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Existence of mild solutions for fractional evolution equations with nonlocal conditions

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Abstract

This paper deals with the existence and uniqueness of mild solutions for a class of fractional evolution equations with nonlocal initial conditions. We present some local growth conditions on a nonlinear part and a nonlocal term to guarantee the existence theorems. An example is given to illustrate the applicability of our results.

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1 Introduction

The differential equations involving fractional derivatives in time have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering and science. Indeed, we can find numerous applications in electrochemistry, control, porous media, electromagnetic processing *etc.* (see [1–5]). Hence, the research on fractional differential equations has become an object of extensive study during recent years; see [6–11] and references therein.

On the other hand, the nonlocal initial condition, in many cases, has much better effect in applications than the traditional initial condition. As remarked by Byszewski and Lakshmikantham (see [12, 13]), the nonlocal initial value problems can be more useful than the standard initial value problems to describe many physical phenomena.

Let X be a Banach space with norm $\|\cdot\|$, and let $T > 0$ be a constant. Consider the existence and uniqueness of mild solutions of fractional evolution equation with nonlocal condition in the form

$$\begin{cases} D^q u(t) + Au(t) = f(t, u(t)) + \int_0^t K(t, s)h(s, u(s)) ds, & t \in J = [0, T], \\ u(0) + g(u) = x_0, \end{cases} \quad (1)$$

where D^q is the Caputo fractional derivative of order $q \in (0, 1)$, the linear operator $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ in X , the functions f , h and g will be specified later. $K \in C(\Delta, \mathbb{R}^+)$, where $\Delta = \{(t, s) | 0 \leq s \leq t \leq T\}$, $\mathbb{R}^+ = [0, +\infty)$. Throughout this paper, we always assume that $K^* = \max_{(t,s) \in \Delta} K(t, s)$.

In some existing articles, the fractional differential equations with nonlocal initial conditions were treated under the hypothesis that the nonlocal term is completely continuous or global Lipschitz continuous. It is obvious that these conditions are not easy to verify in

many cases. To make the things more applicable, in [6] the authors studied the existence and uniqueness of mild solutions of Eq. (1) under the case $K(t, s) \equiv 0$. In their main results, they did not assume the complete continuity of the nonlocal term, but they needed the following assumptions:

(F1) there exist a constant $q_1 \in [0, q)$ and $m \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that $\|f(t, x)\| \leq m(t)$ for all $x \in X$ and almost all $t \in J$;

(F2) there exists a constant $L > 0$ such that $\|g(u) - g(v)\| \leq L\|u - v\|$ for $u, v \in C(J, X)$; and some other conditions.

In this paper, we will improve the conditions (F1) and (F2). We only assume that f and h satisfy local growth conditions (see assumption (H₃)) and g is local Lipschitz continuous (see assumption (H₄)). We will carry out our investigation in the Banach space $X_\alpha := (D(A^\alpha), \|\cdot\|_\alpha)$, $0 < \alpha < 1$, where $D(A^\alpha)$ is the domain of the fractional power of A . Finally, an example is given to illustrate the applicability of our main results. We can see that the main results in [6] cannot be applied to our example.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given on the fractional power of the generator of an analytic semigroup and on the mild solutions of Eq. (1). In Section 3, we study the existence and uniqueness of mild solutions of Eq. (1). In Section 4, we give an example to illustrate the applicability of our results.

2 Preliminaries

Let X be a Banach space with norm $\|\cdot\|$, and let $-A : D(A) \subset X \rightarrow X$ be the infinitesimal generator of an analytic semigroup $S(t)$ ($t \geq 0$) of a uniformly bounded linear operator in X , that is, there exists $M \geq 1$ such that $\|S(t)\| \leq M$ for all $t \geq 0$. Without loss of generality, let $0 \in \rho(A)$. Then for any $\alpha > 0$, we can define $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) dt.$$

$A^{-\alpha} \in \mathcal{L}(X)$ is injective, and A^α can be defined by $A^\alpha = (A^{-\alpha})^{-1}$ with the domain $D(A^\alpha) = A^{-\alpha}(X)$. For $\alpha = 0$, let $A^\alpha = I$.

