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Existence and uniqueness of anti-periodic solutions for prescribed mean curvature Rayleigh equations

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Abstract
By means of the Leray-Schauder degree theory, we establish some sufficient conditions for the existence and uniqueness of anti-periodic solutions for prescribed mean curvature Rayleigh equations.
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1 Introduction

We are concerned with the existence and uniqueness of anti-periodic solutions of the following prescribed mean curvature Rayleigh equation:

$$\left(\frac{x'}{\sqrt{1+x'^2}}\right)' + f\left(t, x'(t)\right) + g\left(t, x(t)\right) = e(t), \tag{1.1}$$

where $e \in C(\mathbb{R},\mathbb{R})$ is T-periodic, and $f, g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ are T-periodic in the first argument, $T > 0$ is a constant.

In recent years, the existence of periodic solutions and anti-periodic solutions for some types of second-order differential equations, especially for the Rayleigh ones, were widely studied (see [1–7]) and the references cited therein). For example, Liu [7] discussed the Rayleigh equation

$$x'' + f\left(t, x'(t)\right) + g\left(t, x(t)\right) = e(t),$$

and established the existence and uniqueness of anti-periodic solutions. At the same time, a kind of prescribed mean curvature equations attracted many people's attention (see [8–11] and the references cited therein). Feng [8] investigated the prescribed mean curvature Liénard equation

$$\left(\frac{x'}{\sqrt{1+x'^2}}\right)' + f\left(x(t)\right)x'(t) + g\left(t, x\left(t-\tau(t)\right)\right) = e(t)$$

and obtained some existence results on periodic solutions. However, to the best of our knowledge, the existence and uniqueness of anti-periodic solution for Eq. (1.1) have not

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been investigated till now. Motivated by [7, 8], we establish some sufficient conditions for the existence and uniqueness of anti-periodic solutions via the Leray-Schauder degree theory.

The rest of the paper is organized as follows. In Section 2, we shall state and prove some basic lemmas. In Section 3, we shall prove the main result. An example will be given to show the applications of our main result in the final section.

2 Preliminaries

We first give the definition of an anti-periodic function. Assume that N is a positive integer. Let $u:\mathbb{R}\to\mathbb{R}^N$ be a continuous function. We call $u(t)$ an anti-periodic function on $\mathbb{R}$ if u satisfies the following condition:

$$u\left(t+\frac{T}{2}\right)=-u(t),\quad \text{for all } t\in\mathbb{R}.$$

Obviously, a $\frac{T}{2}$-anti-periodic function u is a T-periodic function.

Throughout this paper, we will adopt the following notations:

$$C_T^k(\mathbb{R},\mathbb{R}^N):=\{x(t)=(x_1(t),\dots,x_N(t))^T\in C^k(\mathbb{R},\mathbb{R}^N), x \text{ is } T\text{-periodic}\},\quad k=0,1,2,$$
$$|x_i|_2=\left(\int_0^T |x_i(t)|^2 dt\right)^{1/2},\qquad |x_i|_\infty=\max_{t\in[0,T]}|x_i(t)|,\quad i=1,2,\dots,N,$$
$$|x|_\infty=\max\{|x_1|_\infty,|x_2|_\infty,\dots,|x_N|_\infty\},$$
$$C_T^{k,\frac{1}{2}}(\mathbb{R},\mathbb{R}^N):=\left\{x(t)\in C_T^k(\mathbb{R},\mathbb{R}^N), x\left(t+\frac{T}{2}\right)=-x(t),\ \text{for all } t\in\mathbb{R}\right\},$$

which is a linear normal space endowed with the norm $\|\cdot\|$ defined by

$$\|x\|=\max\{|x|_\infty,|x'|_\infty,\dots,|x^{(k)}|_\infty\},\quad \text{for all } x\in C_T^{k,\frac{1}{2}}(\mathbb{R},\mathbb{R}^N).$$

The following lemmas will be useful to prove our main results.

