l_2 with $2l_1 + 2l_2 < 1$ such that

$$\alpha(f(t \times D_1 \times D_2)) \leq l_1\alpha(D_1) + l_2\alpha(D_2), \quad t \in I, D_1, D_2 \subset P \cap \Omega_r.$$

(H₂) There are three positive constants a_1, b_1, c_1 such that

$$\|f(t, x, y)\| \leq \frac{a_1 \|x\| + b_1 \|y\| + c_1}{N}$$

for all $(t, x, y) \in I \times P^2$ and $\frac{a_1}{\pi^4} + \frac{b_1}{\pi^2} < 1$.

(H₃) There is a $\varphi \in P^*$ (P^* denotes the dual cone of P) with $\varphi(x) > 0$ for any $x > \theta$, two nonnegative constants a_2, b_2 and a real number $r_1 > 0$ such that

$$\varphi(f(t, x, y)) \geq a_2\varphi(x) + b_2\varphi(y)$$

for all $(t, x, y) \in I \times (P \cap \Omega_{r_1})^2$ and $\frac{a_2}{\pi^4} + \frac{b_2}{\pi^2} > 1$.

(H₄) There is a $\varphi \in P^*$ with $\varphi(x) > 0$ for any $x > \theta$ and two nonnegative constants a_3, b_3 and a real number $R_1 > 0$ such that

$$\varphi(f(t, x, y)) \geq a_3\varphi(x) + b_3\varphi(y), \quad t \in I, x, y \in P, \|x\|, \|y\| \geq R_1$$

and $\frac{a_3}{\pi^4} + \frac{b_3}{\pi^2} > 1$.

(H₅) There are three positive constants a_4, b_4, r_2 such that

$$\|f(t, x, y)\| \leq \frac{a_4 \|x\| + b_4 \|y\|}{N}$$

for all $(t, x, y) \in I \times (P \cap \Omega_{r_2})^2$ and $\frac{a_4}{\pi^4} + \frac{b_4}{\pi^2} < 1$.

(H₆) There exists $\eta > 0$ such that

$$\sup_{t \in I, x, y \in P \cap \Omega_\eta} \|f(t, x, y)\| < \frac{4\eta}{N}.$$

(H₇) There is a $\varphi \in P^*$ with $\|\varphi\| = 1$ and $\varphi(x) > 0$ for any $x > \theta$ and a real number $\eta > 0$ such that

$$\varphi(f(t, x, y)) \geq \alpha\eta, \quad t \in [\tau, 1 - \tau], x, y \in P, \frac{\tau(1 - \tau)\eta}{30N} \leq \|x\|, \|y\| \leq \eta,$$

$$\text{where } \tau \in (0, \frac{1}{2}), \alpha = (\tau(1 - \tau) \int_\tau^{1-\tau} s(1 - s) ds)^{-1}.$$

Now, let $G(t, s)$ be the Green's function of the linear problem $v'' = 0, t \in (0, 1)$ together with $v(0) = v(1) = 0$, which can be explicitly given by

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1, \\ s(1 - t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Obviously, $G(t, s)$ have the following properties:

$$\begin{aligned} t(1 - t)s(1 - s) \leq G(t, s) \leq t(1 - t), \quad \forall t, s \in [0, 1]; \\ G(t, s) \geq t(1 - t)G(\tau, s), \quad \forall \tau, t, s \in [0, 1]. \end{aligned} \tag{2.1}$$

Set

$$(Su)(t) = \int_0^1 G(t, s)u(s) ds, \quad t \in I, u \in C[I, E].$$

Obviously, $S : C[I, E] \rightarrow C[I, E]$ is continuous.

Let $u = -x''$. Since $x(0) = x(1) = \theta$, we have

$$x(t) = (Su)(t) = \int_0^1 G(t, s)u(s) ds. \tag{2.2}$$

Using the above transformation and (2.2), BVP (1.1) becomes

$$-u''(t) = f(t, (Su)(t), u(t)) \tag{2.3}$$

with

$$u(0) = u(1) = \theta. \tag{2.4}$$

From (2.3) and (2.4), we have

$$u(t) = \int_0^1 G(t, s)f(s, (Su)(s), u(s)) ds.$$

Now, define an operator A on Q by

$$(Au)(t) = \int_0^1 G(t,s)f\left(s, \int_0^1 G(s,\tau)u(\tau) d\tau, u(s)\right) ds. \tag{2.5}$$

The following Lemma 2.3 can be easily obtained.