Lemma 1 ([14]) *A^α has the following properties:*

- (i) $D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A^\alpha x\|$ for $x \in D(A^\alpha)$;
- (ii) $S(t) : X \rightarrow X_\alpha$ for each $t > 0$;
- (iii) $A^\alpha S(t)x = S(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \geq 0$;
- (iv) $A^{-\alpha}$ is a bounded linear operator on X with $D(A^\alpha) = \text{Im}(A^{-\alpha})$;
- (v) If $0 < \alpha \leq \beta$, then $D(A^\beta) \hookrightarrow D(A^\alpha)$.

Let X_α be the Banach space of $D(A^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$. Denote by $C(J, X_\alpha)$ the Banach space of all continuous functions from J into X_α with the supnorm given by $\|u\|_C = \sup_{t \in J} \|u(t)\|_\alpha$ for $u \in C(J, X_\alpha)$. From Lemma 1(iv), since $A^{-\alpha}$ is a bounded linear operator for $\alpha > 0$, we denote by C_α the operator norm of $A^{-\alpha}$ in X , that is, $C_\alpha := \|A^{-\alpha}\|$. For any $t \geq 0$, denote by $S_\alpha(t)$ the restriction of $S(t)$ to X_α . From Lemma 1(ii) and (iii), for any $x \in X_\alpha$, we have

$$\|S(t)x\|_\alpha = \|A^\alpha \cdot S(t)x\| = \|S(t) \cdot A^\alpha x\| \leq \|S(t)\| \cdot \|A^\alpha x\| = \|S(t)\| \cdot \|x\|_\alpha,$$

and

$$\|S(t)x - x\|_\alpha = \|A^\alpha \cdot S(t)x - A^\alpha x\| = \|S(t) \cdot A^\alpha x - A^\alpha x\| \rightarrow 0$$

as $t \rightarrow 0$. Therefore, $S(t)$ ($t \geq 0$) is a strongly continuous semigroup in X_α , and $\|S_\alpha(t)\|_\alpha \leq \|S(t)\|$ for all $t \geq 0$. To prove our main results, the following lemma is needed.

Lemma 2 ([15]) *If $S(t)$ ($t \geq 0$) is a compact semigroup in X , then $S_\alpha(t)$ ($t \geq 0$) is an immediately compact semigroup in X_α , and hence it is immediately norm-continuous.*

For $x \in X$, define two families $\{U(t)\}_{t \geq 0}$ and $\{V(t)\}_{t \geq 0}$ of operators by

$$U(t)x = \int_0^\infty \eta_q(\theta)S(t^q\theta)x \, d\theta, \quad V(t)x = q \int_0^\infty \theta \eta_q(\theta)S(t^q\theta)x \, d\theta, \quad 0 < q < 1,$$

where

$$\eta_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \rho_q(\theta^{-\frac{1}{q}}),$$

$$\rho_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty),$$

where η_q is the probability density function defined on $(0, \infty)$, which has properties $\eta_q(\theta) \geq 0$ for all $\theta \in (0, \infty)$ and

$$\int_0^\infty \eta_q(\theta) \, d\theta = 1, \quad \int_0^\infty \theta \eta_q(\theta) \, d\theta = \frac{1}{\Gamma(1+q)}. \tag{2}$$

The following lemma follows from the results in [6–8, 10].

Lemma 3 *The following properties are valid:*

(i) *For fixed $t \geq 0$ and any $x \in X_\alpha$, we have*

$$\|U(t)x\|_\alpha \leq M \|x\|_\alpha, \quad \|V(t)x\|_\alpha \leq \frac{qM}{\Gamma(1+q)} \|x\|_\alpha = \frac{M}{\Gamma(q)} \|x\|_\alpha.$$

- (ii) *The operators $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.*
- (iii) *If $S(t)$ ($t \geq 0$) is a compact semigroup in X , then $U(t)$ and $V(t)$ are norm-continuous in X for $t > 0$.*
- (iv) *If $S(t)$ ($t \geq 0$) is a compact semigroup in X , then $U(t)$ and $V(t)$ are compact operators in X for $t > 0$.*

In this paper, we adopt the following definition of a mild solution of Eq. (1).

Definition 1 By a mild solution of Eq. (1), we mean a function $u \in C(J, X_\alpha)$ satisfying

$$u(t) = U(t)(x_0 - g(u)) + \int_0^t (t-s)^{q-1} V(t-s) \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) \, d\tau \right] ds$$

for all $t \in J$.

To prove our main results, we also need the following two lemmas.