Lemma 2.1 [12] *If $x\in C_T^1(\mathbb{R},\mathbb{R})$ and $\int_0^T x(t)\,dt=0$, then*

$$\int_0^T |x(t)|^2\,dt\le (T^2/4\pi^2)\int_0^T |x'(t)|^2\,dt$$

(*Wirtinger inequality*) *and*

$$|x|_\infty^2\le (T/12)\int_0^T |x'(t)|^2\,dt$$

(*Sobolev inequality*).

Lemma 2.2 *Suppose that the following condition holds*:

(H_1) $(g(t,x_1)-g(t,x_2))(x_1-x_2)<0$, *for all* $t,x_1,x_2\in\mathbb{R}$ *and* $x_1\neq x_2$.

Then Eq. (1.1) *has at most one* T*-periodic solution*.

Proof Assume that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of Eq. (1.1). Then we obtain

$$\left(\frac{x_i'(t)}{\sqrt{1+x_i'^2(t)}}\right)' + f\bigl(t, x_i'(t)\bigr) + g\bigl(t, x_i(t)\bigr) = e(t), \quad i = 1, 2. \tag{2.1}$$

It is easy to see that $x_i(t) \in C^2[0, T]$ $(i = 1, 2)$. From (2.1), we know

$$\begin{aligned} &\frac{x_1''(t)}{(\sqrt{1+x_1'^2(t)})^3} - \frac{x_2''(t)}{(\sqrt{1+x_2'^2(t)})^3} \\ &\quad + \bigl(f\bigl(t, x_1'(t)\bigr) - f\bigl(t, x_2'(t)\bigr)\bigr) + \bigl(g\bigl(t, x_1(t)\bigr) - g\bigl(t, x_2(t)\bigr)\bigr) = 0. \end{aligned} \tag{2.2}$$

Set $z(t) = x_1(t) - x_2(t)$. Now, we prove

$$z(t) \le 0, \quad \text{for all } t \in \mathbb{R}.$$

Otherwise, we have

$$\max_{t\in\mathbb{R}} z(t) = \max_{t\in[0,T]} z(t) > 0.$$

Then there exists a $t^* \in [0, T]$ such that

$$x_1(t^*) - x_2(t^*) = z(t^*) = \max_{t\in\mathbb{R}} z(t) = \max_{t\in[0,T]} z(t) > 0, \tag{2.3}$$

which implies that

$$z'(t^*) = x_1'(t^*) - x_2'(t^*) = 0, \tag{2.4}$$

and

$$z''(t^*) = x_1''(t^*) - x_2''(t^*) \le 0. \tag{2.5}$$

It follows from (2.2), (2.4) and (2.5) that

$$\begin{aligned} g\bigl(t^*, x_1(t^*)\bigr) - g\bigl(t^*, x_2(t^*)\bigr) &= -\left(\frac{x_1''(t^*)}{(\sqrt{1+x_1'^2(t^*)})^3} - \frac{x_2''(t^*)}{(\sqrt{1+x_2'^2(t^*)})^3}\right) \\ &\quad - \bigl(f\bigl(t^*, x_1'(t^*)\bigr) - f\bigl(t^*, x_2'(t^*)\bigr)\bigr) \\ &= -\frac{1}{(\sqrt{1+x_1'^2(t^*)})^3}\bigl(x_1''(t^*) - x_2''(t^*)\bigr) \\ &= -\frac{1}{(\sqrt{1+x_1'^2(t^*)})^3} z''(t^*) \ge 0. \end{aligned}$$

From (H_1), we get

$$x_1(t^*) - x_2(t^*) \le 0,$$

which contradicts (2.3). Thus,

$$z(t) \le 0, \quad \text{for all } t \in \mathbb{R}.$$

By using a similar argument, we can also show

$$z(t) \ge 0, \quad \text{for all } t \in \mathbb{R}.$$

Hence,

$$x_1(t) - x_2(t) = z(t) = 0, \quad \text{for all } t \in \mathbb{R}.$$

Therefore, Eq. (1.1) has at most one T-periodic solution. The proof is completed. □

To prove the main result of this paper, we shall use a continuation theorem [13, 14] as follows.