Lemma 2.3 *Assume that (H_1) holds. Then $A : Q \rightarrow C^2[I, P]$ and*

- (i) $A : Q \rightarrow Q$ is continuous and bounded;
- (ii) BVP (1.1) has a solution in $C^2[I, P] \cap C^4[(0,1), P]$ if and only if A has a fixed point in Q .

Let

$$(T\psi)(t) = \int_0^1 G(t,s)\psi(s) ds, \quad \forall \psi \in C[I, \mathbb{R}].$$

It is easy to see that $T : C[I, \mathbb{R}] \rightarrow C[I, \mathbb{R}]$ is a u_0 -positive operator with $u_0(t) = \sin \pi t$ and $r(T) = \frac{1}{\pi^2}$.

Lemma 2.4 *Suppose that (H_1) holds. Then for any $l > 0$, $A : Q \cap B_l \rightarrow Q$ is a strict set contraction.*

Proof For any $u \in Q \cap B_l$ and $t \in [0, 1]$, by the expression of S , we have

$$\|(Su)(t)\| \leq \int_0^1 G(t,s)\|u(s)\| ds \leq l,$$

and thus $S(Q \cap B_l) \subset Q \cap B_l$ is continuous and bounded. By the uniformly continuous f and (H_1) , and Lemma 2.1, we have

$$\alpha(f(I \times S(D) \times D)) = \max_{t \in I} \alpha(f(t \times S(D) \times D)) \leq l_1\alpha(S(D)) + l_2\alpha(D).$$

Since f is uniformly continuous and bounded on $I \times (Q \cap \Omega_l)^2$, we see from (2.5) that A is continuous and bounded on $Q \cap B_l$. Let $D \subset Q \cap B_l$, according to (2.5), it is easy to show that the functions $\{Au : u \in D\}$ are uniformly bounded and equicontinuous, and so in [9] we have

$$\alpha(A(D)) = \max_{t \in I} \alpha(A(D(t))),$$

where

$$A(D(t)) = \{Au(t) : u \in D, t \text{ is fixed}\} \subset P \cap \Omega_l.$$

Using the obvious formula

$$\int_0^1 u(t) dt \in \overline{\text{co}}\{u(t) : t \in I\}, \quad \forall u \in C[I, E]$$

and observing $0 \leq G(t, s) \leq 1$, we find

$$\begin{aligned}
 \alpha(A(D(t))) &\leq \alpha(\overline{co}\{G(t, s)f(s, Su(s), u(s)) : u \in D, s \in I\}) \\
 &\leq \alpha(\overline{co}\{f(s, Su(s), u(s)) \cup \theta : u \in D, s \in I\}) \\
 &= \alpha(\{f(s, Su(s), u(s)) \cup \theta : u \in D, s \in I\}) \\
 &\leq \alpha(\{f(s, Su(s), u(s)) : u \in D, s \in I\}) \\
 &\leq \alpha(f(I \times S(B) \times B)) \leq l_1\alpha(S(B)) + l_2\alpha(B) \\
 &\leq l_1\alpha(B) + l_2\alpha(B),
 \end{aligned} \tag{2.6}$$

where $B = \{u(s) | s \in I, u \in D\} \subset P_I$, $S(B) = \{\int_0^1 G(t, s)u(s) ds | t \in I, u \in D\}$.