Lemma 4 *A measurable function $H : [0, T] \rightarrow X$ is Bochner integrable if $|H|$ is Lebesgue integrable.*

Lemma 5 (Krasnoselskii's fixed point theorem) *Let X be a Banach space, let B be a bounded closed and convex subset of X and let Q_1 and Q_2 be mappings from B into X such that $Q_1x + Q_2y \in B$ for every pair $x, y \in B$. If Q_1 is a contraction and Q_2 is completely continuous, then the operator equation $Q_1x + Q_2x = x$ has a solution on B .*

Lemmas 4 and 5, which can be found in many books, are classical.

The following are the basic assumptions of this paper.

- (H₁) $S(t)$ ($t \geq 0$) is a compact operator semigroup in X .
- (H₂) There exists a constant $\beta \in [\alpha, 1]$ such that the functions $f, h : J \times X_\alpha \rightarrow X_\beta$ satisfy the following conditions:
 - (i) For each $x \in X_\alpha$, the functions $f(\cdot, x), h(\cdot, x)$ are measurable.
 - (ii) For each $t \in J$, the functions $f(t, \cdot), h(t, \cdot)$ are continuous.
- (H₃) For $t \in J$ and $r > 0$, there exist positive functions φ_r satisfying $\frac{\varphi_r(\cdot)}{(t-\cdot)^{1-q}} \in L^1([0, t], \mathbb{R}^+)$ and $\phi_r \in L^1(J, \mathbb{R}^+)$ such that

$$\sup_{\|x\|_\alpha \leq r} \|f(t, x)\|_\beta \leq \varphi_r(t), \quad \sup_{\|x\|_\alpha \leq r} \|h(t, x)\|_\beta \leq \phi_r(t),$$

and there are positive constants σ_1 and σ_2 such that

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \int_0^t \frac{\varphi_r(s)}{(t-s)^{1-q}} ds \leq \sigma_1 < +\infty, \quad \liminf_{r \rightarrow +\infty} \frac{1}{r} \int_0^t \frac{\int_0^s \phi_r(\tau) d\tau}{(t-s)^{1-q}} ds \leq \sigma_2 < +\infty,$$

- (H₄) $g : C(J, X_\alpha) \rightarrow X_\alpha$ and for $r > 0$, there exists a positive constant L such that

$$\|g(u) - g(v)\|_\alpha \leq L \|u - v\|_C$$

for all $u, v \in B_r := \{u \in C(J, X_\alpha) : \|u(t)\|_\alpha \leq r, t \in J\}$.

3 Main results

In this section, we introduce the existence and uniqueness theorems of mild solutions of Eq. (1). The discussions are based on fractional calculus and Krasnoselskii's fixed point theorem. Our main results are as follows.

Theorem 1 *Suppose that the assumptions (H₁)-(H₄) hold. If $x_0 \in X_\alpha$ and the following inequality holds:*

$$ML + \frac{MC_{\beta-\alpha}}{\Gamma(q)} (\sigma_1 + K^* \sigma_2) < 1, \tag{3}$$

then Eq. (1) has at least one mild solution on J .

Proof Define two operators Q_1 and Q_2 as follows:

$$(Q_1u)(t) = U(t)(x_0 - g(u)), \quad t \in J,$$

$$(Q_2u)(t) = \int_0^t (t-s)^{q-1} V(t-s) \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau \right] ds, \quad t \in J.$$

Obviously, u is a mild solution of Eq. (1) if and only if u is a solution of the operator equation $u = Q_1u + Q_2u$ on J . To prove the operator equation $u = Q_1u + Q_2u$ has solutions, we first show that there is a positive number r such that $Q_1u + Q_2v \subset B_r$ for every pair $u, v \in B_r$. If this were not the case, then for each $r > 0$, there would exist $u_r, v_r \in B_r$ and $t_r \in J$ such that $\|(Q_1u_r)(t_r) + (Q_2v_r)(t_r)\|_\alpha > r$. Thus, from Lemma 3, (H₃) and (H₄), we have

$$\begin{aligned} r &< \|(Q_1u_r)(t_r) + (Q_2v_r)(t_r)\|_\alpha \leq \|U(t_r)(x_0 - g(u_r))\|_\alpha \\ &\quad + \int_0^{t_r} (t_r - s)^{q-1} \left\| V(t_r - s) \left[f(s, v_r(s)) + \int_0^s K(s, \tau) h(\tau, v_r(\tau)) d\tau \right] \right\|_\alpha ds \\ &\leq M \|x_0 - [g(u_r) - g(0)] - g(0)\|_\alpha \\ &\quad + \frac{M}{\Gamma(q)} \int_0^{t_r} (t_r - s)^{q-1} \|A^{\alpha-\beta} \cdot A^\beta [f(s, v_r(s)) + \int_0^s K(s, \tau) h(\tau, v_r(\tau)) d\tau]\|_\alpha ds \\ &\leq M \|x_0\|_\alpha + MLr + M \|g(0)\|_\alpha \\ &\quad + \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^{t_r} (t_r - s)^{q-1} \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds. \end{aligned}$$