Lemma 2.3 *Let Ω be open bounded in a linear normal space X. Suppose that $\widetilde{f}$ is a complete continuous field on $\overline{\Omega}$. Moreover, assume that the Leray-Schauder degree*

$$\deg\{\widetilde{f}, \Omega, p\} \ne 0, \quad \textit{for } p \in X \setminus \widetilde{f}(\partial\Omega).$$

Then the equation $\widetilde{f}(x) = p$ has at least one solution in Ω.

3 Main result

In this section, we present and prove our main result concerning the existence and uniqueness of anti-periodic solutions of Eq. (1.1).

Theorem 3.1 *Let (H_1) hold. Moreover, assume that the following conditions hold*:

(H_2) *there exists $l > 0$ such that*

$$\left|g(t,x_1) - g(t,x_2)\right| \le l|x_1 - x_2|, \quad \textit{for all } t, x_1, x_2 \in \mathbb{R};$$

(H_3) *there exists $\beta, \gamma > 0$ such that*

$$\gamma \le \liminf_{|x|\to\infty} \frac{f(t,x)}{x} \le \limsup_{|x|\to\infty} \frac{f(t,x)}{x} \le \beta, \quad \textit{uniformly in } t \in \mathbb{R};$$

(H_4) *for all $t, x \in \mathbb{R}$,*

$$f\left(t + \frac{T}{2}, -x\right) = -f(t,x), \qquad g\left(t + \frac{T}{2}, -x\right) = -g(t,x), \qquad e\left(t + \frac{T}{2}\right) = -e(t).$$

Then Eq. (1.1) *has a unique anti-periodic solution for $l \cdot \frac{T}{2\pi} < \gamma$.*

Proof Rewrite Eq. (1.1) in the equivalent form:

$$\begin{cases} x_1'(t) = \psi(x_2(t)) = \frac{x_2(t)}{\sqrt{1 - x_2^2(t)}}, \\ x_2'(t) = -f(t, \psi(x_2(t))) - g(t, x_1(t)) + e(t), \end{cases} \tag{3.1}$$

where $\psi(x) = \frac{x}{\sqrt{1-x^2}}$. Now, we consider the auxiliary equation of (3.1),

$$\begin{cases} x_1'(t) = \lambda \frac{x_2(t)}{\sqrt{1-x_2^2(t)}} = \lambda\psi(x_2(t)), \\ x_2'(t) = -\lambda f(t,\psi(x_2(t))) - \lambda g(t,x_1(t)) + \lambda e(t), \end{cases} \tag{3.2}$$

where $\lambda \in (0,1]$ is a parameter. Set

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad Q_1\bigl(t,x_1(t),x_2(t)\bigr) = \begin{pmatrix} \psi(x_2(t)) \\ -f(t,\psi(x_2(t))) - g(t,x_1(t)) + e(t) \end{pmatrix}.$$

Then Eq. (3.2) can be reduced to the equation as follows:

$$x'(t) = \lambda Q_1\bigl(t,x_1(t),x_2(t)\bigr).$$

By Lemma 2.2 and condition (H_1), it is easy to see that Eq. (1.1) has at most one anti-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Eq. (1.1) has at least one anti-periodic solution. To do this, we shall apply Lemma 2.3. Firstly, we will prove that the set of all possible anti-periodic solutions of Eq. (3.2) is bounded.

Let $x(t) = (x_1(t),x_2(t))^T \in C_T^{1,\frac{1}{2}}(\mathbb{R},\mathbb{R}^2)$ be an arbitrary possible anti-periodic solution of Eq. (3.2). Then $x_1(t) \in C_T^{1,\frac{1}{2}}(\mathbb{R},\mathbb{R})$. Thus, we have

$$\begin{aligned} \int_0^T x_1(t)\,dt &= \int_0^{\frac{T}{2}} x_1(t)\,dt + \int_{\frac{T}{2}}^T x_1(t)\,dt \\ &= \int_0^{\frac{T}{2}} x_1(t)\,dt + \int_0^{\frac{T}{2}} x_1\left(t + \frac{T}{2}\right)dt = 0. \end{aligned}$$