From the fact of [9], we know

$$\alpha(B) \leq 2\alpha(D). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\alpha(A(D(t))) \leq 2(l_1 + l_2)\alpha(D),$$

and consequently, A is a strict set contraction on $Q \cap B_l$ because $2(l_1 + l_2) < 1$. □

3 Main results

Theorem 3.1 *Let a cone P be normal and condition (H_1) be satisfied. If (H_2) and (H_3) or (H_4) and (H_5) are satisfied, then BVP (1.1) has at least one positive solution.*

Proof Set

$$Q_1 = \{u \in Q : u(t) \geq t(1 - t)u(s), \forall t, s \in [0, 1]\}.$$

It is clear that Q_1 is a cone of the Banach space $C[I, E]$ and $Q_1 \subset Q$. For any $u \in Q_1$, by (2.1), we can obtain $A(Q) \subset Q_1$, then

$$A(Q_1) \subset Q_1.$$

We first assume that (H_2) and (H_3) are satisfied. Let

$$W = \{u \in Q_1 | Au \geq u\}.$$

In the following, we prove that W is bounded.

For any $u \in W$, we have $\theta \leq u \leq Au$, that is, $\theta \leq u(t) \leq Au(t)$, $t \in I$. And so $\|u(t)\| \leq N\|Au(t)\|$, set $v(t) = \|u(t)\|$, by (H_2)

$$\begin{aligned}
 v(t) &\leq N\|Au(t)\| \\
 &\leq N \int_0^1 G(t, s) \left\| f\left(s, \int_0^1 G(s, \tau)u(\tau) d\tau, u(s)\right) \right\| ds
 \end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 G(t,s) \left(a_1 \left\| \int_0^1 G(s,\tau) u(\tau) d\tau \right\| + b_1 \|u(s)\| + c_1 \right) ds \\ &\leq \int_0^1 G(t,s) \left(a_1 \int_0^1 G(s,\tau) \|u(\tau)\| d\tau + b_1 \|u(s)\| \right) ds + c_1. \end{aligned} \tag{3.1}$$

For $\psi \in C[I, \mathbb{R}]$, let $L_1\psi = a_1 T^2\psi + b_1 T\psi$, then $L_1 : C[I, \mathbb{R}] \rightarrow C[I, \mathbb{R}]$ is a bounded linear operator. From (3.1), one deduces that

$$(I - L_1)v(t) \leq c_1.$$

Since π^2 is the first eigenvalue of T , by (H_2) , the first eigenvalue of L_1 , $r(L_1) = \frac{a_1}{\pi^4} + \frac{b_1}{\pi^2} < 1$. Therefore, by [14], the inverse operator $(I - L_1)^{-1}$ exists and

$$(I - L_1)^{-1} = I + L_1 + L_1^2 + \dots + L_1^n + \dots.$$

It follows from $L_1(K) \subset K$ that $(I - L_1)^{-1}(K) \subset K$. So, we know that $v(t) \leq (I - L_1)^{-1}c_1$, $t \in [0, 1]$ and W is bounded.

Taking $R_2 > \max\{r_1, \sup W\}$, we have

$$Au \not\leq u, \quad \forall u \in \partial B_{R_2} \cap Q_1. \tag{3.2}$$

Next, we are going to verify that for any $r_3 \in (0, r_1)$,

$$Au \not\leq u, \quad \forall u \in \partial B_{r_3} \cap Q_1. \tag{3.3}$$

If this is false, then there exists $u_1 \in \partial B_{r_3} \cap Q_1$ such that $Au_1 \leq u_1$. This together with (H_3) yields

$$\begin{aligned} \varphi(u_1(t)) &\geq \varphi(Au_1(t)) = \varphi\left(\int_0^1 G(t,s)f(s, (Su_1)(s), u_1(s)) ds\right) \\ &= \int_0^1 G(t,s)\varphi(f(s, (Su_1)(s), u_1(s))) ds \\ &\geq \int_0^1 G(t,s)(a_2\varphi((Su_1)(s)) + b_2\varphi(u_1(s))) ds \\ &= \int_0^1 G(t,s)\left(a_2 \int_0^1 G(s,\tau)\varphi(u_1(\tau)) d\tau + b_2\varphi(u_1(s))\right) ds \\ &= (a_2 T^2 + b_2 T)\varphi(u_1(t)), \quad t \in I. \end{aligned}$$