Dividing on both sides by r and taking the lower limit as $r \rightarrow +\infty$, we have

$$ML + \frac{MC_{\beta-\alpha}}{\Gamma(q)} (\sigma_1 + K^* \sigma_2) \geq 1,$$

which contradicts (3). Hence, for some $r > 0$, $Q_1u + Q_2v \subset B_r$ for every pair $u, v \in B_r$.

The next proof will be given in two steps.

Step 1. Q_1 is a contraction on B_r .

For any $u, v \in B_r$ and $t \in J$, according to Lemma 3 and assumption (H₄), we have

$$\|(Q_1u)(t) - (Q_1v)(t)\|_\alpha = \|U(t)[g(v) - g(u)]\|_\alpha \leq M \|g(u) - g(v)\|_\alpha \leq ML \|u - v\|_C,$$

which implies that $\|Q_1u - Q_1v\|_C \leq ML \|u - v\|_C$. It follows from (3) that $ML < 1$, hence Q_1 is a contraction on B_r .

Step 2. Q_2 is a completely continuous operator on B_r .

We first prove that Q_2 is continuous on B_r . Let $\{u_n\} \subset B_r$ with $u_n \rightarrow u \in B_r$ as $n \rightarrow \infty$. Then for any $t \in J, s \in [0, t]$, by assumption (H₂), we have

$$f(s, u_n(s)) \rightarrow f(s, u(s)), \quad h(s, u_n(s)) \rightarrow h(s, u(s))$$

as $n \rightarrow \infty$, and from assumption (H₃), we have

$$\begin{aligned} \|f(s, u_n(s)) - f(s, u(s))\|_\beta &\leq 2\varphi_r(s), & \|h(s, u_n(s)) - h(s, u(s))\|_\beta &\leq 2\phi_r(s), \\ \int_0^s K(s, \tau) \|h(\tau, u_n(\tau)) - h(\tau, u(\tau))\|_\beta d\tau &\leq 2K^* \|\phi_r\|_{L^1}. \end{aligned}$$

This together with the Lebesgue dominated convergence theorem gives that

$$\begin{aligned} & \| (Q_2 u_n)(t) - (Q_2 u)(t) \|_\alpha \\ & \leq \int_0^t (t-s)^{q-1} \| V(t-s) \cdot [f(s, u_n(s)) - f(s, u(s))] \|_\alpha ds \\ & \quad + \int_0^t (t-s)^{q-1} \left\| V(t-s) \cdot \int_0^s K(s, \tau) [h(\tau, u_n(\tau)) - h(\tau, u(\tau))] d\tau \right\|_\alpha ds \\ & \leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| A^{\alpha-\beta} \cdot A^\beta [f(s, u_n(s)) - f(s, u(s))] \| ds \\ & \quad + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\| A^{\alpha-\beta} \int_0^s K(s, \tau) \cdot A^\beta [h(\tau, u_n(\tau)) - h(\tau, u(\tau))] d\tau \right\| ds \\ & \leq \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, u_n(s)) - f(s, u(s)) \|_\beta ds \\ & \quad + \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s K(s, \tau) \| h(\tau, u_n(\tau)) - h(\tau, u(\tau)) \|_\beta d\tau ds \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence, $\lim_{n \rightarrow \infty} \| Q_2 u_n - Q_2 u \|_C = 0$. This means that Q_2 is continuous on B_r .

Next, we will show that the set $\{Q_2 u : u \in B_r\}$ is relatively compact. It suffices to show that the family of functions $\{Q_2 u : u \in B_r\}$ is uniformly bounded and equicontinuous, and for any $t \in J$, the set $\{(Q_2 u)(t) : u \in B_r\}$ is relatively compact.