It follows from Lemma 2.1 that

$$|x_1|_\infty \le \sqrt{\frac{T}{12}}\bigl|x_1'\bigr|_2.$$

Obviously, Eq. (3.2) is equivalent to the following equation:

$$\left(\frac{\frac{1}{\lambda}x_1'(t)}{\sqrt{1+\frac{1}{\lambda^2}x_1'^2(t)}}\right)' + \lambda f\left(t,\frac{1}{\lambda}x_1'(t)\right) + \lambda g\bigl(t,x_1(t)\bigr) = \lambda e(t). \tag{3.3}$$

Multiplying (3.3) by x_1' and integrating from 0 to T, we have

$$\lambda\int_0^T f\left(t,\frac{1}{\lambda}x_1'(t)\right)x_1'(t)\,dt + \lambda\int_0^T g\bigl(t,x_1(t)\bigr)x_1'(t)\,dt = \lambda\int_0^T e(t)x_1'(t)\,dt. \tag{3.4}$$

Since $l \cdot \frac{T}{2\pi} < \gamma$, there exists a constant $\varepsilon > 0$ such that

$$l \cdot \frac{T}{2\pi} < \gamma - \varepsilon. \tag{3.5}$$

For such a $\varepsilon > 0$, in view of (H_3), there exists $M_1 \geq 0$ such that for all $t, x \in \mathbb{R}$, $xf(t,x) \geq (\gamma - \varepsilon)x^2 - M_1$. Hence,

$$\begin{aligned}\left|\lambda \int_0^T f\left(t, \frac{1}{\lambda}x_1'(t)\right)x_1'(t)\,dt\right| &\geq \lambda^2 \int_0^T f\left(t, \frac{1}{\lambda}x_1'(t)\right)\frac{x_1'(t)}{\lambda}\,dt \\ &\geq (\gamma - \varepsilon)\int_0^T \left|x_1'(t)\right|^2 dt - \lambda^2 M_1. \end{aligned} \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$\begin{aligned}(\gamma - \varepsilon)\int_0^T \left|x_1'(t)\right|^2 dt &\leq \left|\lambda \int_0^T f\left(t, \frac{1}{\lambda}x_1'(t)\right)x_1'(t)\,dt\right| + \lambda^2 M_1 \\ &\leq \left|\int_0^T g\big(t, x_1(t)\big)x_1'(t)\,dt\right| + \left|\int_0^T e(t)x_1'(t)\,dt\right| + M_1 \\ &\leq \int_0^T \left|g\big(t, x_1(t)\big) - g(t,0)\right|\left|x_1'(t)\right| dt \\ &\quad + \int_0^T \big(\left|g(t,0)\right| + \left|e(t)\right|\big)\left|x_1'(t)\right| dt + M_1 \\ &\leq l\int_0^T \left|x_1(t)\right|\left|x_1'(t)\right| dt \\ &\quad + \max_{t\in[0,T]}\left\{\left|g(t,0)\right| + \left|e(t)\right|\right\}\int_0^T \left|x_1'(t)\right| dt + M_1.\end{aligned}$$

For $u, v \in C([a,b], \mathbb{R})$, we have the Schwarz inequality

$$\int_a^b \left|u(x)\right|\left|v(x)\right| dx \leq \left(\int_a^b \left|u(x)\right|^2 dx\right)^{\frac{1}{2}}\left(\int_a^b \left|v(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned}(\gamma - \varepsilon)\int_0^T \left|x_1'(t)\right|^2 dt &\leq l\left(\int_0^T \left|x_1(t)\right|^2 dt\right)^{\frac{1}{2}} \cdot \left(\int_0^T \left|x_1'(t)\right|^2 dt\right)^{\frac{1}{2}} \\ &\quad + \max_{t\in[0,T]}\left\{\left|g(t,0)\right| + \left|e(t)\right|\right\}\sqrt{T}\left(\int_0^T \left|x_1'(t)\right|^2 dt\right)^{\frac{1}{2}} + M_1 \\ &= l|x_1|_2\left|x_1'\right|_2 + \sqrt{T}\left|x_1'\right|_2 \max_{t\in[0,T]}\left\{\left|g(t,0)\right| + \left|e(t)\right|\right\} + M_1 \\ &\leq l \cdot \frac{T}{2\pi}\left|x_1'\right|_2^2 + \sqrt{T}\left|x_1'\right|_2 \max_{t\in[0,T]}\left\{\left|g(t,0)\right| + \left|e(t)\right|\right\} + M_1.\end{aligned} \tag{3.7}$$