For $\psi \in C[I, \mathbb{R}]$, let $L_2\psi = a_2 T^2\psi + b_2 T\psi$, then the above inequality can be written in the form

$$\varphi(u_1(t)) \geq L_2(\varphi(u_1(t))). \tag{3.4}$$

It is easy to see that

$$\varphi(u_1(t)) \neq 0, \quad t \in I.$$

In fact, $\varphi(u_1(t)) = 0$ implies $u_1(t) = \theta$ for $t \in I$, and consequently, $\|u_1\|_C = 0$ in contradiction to $\|u_1\|_C = r_3$. Now, notice that L_2 is a u_0 -positive operator with $u_0(t) = \sin \pi t$, then by Lemma 2.2, we have $\varphi(u_1(t)) = \mu_0 \sin \pi t$ for some $\mu_0 > 0$. This together with $(\frac{a_2}{\pi^4} + \frac{b_2}{\pi^2}) \sin \pi t = L_2(\sin \pi t)$ and (3.4) implies that

$$\mu_0 \sin \pi t = \varphi(u_1(t)) \geq L_2(\varphi(u_1(t))) = L_2(\mu_0 \sin \pi t) = \mu_0 \left(\frac{a_2}{\pi^4} + \frac{b_2}{\pi^2} \right) \sin \pi t,$$

which is a contradiction to $\frac{a_2}{\pi^4} + \frac{b_2}{\pi^2} > 1$. So, (3.3) holds.

By Lemma 2.4, A is a strict set contraction on $Q_{r_3, R_2} = \{u \in Q_1 : r_3 \leq \|u\|_C \leq R_2\}$. Observing (3.2) and (3.3) and using Theorem 2.1, we see that A has a fixed point on Q_{r_3, R_2} .

Next, in the case that (H_4) and (H_5) are satisfied, by the method as in establishing (3.3), we can assert from (H_5) that for any $r_4 \in (0, r_2)$,

$$Au \not\geq u, \quad \forall u \in \partial B_{r_4} \cap Q_1. \tag{3.5}$$

Let

$$(L_3 v)(t) = \int_0^1 G(t, s) \left(a_3 \int_0^1 G(s, \tau) v(\tau) d\tau + b_3 v(s) \right) ds, \quad v \in C[I, \mathbb{R}].$$

It is clear that $L_3 : C[I, \mathbb{R}] \rightarrow C[I, \mathbb{R}]$ is a completely continuous linear u_0 -operator with $u_0(t) = \sin \pi t$ and $L_3 : K_1 \rightarrow K_1$ in which $K_1 = \{v \in K \subset C[I, \mathbb{R}] : v(t) \geq t(1-t)v(s), \forall t, s \in I\}$. In addition, the spectral radius $r(L_3) = \frac{a_3}{\pi^4} + \frac{b_3}{\pi^2}$ and $\sin \pi t$ is the positive eigenfunction of L_3 corresponding to its first eigenvalue $\lambda_1 = (r(L_3))^{-1}$.

Let

$$(L_\delta v)(t) = \int_\delta^{1-\delta} G(t, s) \left(a_3 \int_0^1 G(s, \tau) v(\tau) d\tau + b_3 v(s) \right) ds, \quad v \in C[I, \mathbb{R}],$$

where $\delta \in (0, \frac{1}{2})$. It is clear that $L_\delta : C[I, \mathbb{R}] \rightarrow C[I, \mathbb{R}]$ is a completely continuous linear u_0 -operator with $u_0(t) = t(1-t)$ and $L_\delta(K_1) \subset K_1$. Thus, the spectral radius $r(L_\delta) \neq 0$ and L_δ has a positive eigenfunction corresponding to its first eigenvalue $\lambda_\delta = (r(L_\delta))^{-1}$.