For any $u \in B_r$, we have $\|Q_2 u\|_C \leq r$ for some $r > 0$, which means that $\{Q_2 u : u \in B_r\}$ is uniformly bounded. In what follows, we show that $\{Q_2 u : u \in B_r\}$ is a family of equicontinuous functions. For $t \in [0, T)$, we have

$$\| (Q_2 u)(t) - (Q_2 u)(0) \|_\alpha \leq \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds.$$

Hence, it is only necessary to consider $t > 0$. For $0 < t_1 < t_2 \leq T$, from Lemma 3 and assumption (H_3) , we have

$$\begin{aligned} & \| (Q_2 u)(t_2) - (Q_2 u)(t_1) \|_\alpha \\ & \leq \left\| \int_0^{t_1} (t_2-s)^{q-1} [V(t_2-s) - V(t_1-s)] \cdot \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau \right] ds \right\|_\alpha \\ & \quad + \left\| \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] V(t_1-s) \right. \\ & \quad \times \left. \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau \right] ds \right\|_\alpha \\ & \quad + \left\| \int_{t_1}^{t_2} (t_2-s)^{q-1} V(t_2-s) \cdot \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau \right] ds \right\|_\alpha \\ & \leq \int_0^{t_1} (t_2-s)^{q-1} \left\| A^{\alpha-\beta} [V(t_2-s) - V(t_1-s)] \right. \\ & \quad \times \left. A^\beta \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau \right] \right\| ds \\ & \quad + \frac{M}{\Gamma(q)} \int_0^{t_1} |(t_2-s)^{q-1} - (t_1-s)^{q-1}| \end{aligned}$$

$$\begin{aligned}
 & \times \left\| A^{\alpha-\beta} \cdot A^\beta \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau \right] \right\| ds \\
 & + \frac{M}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left\| A^{\alpha-\beta} \cdot A^\beta \left[f(s, u(s)) + \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau \right] \right\| ds \\
 & \leq C_{\beta-\alpha} \int_0^{t_1} (t_2 - s)^{q-1} \|V(t_2 - s) - V(t_1 - s)\| \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds \\
 & + \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds \\
 & + \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds \\
 & \triangleq I_1 + I_2 + I_3,
 \end{aligned}$$

where

$$\begin{aligned}
 I_1 &= C_{\beta-\alpha} \int_0^{t_1} (t_2 - s)^{q-1} \|V(t_2 - s) - V(t_1 - s)\| \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds, \\
 I_2 &= \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds, \\
 I_3 &= \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds.
 \end{aligned}$$

For any $\epsilon \in (0, t_1)$, we have

$$\begin{aligned}
 I_1 &\leq C_{\beta-\alpha} \int_0^{t_1-\epsilon} (t_2 - s)^{q-1} \|V(t_2 - s) - V(t_1 - s)\| \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds \\
 &+ C_{\beta-\alpha} \int_{t_1-\epsilon}^{t_1} (t_2 - s)^{q-1} \|V(t_2 - s) - V(t_1 - s)\| \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds \\
 &\leq C_{\beta-\alpha} \int_0^{t_1-\epsilon} (t_1 - s)^{q-1} \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds \cdot \sup_{s \in [0, t_1-\epsilon]} \|V(t_2 - s) - V(t_1 - s)\| \\
 &+ \frac{2MC_{\beta-\alpha}}{\Gamma(q)} \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{q-1} \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds.
 \end{aligned}$$

It follows from Lemma 3 that $I_1 \rightarrow 0$ as $t_2 \rightarrow t_1$ and $\epsilon \rightarrow 0$ independently of $u \in B_r$. From the expressions of I_2 and I_3 , it is clear that $I_2 \rightarrow 0$ and $I_3 \rightarrow 0$ as $t_2 \rightarrow t_1$ independently of $u \in B_r$. Therefore, we prove that $\{Q_2 u : u \in B_r\}$ is a family of equicontinuous functions.

It remains to prove that for any $t \in J$, the set $W(t) := \{(Q_2 u)(t) : u \in B_r\}$ is relatively compact.