From (3.5) and (3.7), we know that there exists a constant $D_1 > 0$ such that

$$\left|x_1'\right|_2 \leq D_1, \quad \text{and} \quad |x_1|_\infty \leq D_1. \tag{3.8}$$

By the first equation of (3.2), we have

$$\int_0^T \frac{x_2(t)}{\sqrt{1 - x_2^2(t)}}\,dt = 0.$$

Then there exists $\eta \in [0,T]$ such that $x_2(\eta) = 0$. It follows that $x_2(t) = x_2(\eta) + \int_\eta^t x_2'(s)\,ds$, and so

$$|x_2|_\infty \le \int_0^T \left|x_2'(t)\right| dt.$$

According to (H_3), we know there exists $M_2 \ge 0$ such that for all $t, x \in \mathbb{R}$,

$$\left|f(t,x)\right| \le (\beta+1)|x| + M_2.$$

From the second equation of (3.2), we get

$$\begin{aligned}
\int_0^T \left|x_2'(t)\right| dt &\le \int_0^T \lambda \left|f\left(t, \frac{1}{\lambda}x_1'(t)\right)\right| dt + \int_0^T \left|g\big(t, x_1(t)\big)\right| dt + \int_0^T \left|e(t)\right| dt \\
&\le (\beta+1)\int_0^T \left|x_1'(t)\right| dt + \lambda M_2 T + \int_0^T \left|g\big(t, x_1(t)\big)\right| dt + \int_0^T \left|e(t)\right| dt \\
&\le (\beta+1)\sqrt{T}\left|x_1'\right|_2 + \int_0^T \left|g\big(t, x_1(t)\big)\right| dt + \int_0^T \left|e(t)\right| dt + M_2 T.
\end{aligned}$$

From (3.8), we know that there exists a constant $k > 0$ such that

$$\left|g(t, x_1(t)\right| \le k, \quad \forall t \in [0,T].$$

Thus,

$$\int_0^T \left|x_2'(t)\right| dt \le (\beta+1)\sqrt{T}D_1 + kT + T\max_{t\in[0,T]}\left|e(t)\right| + M_2 T,$$

which implies that there exists a constant $D_2 > 0$ such that

$$|x_2|_\infty \le D_2.$$

Let

$$M = \max\{D_1, D_2\} + 1. \tag{3.9}$$

Set

$$\Omega = \left\{x \in C_T^{0,\frac{1}{2}}\left(\mathbb{R}, \mathbb{R}^2\right) = X : \|x\| < M\right\}.$$

Then Eq. (3.2) has no anti-periodic solution on $\partial\Omega$ for $\lambda \in (0,1]$.

Next, we consider the Fourier series expansions of two functions $x_j(t) \in C_T^{k,\frac{1}{2}}(\mathbb{R},\mathbb{R})$ $(j = 1,2)$. We have

$$x_j(t) = \sum_{i=0}^{\infty}\left[a_{2i+1}^j \cos\frac{2\pi(2i+1)t}{T} + b_{2i+1}^j \sin\frac{2\pi(2i+1)t}{T}\right].$$

Define an operator $L_1 : C_T^{k,\frac{1}{2}}(\mathbb{R},\mathbb{R}) \to C_T^{k+1,\frac{1}{2}}(\mathbb{R},\mathbb{R})$ by setting

$$\begin{aligned}(L_1x_j)(t) &= \int_0^t x_j(s)\,ds - \frac{T}{2\pi}\sum_{i=0}^{\infty}\frac{b^j_{2i+1}}{2i+1} \\ &= \frac{T}{2\pi}\sum_{i=0}^{\infty}\left[\frac{a^j_{2i+1}}{2i+1}\sin\frac{2\pi(2i+1)t}{T} - \frac{b^j_{2i+1}}{2i+1}\cos\frac{2\pi(2i+1)t}{T}\right].\end{aligned}$$