Take $\delta_n \in (0, 1/2)$ ($n = 1, 2, \dots$) satisfying $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n \geq \dots$ and $\delta_n \rightarrow 0$ ($n \rightarrow \infty$). For $m > n$, $v \in K_1$, we have

$$(L_{\delta_n} v)(t) \leq (L_{\delta_m} v)(t) \leq (L_3 v)(t), \quad \forall t \in I.$$

By [12], we have $r(L_{\delta_n}) \leq r(L_{\delta_m}) \leq r(L_3)$. Let $\lambda_{\delta_n} = (r(L_{\delta_n}))^{-1}$, by Gelfand's formula, we have $\lambda_{\delta_n} \geq \lambda_{\delta_m} \geq \lambda_1$. Let $\lambda_{\delta_n} \rightarrow \tilde{\lambda}_1$ as $n \rightarrow \infty$.

In the following, we prove that $\tilde{\lambda}_1 = \lambda_1$.

Let v_{δ_n} be the positive eigenfunction of T_{δ_n} corresponding to λ_{δ_n} , i.e.,

$$\begin{aligned} v_{\delta_n}(t) &= \lambda_{\delta_n} (L_\delta v_{\delta_n})(t) \\ &= \lambda_{\delta_n} \int_\delta^{1-\delta_n} G(t, s) \left(a_3 \int_0^1 G(s, \tau) v_{\delta_n}(\tau) d\tau + b_3 v_{\delta_n}(s) \right) ds, \end{aligned} \tag{3.6}$$

satisfying $\|v_{\delta_n}\| = 1$. Without loss of generality, by standard argument, we may suppose by the Arzela-Ascoli theorem and $\lambda_{\delta_n} \rightarrow \tilde{\lambda}_1$ that $v_{\delta_n} \rightarrow \tilde{v}_0$ as $n \rightarrow \infty$. Thus, $\|\tilde{v}_0\| = 1$ and by (3.6), we have

$$\tilde{v}_0(t) = \tilde{\lambda}_1 \int_0^1 G(t,s) \left(a_3 \int_0^1 G(s,\tau) \tilde{v}_0(\tau) d\tau + b_3 \tilde{v}_0(s) \right) ds,$$

that is, $\tilde{v}_0(t) = \tilde{\lambda}_1 (L_3 \tilde{v}_0)(t)$. This together with Lemma 2.2 guarantees that $\tilde{\lambda}_1 = \lambda_1$.

By the above argument, it is easy to see that there exists a $\delta \in (0, \frac{1}{2})$ such that

$$1 < r(L_\delta) < \frac{a_3}{\pi^4} + \frac{b_3}{\pi^2}.$$

Choose

$$R_3 > \max \left\{ R_1, r_2, \frac{30NR_1}{\delta(1-\delta)} \right\}. \tag{3.7}$$

Now, we assert that

$$Au \not\leq u, \quad \forall u \in Q_1, \|u\|_C = R_3. \tag{3.8}$$

If this is not true, then there exists $u_2 \in Q_1$ with $\|u\|_C = R_3$ such that $Au_2 \leq u_2$, then $Au_2(t) \leq u_2(t)$. Moreover, by the definition of Q_1 , we know

$$\begin{aligned} u_2(t) &\geq t(1-t)u_2(s), \quad \forall t, s \in I, \\ (Su_2)(t) &= \int_0^1 G(t,s)u_2(s) ds \geq t(1-t) \int_0^1 s^2(1-s)^2 ds \cdot u_2(\tau), \quad \forall t, \tau \in I. \end{aligned}$$

Thus, $N\|u_2(t)\| \geq t(1-t)\|u_2\|_C$, $N\|(Su_2)(t)\| \geq \frac{t(1-t)}{30}\|u_2\|_C$, which implies by (3.7) we have

$$\min_{t \in [\delta, 1-\delta]} \|u(t)\| \geq \frac{\delta(1-\delta)}{N} \|u\|_C \geq R_1,$$

and

$$\min_{t \in [\delta, 1-\delta]} \|(Su)(t)\| \geq \frac{\delta(1-\delta)}{30N} \|u\|_C \geq R_1.$$