Obviously, $W(0)$ is relatively compact in X_α . Let $0 < t \leq T$ be fixed. For each $\delta \in (0, t)$, $\epsilon > 0$ and $u \in B_r$, we define an operator $Q_2^{\delta, \epsilon}$ by

$$\begin{aligned}
 (Q_2^{\delta, \epsilon} u)(t) &= \int_0^{t-\delta} (t - s)^{q-1} \int_\epsilon^\infty q\theta \eta_q(\theta) S((t - s)^q \theta) f(s, u(s)) d\theta ds \\
 &+ \int_0^{t-\delta} (t - s)^{q-1} \int_\epsilon^\infty q\theta \eta_q(\theta) S((t - s)^q \theta) \int_0^s K(s, \tau) h(\tau, u(\tau)) d\tau d\theta ds \\
 &= S(\delta^q \epsilon) \left[\int_0^{t-\delta} (t - s)^{q-1} \int_\epsilon^\infty q\theta \eta_q(\theta) S((t - s)^q \theta - \delta^q \epsilon) f(s, u(s)) d\theta ds \right.
 \end{aligned}$$

$$\begin{aligned} & + \int_0^{t-\delta} (t-s)^{q-1} \int_{\epsilon}^{\infty} q\theta\eta_q(\theta)S((t-s)^q\theta - \delta^q\epsilon) \\ & \times \int_0^s K(s,\tau)h(\tau,u(\tau))d\tau d\theta ds \Big]. \end{aligned}$$

Then the sets $W_{\delta,\epsilon}(t) := \{(Q_2^{\delta,\epsilon}u)(t) : u \in B_r\}$ are relatively compact in X_α since by Lemma 2, the operator $S_\alpha(t)$ is compact for $t > 0$ in X_α . Moreover, for every $u \in B_r$, from Lemma 3 and assumption (H₃), we have

$$\begin{aligned} & \|(Q_2u)(t) - (Q_2^{\delta,\epsilon}u)(t)\|_\alpha \\ & \leq \left\| \int_0^t (t-s)^{q-1} \int_0^\epsilon q\theta\eta_q(\theta)S((t-s)^q\theta) \right. \\ & \quad \times \left[f(s,u(s)) + \int_0^s K(s,\tau)h(\tau,u(\tau))d\tau \right] d\theta ds \Big\|_\alpha \\ & \quad + \left\| \int_{t-\delta}^t (t-s)^{q-1} \int_\epsilon^\infty q\theta\eta_q(\theta)S((t-s)^q\theta) \right. \\ & \quad \times \left[f(s,u(s)) + \int_0^s K(s,\tau)h(\tau,u(\tau))d\tau \right] d\theta ds \Big\|_\alpha \\ & \leq \int_0^t (t-s)^{q-1} \left\| A^{\alpha-\beta} \int_0^\epsilon q\theta\eta_q(\theta)S((t-s)^q\theta) d\theta \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] \right\| ds \\ & \quad + \int_{t-\delta}^t (t-s)^{q-1} \left\| A^{\alpha-\beta} \int_\epsilon^\infty q\theta\eta_q(\theta)S((t-s)^q\theta) d\theta \cdot \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] \right\| ds \\ & \leq qMC_{\beta-\alpha} \int_0^t (t-s)^{q-1} \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds \cdot \int_0^\epsilon \theta\eta_q(\theta) d\theta \\ & \quad + \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_{t-\delta}^t (t-s)^{q-1} \left[\varphi_r(s) + K^* \int_0^s \phi_r(\tau) d\tau \right] ds. \end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $W(t)$ for $t \in (0, T]$ and since it is compact at $t = 0$, we have the relative compactness of $W(t)$ in X_α for all $t \in J$.

Therefore, the set $\{Q_2u : u \in B_r\}$ is relatively compact by the Ascoli-Arzelà theorem. Thus, the continuity of Q_2 and relative compactness of the set $\{Q_2u : u \in B_r\}$ imply that Q_2 is a completely continuous operator. Hence, Krasnoselskii's fixed point theorem shows that the operator equation $u = Q_1u + Q_2u$ has a solution on B_r . Therefore, Eq. (1) has at least one mild solution. The proof is completed. \square

The following existence and uniqueness theorem for Eq. (1) is based on the Banach contraction principle. We will also need the following assumptions.

- (H₅) There exists a constant $\beta \in [\alpha, 1]$ such that the functions $f, h : J \times X_\alpha \rightarrow X_\beta$ are strongly measurable.
- (H₆) For $r > 0$, there exist functions $\rho_1, \rho_2 \in L^1(J, \mathbb{R}^+)$ such that

$$\begin{aligned} & \|f(t,x) - f(t,y)\|_\beta \leq \rho_1(t)\|x - y\|_\alpha, \\ & \|h(t,x) - h(t,y)\|_\beta \leq \rho_2(t)\|x - y\|_\alpha \end{aligned}$$

for any $x, y \in B_r$ and $t \in J$.