Then

$$\frac{d}{dt}(L_1x_j)(t) = x_j(t),$$

and

$$\begin{aligned}\left|(L_1x_j)(t)\right| &\le \int_0^T \left|x_j(s)\right|ds + \frac{T}{2\pi}\sum_{i=0}^{\infty}\frac{|b^j_{2i+1}|}{2i+1} \\ &\le T|x_j|_\infty + \frac{T}{2\pi}\left(\sum_{i=0}^{\infty}\left(b^j_{2i+1}\right)^2\right)^{\frac{1}{2}}\left(\sum_{i=0}^{\infty}\frac{1}{(2i+1)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Since

$$\left(\sum_{i=0}^{\infty}\frac{1}{(2i+1)^2}\right)^{\frac{1}{2}} = \frac{\pi}{2\sqrt{2}},$$

and

$$\int_0^T \left|x_j(s)\right|^2 ds = \frac{T}{2}\sum_{i=0}^{\infty}\left[\left(a^j_{2i+1}\right)^2 + \left(b^j_{2i+1}\right)^2\right],$$

we obtain

$$\begin{aligned}\left|(L_1x_j)(t)\right| &\le T|x_j|_\infty + \frac{T}{4\sqrt{2}}\left(\sum_{i=0}^{\infty}\left[\left(a^j_{2i+1}\right)^2 + \left(b^j_{2i+1}\right)^2\right]\right)^{\frac{1}{2}} \\ &\le T|x_j|_\infty + \frac{T}{4\sqrt{2}}\left(\frac{2}{T}\int_0^T \left|x_j(s)\right|^2 ds\right)^{\frac{1}{2}} \\ &\le \left(T + \frac{T}{4}\right)|x_j|_\infty.\end{aligned}$$

Define $L : C_T^{k,\frac{1}{2}}(\mathbb{R},\mathbb{R}^2) \to C_T^{k+1,\frac{1}{2}}(\mathbb{R},\mathbb{R}^2)$ by setting

$$(Lx)(t) = L\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix} = \begin{pmatrix}(L_1x_1)(t)\\(L_1x_2)(t)\end{pmatrix}.$$

Then $|Lx|_\infty \le (T + \frac{T}{4})|x|_\infty$, and thus L is continuous.

For any $x(t) \in C_T^{0,\frac{1}{2}}(\mathbb{R},\mathbb{R}^2)$, we know from (H_4) that

$$Q_1\left(t+\frac{T}{2},x_1\left(t+\frac{T}{2}\right),x_2\left(t+\frac{T}{2}\right)\right) = -Q_1\big(t,x_1(t),x_2(t)\big).$$

Therefore, $Q_1(t,x_1(t),x_2(t)) \in C_T^{0,\frac{1}{2}}(\mathbb{R},\mathbb{R}^2)$. Define an operator $F_\mu : \overline{\Omega} \to C_T^{1,\frac{1}{2}}(\mathbb{R},\mathbb{R}^2) \subset X$ by setting

$$F_\mu(x) = \mu L\big(Q_1\big(t,x_1(t),x_2(t)\big)\big), \quad \mu \in [0,1].$$

It is easy to see that F_μ is a compact homotopy, and the fixed point of F_1 on $\overline{\Omega}$ is the anti-periodic of Eq. (3.1).