So, by (H_4) , we get

$$\begin{aligned} \varphi(u_2(t)) &\geq \varphi((Au_2)(t)) \\ &= \varphi \left(\int_0^1 G(t,s) f(s, (Su_2)(s), u_2(s)) ds \right) \\ &\geq \int_\delta^{1-\delta} G(t,s) \varphi(f(s, (Su_2)(s), u_2(s))) ds \\ &\geq \int_\delta^{1-\delta} G(t,s) (a_3 \varphi((Su_2)(s)) + b_3 \varphi(u_2(s))) ds \end{aligned}$$

$$\begin{aligned} &\geq \int_{\delta}^{1-\delta} G(t,s) \left(a_3 \int_0^1 G(s,\tau) \varphi(u_2(\tau)) d\tau + b_3 \varphi(u_2(s)) \right) ds \\ &= L_{\delta}(\varphi(u_2(t))). \end{aligned}$$

It is easy to see that

$$\varphi(u_2(t)) \neq 0, \quad t \in I.$$

In fact, $\varphi(u_2(t)) = 0$ implies $u_2(t) = \theta$ for $t \in I$, and consequently, $\|u_2\|_C = 0$ in contradiction to $\|u_2\|_C = R_3$. Now, notice that L_{δ} is a u_0 -positive operator with $u_0(t) = t(1-t)$. Then by Lemma 2.2, we have $\varphi(u_2(t)) = \mu_0 v_{\delta}(t)$ for some $\mu_0 > 0$, where v_{δ} is the positive eigenfunction of L_{δ} corresponding to λ_{δ} . This together with $r(L_{\delta})v_{\delta}(t) = L_{\delta}(v_{\delta}(t))$ implies that

$$\mu_0 v_{\delta}(t) = \varphi(u_2(t)) \geq L_{\delta}(\varphi(u_2(t))) = L_{\delta}(\mu_0 v_{\delta}(t)) = \mu_0 r(L_{\delta})v_{\delta}(t),$$

which is a contradiction to $r(L_{\delta}) > 1$. So, (3.8) holds.

By Lemma 2.4, A is a strict set contraction on $Q_{r_4,R_3} = \{u \in Q_1 : r_4 \leq \|u\|_C \leq R_3\}$. Observing (3.5) and (3.8) and using Theorem 2.1, we see that A has a fixed point on Q_{r_4,R_3} . This together with Lemma 2.3 implies that BVP (1.1) has at least one positive solution. \square

Theorem 3.2 *Let a cone P be normal. Suppose that conditions (H_1) , (H_3) , (H_4) and (H_6) are satisfied. Then BVP (1.1) has at least two positive solutions.*

Proof We can take the same $Q_1 \subset C[I,E]$ as in Theorem 3.1. As in the proof of Theorem 3.1, we can also obtain that $A(Q_1) \subset Q_1$. And we choose r_5, R_4 with $R_4 > \eta > r_5 > 0$ such that

$$Au \not\leq u, \quad \forall u \in \partial B_{r_5} \cap Q_1, \tag{3.9}$$

$$Au \not\leq u, \quad \forall u \in \partial B_{R_4} \cap Q_1. \tag{3.10}$$

On the other hand, it is easy to see that

$$Au \not\geq u, \quad \forall u \in \partial B_{\eta} \cap Q_1. \tag{3.11}$$

In fact, if there exists $u_3 \in Q_1$ with $\|u_3\|_C = \eta$ such that $Au_3 \geq u_3$, then observing $\max_{t,s \in I} G(t,s) = \frac{1}{4}$ and $\|Su_2\|_C \leq \eta$, we get

$$\begin{aligned} \theta \leq u_3(t) &\leq \int_0^1 G(t,s) f(s, (Su_3)(s), u_3(s)) ds \\ &\leq \frac{1}{4} \int_0^1 f(s, (Su_3)(s), u_3(s)) ds, \quad \forall t \in I, \end{aligned}$$

and so

$$\|u_3(t)\| \leq \frac{N}{4} \int_0^1 \|f(s, (Su_3)(s), u_3(s))\| ds \leq \frac{1}{4} NM, \quad t \in I, \tag{3.12}$$

where, by virtue of (H_5) ,

$$M = \sup_{t \in I, x, y \in P \cap \Omega_\eta} \|f(t, x, y)\| < \frac{4\eta}{N}. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\eta = \|u_3\|_C \leq \frac{4\eta}{N} M < \eta,$$

a contradiction. Thus (3.11) is true.