Theorem 2 *Let the assumptions (H₃)-(H₆) be satisfied. If $x_0 \in X_\alpha$ and the inequalities (3) and*

$$\mu \triangleq ML + \frac{MT^q C_{\beta-\alpha}}{\Gamma(q+1)} (\|\rho_1\|_{L^1} + K^* \|\rho_2\|_{L^1}) < 1 \tag{4}$$

hold, then Eq. (1) has a unique mild solution.

Proof From Lemma 4 and assumption (H₅), it is easy to see that $(t-s)^{q-1}V(t-s)[f(s, u(s)) + \int_0^s K(s, \tau)h(\tau, u(\tau)) d\tau]$ is Bochner integrable with respect to $s \in [0, t]$ for all $t \in J$. For any $u \in B_r$, we define an operator Q by

$$\begin{aligned} (Qu)(t) &= U(t)(x_0 - g(u)) + \int_0^t (t-s)^{q-1}V(t-s) \\ &\quad \times \left[f(s, u(s)) + \int_0^s K(s, \tau)h(\tau, u(\tau)) d\tau \right] ds, \quad t \in J. \end{aligned}$$

According to the proof of Theorem 1, we know that $Q(B_r) \subset B_r$ for some $r > 0$. For any $u, v \in B_r$ and $t \in J$, from Lemma 3, assumptions (H₄) and (H₆), we have

$$\begin{aligned} &\| (Qu)(t) - (Qv)(t) \|_\alpha \\ &\leq \| U(t)[g(v) - g(u)] \|_\alpha \\ &\quad + \int_0^t (t-s)^{q-1} \| V(t-s)[f(s, u(s)) - f(s, v(s))] \|_\alpha ds \\ &\quad + \int_0^t (t-s)^{q-1} \left\| V(t-s) \int_0^s K(s, \tau)[h(\tau, u(\tau)) - h(\tau, v(\tau))] d\tau \right\|_\alpha ds \\ &\leq ML \|u - v\|_C + \frac{qMC_{\beta-\alpha}}{\Gamma(q+1)} \int_0^t (t-s)^{q-1} \rho_1(s) \|u(s) - v(s)\|_\alpha ds \\ &\quad + \frac{qMC_{\beta-\alpha}K^*}{\Gamma(q+1)} \int_0^t (t-s)^{q-1} \int_0^s \rho_2(\tau) \|u(\tau) - v(\tau)\|_\alpha d\tau ds \\ &\leq ML \|u - v\|_C + \frac{qMC_{\beta-\alpha}}{\Gamma(q+1)} \int_0^t (t-s)^{q-1} ds \cdot \|\rho_1\|_{L^1} \cdot \|u - v\|_C \\ &\quad + \frac{qMC_{\beta-\alpha}K^*}{\Gamma(q+1)} \int_0^t (t-s)^{q-1} ds \cdot \|\rho_2\|_{L^1} \cdot \|u - v\|_C \\ &= \left[ML + \frac{MT^q C_{\beta-\alpha}}{\Gamma(q+1)} (\|\rho_1\|_{L^1} + K^* \|\rho_2\|_{L^1}) \right] \cdot \|u - v\|_C. \end{aligned}$$

Thus,

$$\|Qu - Qv\|_C \leq \mu \|u - v\|_C,$$

which means that Q is a contraction according to (4). By applying the Banach contraction principle, we know that Q has a unique fixed point on B_r , which is the unique mild solution of Eq. (1). This completes the proof. \square

4 An example

Let $X = L^2[0, \pi]$ equip with its natural norm $\|\cdot\|_2$ and inner product $\langle \cdot, \cdot \rangle$. Consider the following system:

$$\begin{cases} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = u(t, x) \sin t^{\frac{1}{2}} + \int_0^t K(t, s) u(s, x) \cos s \, ds, \\ t \in [0, T], x \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, \quad t \in [0, T], \\ u(0, x) + \sum_{i=1}^m \int_0^\pi K_0(x, y) u(t_i, y) \, dy = u_0(x), \quad x \in [0, \pi], \end{cases} \quad (5)$$

where $T > 0$ is a constant, $0 < t_1 < \dots < t_m < T$, $u_0 \in X$ and K_0 will be specified later.