Define a homotopic field as follows:

$$H_\mu(x) : \overline{\Omega} \times [0,1] \to C_T^{0,\frac{1}{2}}\big(\mathbb{R},\mathbb{R}^2\big), \quad H_\mu(x) = x - F_\mu(x).$$

From (3.9), we have

$$H_\mu(\partial\Omega) \neq 0, \quad \mu \in [0,1].$$

Using the homotopy invariance property of degree, we obtain

$$\deg\big\{x - F_1(x), \Omega, 0\big\} = \deg\{x, \Omega, 0\} \neq 0.$$

Till now, we have proved that Ω satisfies all the requirements in Lemma 2.3. Consequently, $x - F_1(x) = 0$ has at least one solution in Ω, *i.e.*, F_1 has a fixed point $x(t) = (x_1(t),x_2(t))^T$ on $\overline{\Omega}$. Therefore, Eq. (1.1) has at least one anti-periodic solution $x_1(t)$. This completes the proof. □

4 An example

In this section, we shall construct an example to show the applications of Theorem 3.1.

Example 4.1 Let $f(t,x) = (1 + \frac{1}{2}\sin^2 t)\frac{x^3}{\sqrt{1+x^4}}$, $g(t,x) = -(1 + \sin^4 t) \cdot \frac{x}{3}$. Then the prescribed mean curvature Rayleigh equation

$$\left(\frac{x'}{\sqrt{1+x'^2}}\right)' + f\big(t,x'(t)\big) + g\big(t,x(t)\big) = \cos t \tag{4.1}$$

has a unique anti-periodic solution with period 2π.

Proof Let $T = 2\pi$. From the definitions of $f(t,x)$ and $g(t,x)$, we can easily check that conditions (H_1) and (H_4) hold. Moreover, it is easy to see that (H_2) holds for $l = \frac{2}{3}$ and (H_3) holds for $\gamma = 1$, $\beta = \frac{3}{2}$. Since $l \cdot \frac{T}{2\pi} < \gamma$, we know from Theorem 3.1 that Eq. (4.1) has a unique anti-periodic solution with period 2π. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors, AA and MHA, contributed to each part of this work equally and read and approved the final version of the manuscript.

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References

1. Lu, S, Gui, Z: On the existence of periodic solutions to p-Laplacian Rayleigh differential equation with a delay. J. Math. Anal. Appl. **325**, 685-702 (2007)
2. Zong, M, Liang, H: Periodic solutions for Rayleigh type p-Laplacian equation with deviating arguments. Appl. Math. Lett. **20**, 43-47 (2007)
3. Gao, H, Liu, B: Existence and uniqueness of periodic solutions for forced Rayleigh-type equations. Appl. Math. Comput. **211**, 148-154 (2009)
4. Ma, T, Wang, Z: A continuation lemma and its applications to periodic solutions of Rayleigh differential equations with subquadratic potential conditions. J. Math. Anal. Appl. **385**, 1107-1118 (2012)
5. Yu, Y, Shao, J, Yue, G: Existence and uniqueness of anti-periodic solutions for a kind of Rayleigh equation with two deviating arguments. Nonlinear Anal. **71**, 4689-4695 (2009)
6. Lv, X, Yan, P, Liu, D: Anti-periodic solutions for a class of nonlinear second-order Rayleigh equations with delays. Commun. Nonlinear Sci. Numer. Simul. **15**, 3593-3598 (2010)
7. Liu, B: Anti-periodic solutions for forced Rayleigh-type equations. Nonlinear Anal., Real World Appl. **10**, 2850-2856 (2009)
8. Feng, M: Periodic solutions for prescribed mean curvature Liénard equation with a deviating argument. Nonlinear Anal., Real World Appl. **13**, 1216-1223 (2012)
9. Bonheure, D, Habets, P, Obersnel, F, Omari, P: Classical and non-classical solutions of a prescribed curvature equation. J. Differ. Equ. **243**, 208-237 (2007)
10. Pan, H: One-dimensional prescribed mean curvature equation with exponential nonlinearity. Nonlinear Anal. **70**, 999-1010 (2009)
11. Benevieria, P, do Ó, J, Medeiros, E: Periodic solutions for nonlinear systems with mean curvature-like operators. Nonlinear Anal. **65**, 1462-1475 (2006)
12. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Application of Mathematical Science, vol. 74. Springer, New York (1989)
13. Gaines, R, Mawhin, J: Coincidence Degree and Nonlinear Differential Equations. Lecture Notes in Mathematics, vol. 568. Springer, Berlin (1977)
14. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)

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