By Lemma 2.4, A is a strict set contraction on $Q_{r_5, \eta} = \{u \in Q_1 \mid r_5 \leq \|u\|_C \leq \eta\}$, and also on $Q_{\eta, R_4} = \{u \in Q_1 \mid \eta \leq \|u\|_C \leq R_4\}$. Observing (3.9), (3.10), (3.11) and applying, respectively, Theorem 2.1 to A , $Q_{r_5, \eta}$ and Q_{η, R_4} , we assert that there exist $u_1 \in Q_{r_5, \eta}$ and $u_2 \in Q_{\eta, R_4}$ such that $Au_1 = u_1$ and $Au_2 = u_2$ and, by Lemma 2.3 and (3.11), Su_1, Su_2 are positive solutions of BVP (1.1). \square

Theorem 3.3 *Let a cone P be normal. Suppose that conditions (H_1) , (H_2) and (H_5) and (H_7) are satisfied. Then BVP (1.1) has at least two positive solutions.*

Proof We can take the same $Q_1 \subset C[I, E]$ as in Theorem 3.1. As in the proof of Theorem 3.1, we can also obtain that $A(Q_1) \subset Q_1$. And we choose r_6, R_5 with $R_5 > \eta > r_6 > 0$ such that

$$Au \not\leq u, \quad \forall u \in \partial B_{r_6} \cap Q_1, \tag{3.14}$$

$$Au \not\leq u, \quad \forall u \in \partial B_{R_5} \cap Q_1. \tag{3.15}$$

On the other hand, it is easy to see that

$$Au \not\leq u, \quad \forall u \in \partial B_\eta \cap Q_1. \tag{3.16}$$

In fact, if there exists $u_4 \in Q_1$ with $\|u_4\|_C = \eta$ such that $Au_4 \geq u_4$, then

$$\theta \leq (Au_4)(t) \leq u_4(t), \quad t \in I.$$

Observing $u_4(t) \in Q_1$ and $\frac{t(1-t)}{30} u_4(s) \leq (Su_4)(t) \leq u_4(s)$, we get

$$\begin{aligned} \eta &= \|\varphi\| \|u_4\|_C \geq \varphi(u_4(t)) \geq \varphi((Au_4)(t)) \\ &= \int_0^1 G(t, s) \varphi(f(s, (Su_4)(s), u_4(s))) \, ds \\ &\geq \int_\tau^{1-\tau} G(t, s) \varphi(f(s, (Su_4)(s), u_4(s))) \, ds \\ &> \alpha \eta \tau (1 - \tau) \int_\tau^{1-\tau} s(1 - s) \, ds = \eta, \end{aligned}$$

which is a contradiction. Hence, (3.16) holds.

By Lemma 2.4, A is a strict set contraction on $Q_{r_6, \eta} = \{u \in Q_1 | r_6 \leq \|u\|_C \leq \eta\}$ and also on $Q_{\eta, R_5} = \{u \in Q_1 | \eta \leq \|u\|_C \leq R_5\}$. Observing (3.14), (3.15), (3.16) and applying, respectively, Theorem 2.1 to A , $Q_{r_6, \eta}$ and Q_{η, R_5} , we assert that there exist $u_1 \in Q_{r_6, \eta}$ and $u_2 \in Q_{\eta, R_5}$ such that $Au_1 = u_1$ and $Au_2 = u_2$ and, by Lemma 2.3 and (3.16), Su_1, Su_2 are positive solutions of BVP (1.1). \square

4 One example

Now, we consider an example to illustrate our results.