Let the operator $A : D(A) \subset X \rightarrow X$ be defined by

$$D(A) := \left\{ v \in X : v', v'' \in X, v(0) = v(\pi) = 0 \right\}, \quad Au = -\frac{\partial^2 u}{\partial x^2}.$$

Then $-A$ generates a compact analytic semigroup $S(\cdot)$ of uniformly bounded linear operators and $\|S(t)\|_{\mathcal{L}(X)} \leq e^{-t}$ for all $t \geq 0$. Hence, we take $M = 1$. Moreover, the eigenvalues of A are n^2 , $n \in \mathbb{N}$ and the corresponding normalized eigenvectors are $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \dots$. The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} \xi = \sum_{n=1}^{\infty} n \langle \xi, z_n \rangle z_n$$

for each $\xi \in D(A^{\frac{1}{2}}) := \{v \in X : \sum_{n=1}^{\infty} n \langle v, z_n \rangle z_n \in X\}$ and $\|A^{-\frac{1}{2}}\|_{\mathcal{L}(X)} = 1$.

Lemma 6 ([16]) *If $\xi \in D(A^{\frac{1}{2}})$, then ξ is absolutely continuous, $\xi' \in X$ and $\|\xi'\|_2 = \|A^{\frac{1}{2}} \xi\|_2$.*

Let $X_{\frac{1}{2}} = (D(A^{\frac{1}{2}}), \|\cdot\|_{\frac{1}{2}})$, where $\|\xi\|_{\frac{1}{2}} := \|A^{\frac{1}{2}} \xi\|_X$ for all $\xi \in D(A^{\frac{1}{2}})$. Assume that

(P₁) The function $K_0 \in L^2([0, \pi] \times [0, \pi], \mathbb{R})$, $K_0(0, y) = K_0(\pi, y) = 0$, and the partial derivative $(x, y) \mapsto \frac{\partial}{\partial x} K_0(x, y)$ belongs to $L^2([0, \pi] \times [0, \pi], \mathbb{R})$.

Define

$$\begin{aligned} f(t, u(t))(x) &= u(t, x) \sin t^{\frac{1}{2}}, \\ h(t, u(t))(x) &= u(t, x) \cos t, \\ g(u)(x) &= \sum_{i=1}^m \int_0^\pi K_0(x, y) u(t_i, y) \, dy. \end{aligned}$$

Let $\xi \in X_{\frac{1}{2}}$, it follows from

$$\begin{aligned} \langle f(t, \xi), z_n \rangle &= \int_0^\pi u(t, x) \sin t^{\frac{1}{2}} \cdot \sqrt{\frac{2}{\pi}} \sin(nx) \, dx \\ &= \frac{1}{n} \int_0^\pi \left[\frac{\partial}{\partial x} u(t, x) \right] \sin t^{\frac{1}{2}} \cdot \sqrt{\frac{2}{\pi}} \cos(nx) \, dx \end{aligned}$$

and

$$\begin{aligned} \langle h(t, \xi), z_n \rangle &= \int_0^\pi u(t, x) \cos t \cdot \sqrt{\frac{2}{\pi}} \sin(nx) dx \\ &= \frac{1}{n} \int_0^\pi \left[\frac{\partial}{\partial x} u(t, x) \right] \cos t \cdot \sqrt{\frac{2}{\pi}} \cos(nx) dx \end{aligned}$$

that f and h are functions from $[0, T] \times X_{\frac{1}{2}}$ into $X_{\frac{1}{2}}$ and they are continuous. Moreover, a similar computation of [17] together with Lemma 6 and assumption (P_1) shows that $g(u) \in X_{\frac{1}{2}}$ whenever $u \in C([0, T], X_{\frac{1}{2}})$.

Then for any $r > 0$, we see that the assumptions (H_3) - (H_6) are satisfied with

$$\begin{aligned} \varphi_r(t) &= r \sin t^{\frac{1}{2}}, & \phi_r(t) &= r \cos t, & \sigma_1 &= \sigma_2 = 2T^{\frac{1}{2}}, \\ L &= (n+1) \left(\int_0^\pi \int_0^\pi \left[\frac{\partial}{\partial x} K_0(x, y) \right]^2 dx dy \right)^{\frac{1}{2}}, & \rho_1(t) &= \rho_2(t) = 1. \end{aligned}$$

Thus, the system (5) has at least one mild solution due to Theorem 1 provided that $L + \frac{2T^{\frac{1}{2}}}{\sqrt{\pi}}(1+K^*) < 1$. And by Theorem 2, this mild solution of the system (5) is unique on $[0, T] \times [0, \pi]$.

Competing interests

The author declares that they have no competing interests.

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