Example 4.1 Consider the following boundary value problem of the finite system of scalar differential equations:

$$\begin{cases} x_n^{(4)}(t) = f_n(t, x(t), -x''(t)), & t \in (0, 1), \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases} \tag{4.1}$$

where

$$f_n(t, x, y) = g(y_n) + (1 + \sin 2\pi t) \frac{x_n}{100(1 + x_n + y_{n+1})}, \quad n = 1, 2, \dots, 9, \tag{4.2}$$

$$f_{10}(t, x, y) = g(y_{10}) + (1 + \sin 2\pi t) \frac{x_{10}}{100(1 + x_{10} + y_1)},$$

$$x = (x_1, x_2, \dots, x_{10}), \quad y = (y_1, y_2, \dots, y_{10}),$$

$$g(u) = \begin{cases} 100u, & 0 \leq u \leq 0.02, \\ 2, & 0.02 \leq u \leq 0.51, \\ 200u - 100, & 0.51 \leq u \leq 1, \\ 100u, & u \geq 1. \end{cases} \tag{4.3}$$

Claim (4.1) has at least two positive solutions $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_{10}^*(t))$ and $x^{**}(t) = (x_1^{**}(t), x_2^{**}(t), \dots, x_{10}^{**}(t))$ such that

$$0 < \max_{1 \leq i \leq 10, t \in [0, 1]} |x_i^*(t)| < 0.51 < \max_{1 \leq i \leq 10, t \in [0, 1]} |x_i^{**}(t)|.$$

Proof Let $E = \mathbb{R}^{10} = \{x = (x_1, x_2, \dots, x_{10}) : x_n \in \mathbb{R}, n = 1, 2, \dots, 10\}$ with the norm $\|x\| = \max_{1 \leq n \leq 10} |x_n|$, and $P = \{x = (x_1, x_2, \dots, x_{10}) : x_n \geq 0, n = 1, 2, \dots, 10\}$. Then P is a normal cone in E , and the normal constant is $N = 1$. System (4.1) can be regarded as a boundary value problem of (1.1) in E , where $x = (x_1, x_2, \dots, x_{10}), y = (y_1, y_2, \dots, y_{10})$,

$$f(t, x, y) = (f_1(t, x, y), f_2(t, x, y), \dots, f_{10}(t, x, y)).$$

Evidently, $f : I \times P^2 \rightarrow P$ is continuous. In this case, condition (H_1) is automatically satisfied. Since $\alpha(f(t, D_1, D_2))$ is identical zero for any $t \in I$ and $D_1, D_2 \subset P \cap \Omega_t$. Obviously, $P^\circ = P$, so we may choose $\varphi = (1, 1, \dots, 1)$, then for any $x, y \in P$, we have

$$\varphi(f(t, x, y)) = \sum_{n=1}^{10} f_n(t, x, y).$$

Noticing $\|x\| \leq \varphi(x) \leq 10\|x\|$, we have

$$\varphi(f(t, x, y)) = \sum_{n=1}^{10} f_n(t, x, y) \geq 100\|y\| \geq 10\varphi(y)$$

for all $(t, x, y) \in I \times P^2$ with $\|x\| \leq 0.02$ and $\|y\| \leq 0.02$, and

$$\varphi(f(t, x, y)) = \sum_{n=1}^{10} f_n(t, x, y) \geq 100\|y\| \geq 10\varphi(y),$$

for all $(t, x, y) \in I \times P^2$ with $\|x\| \geq 1$ and $\|y\| \geq 1$. So, the conditions (H_3) and (H_4) are satisfied with $a_2 = a_3 = 0$ and $b_2 = b_3 = 10$.

Choosing $\eta = 0.51$ for $t \in I$ and $x, y \in P$ with $\|x\|, \|y\| \leq 0.51$, we have

$$\|f(t, x, y)\| \leq 2.0102 < 2.04 = 4\eta.$$

So, condition (H_6) is satisfied. Thus, our conclusion follows from Theorem 3.2. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors typed, read and approved the final manuscript.

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Acknowledgements

The authors would like to thank the referees for carefully reading this article and making valuable comments and suggestions. This work is supported by the Foundation items: NSFC (10971179), NSF (BS2010SF023, BS2012SF022) of Shandong Province.

Received: 2 June 2012 Accepted: 24 September 2012 Published: 9 October 2012

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doi:10.1186/1687-2770-2012-107

Cite this article as: Cui and Sun: Existence of multiple positive solutions for fourth-order boundary value problems in Banach spaces. *Boundary Value Problems* 2012 **2012**:107.