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Multiple positive solutions for semilinear elliptic systems involving subcritical nonlinearities in \mathbb{R}^N

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Abstract

In this paper, we investigate the effect of the coefficient $f(x)$ of the subcritical nonlinearity. Under some assumptions, for sufficiently small $\varepsilon, \lambda, \mu > 0$, there are at least $k (\geq 1)$ positive solutions of the semilinear elliptic systems

$$\begin{cases} -\varepsilon^2 \Delta \bar{u} + \bar{u} = \lambda g(x) |\bar{u}|^{q-2} \bar{u} + \frac{\alpha}{\alpha+\beta} f(x) |\bar{u}|^{\alpha-2} \bar{u} |\bar{v}|^\beta & \text{in } \mathbb{R}^N; \\ -\varepsilon^2 \Delta \bar{v} + \bar{v} = \mu h(x) |\bar{v}|^{q-2} \bar{v} + \frac{\beta}{\alpha+\beta} f(x) |\bar{u}|^\alpha |\bar{v}|^{\beta-2} \bar{v} & \text{in } \mathbb{R}^N; \\ \bar{u}, \bar{v} \in H^1(\mathbb{R}^N), \end{cases}$$

where $\alpha > 1, \beta > 1, 2 < q < p = \alpha + \beta < 2^* = 2N/(N-2)$ for $N \geq 3$.

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1 Introduction

For $N \geq 3, \alpha > 1, \beta > 1$ and $2 < q < p = \alpha + \beta < 2^* = 2N/(N-2)$, we consider the semilinear elliptic systems

$$\begin{cases} -\varepsilon^2 \Delta \bar{u} + \bar{u} = \lambda g(x) |\bar{u}|^{q-2} \bar{u} + \frac{\alpha}{\alpha+\beta} f(x) |\bar{u}|^{\alpha-2} \bar{u} |\bar{v}|^\beta & \text{in } \mathbb{R}^N; \\ -\varepsilon^2 \Delta \bar{v} + \bar{v} = \mu h(x) |\bar{v}|^{q-2} \bar{v} + \frac{\beta}{\alpha+\beta} f(x) |\bar{u}|^\alpha |\bar{v}|^{\beta-2} \bar{v} & \text{in } \mathbb{R}^N; \\ \bar{u} > 0, \quad \bar{v} > 0, \end{cases} \quad (\bar{E}_{\varepsilon, \lambda, \mu})$$

where $\varepsilon, \lambda, \mu > 0$.

Let f, g and h satisfy the following conditions:

- (A1) f is a positive continuous function in \mathbb{R}^N and $\lim_{|x| \rightarrow \infty} f(x) = f_\infty > 0$.
- (A2) there exist k points a^1, a^2, \dots, a^k in \mathbb{R}^N such that

$$f(a^i) = \max_{x \in \mathbb{R}^N} f(x) = 1 \quad \text{for } 1 \leq i \leq k,$$

and $f_\infty < 1$.

- (A3) $g, h \in L^m(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where $m = (\alpha + \beta)/(\alpha + \beta - q)$, and $g, h \not\equiv 0$.

In [1], if Ω is a smooth and bounded domain in \mathbb{R}^N ($N \leq 3$), they considered the following system:

$$\begin{cases} \varepsilon^2 \Delta \bar{u} - \lambda_1 \bar{u} = \mu_1 \bar{u}^3 + \beta \bar{u} \bar{v}^2 & \text{in } \Omega; \\ \varepsilon^2 \Delta \bar{v} - \lambda_2 \bar{v} = \mu_2 \bar{v}^3 + \beta \bar{u}^2 \bar{v} & \text{in } \Omega; \\ \bar{u} > 0, \quad \bar{v} > 0, \end{cases}$$

and proved the existence of a least energy solution in Ω for sufficiently small $\varepsilon > 0$ and $\beta \in (-\infty, \beta_0)$. Lin and Wei also showed that this system has a least energy solution in \mathbb{R}^N for $\varepsilon = 1$ and $\beta \in (0, \beta_0)$. In this paper, we study the effect of $f(z)$ of $(\bar{E}_{\varepsilon, \lambda, \mu})$. Recently, many authors [2–5] considered the elliptic systems with subcritical or critical exponents, and they proved the existence of a least energy positive solution or the existence of at least two positive solutions for these problems. In this paper, we construct the k compact Palais-Smale sequences which are suitably localized in correspondence of k maximum points of f . Then we could show that under some assumptions (A1)–(A3), for sufficiently small $\varepsilon, \lambda, \mu > 0$, there are at least k (≥ 1) positive solutions of the elliptic system $(E_{\varepsilon, \lambda, \mu})$. By the change of variables

$$x = \varepsilon z, \quad u(z) = \bar{u}(\varepsilon z) \quad \text{and} \quad v(z) = \bar{v}(\varepsilon z),$$

System $(\bar{E}_{\varepsilon, \lambda, \mu})$ is transformed to

$$\begin{cases} -\Delta u + u = \lambda g(\varepsilon z) |u|^{q-2} u + \frac{\alpha}{\alpha + \beta} f(\varepsilon z) |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^N; \\ -\Delta v + v = \mu h(\varepsilon z) |v|^{q-2} v + \frac{\beta}{\alpha + \beta} f(\varepsilon z) |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N; \\ u > 0, \quad v > 0. \end{cases} \quad (E_{\varepsilon, \lambda, \mu})$$

Let $H = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ be the space with the standard norm

$$\|(u, v)\|_H = \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dz + \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) dz \right]^{1/2}.$$

Associated with the problem $(E_{\varepsilon, \lambda, \mu})$, we consider the C^1 -functional $J_{\varepsilon, \lambda, \mu}$, for $(u, v) \in H$,

$$\begin{aligned} J_{\varepsilon, \lambda, \mu}(u, v) &= \frac{1}{2} \|(u, v)\|_H^2 - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz \\ &\quad - \frac{1}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz. \end{aligned}$$

Actually, the weak solution $(u, v) \in H$ of $(E_{\varepsilon, \lambda, \mu})$ is the critical point of the functional $J_{\varepsilon, \lambda, \mu}$, that is, $(u, v) \in H$ satisfies

$$\begin{aligned} &\int_{\mathbb{R}^N} (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 + u \varphi_1 + v \varphi_2) dz \\ &\quad - \lambda \int_{\mathbb{R}^N} g(\varepsilon z) |u|^{q-2} u \varphi_1 dz - \mu \int_{\mathbb{R}^N} h(\varepsilon z) |v|^{q-2} v \varphi_2 dz \\ &\quad - \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^{\alpha-2} u |v|^\beta \varphi_1 dz - \frac{\beta}{\alpha + \beta} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^{\beta-2} v \varphi_2 dz = 0 \end{aligned}$$

for any $(\varphi_1, \varphi_2) \in H$.

We consider the Nehari manifold

$$\mathbf{M}_{\varepsilon,\lambda,\mu} = \{(u, v) \in H \setminus \{(0, 0)\} \mid \langle J'_{\varepsilon,\lambda,\mu}(u, v), (u, v) \rangle = 0\}, \tag{1.1}$$

where

$$\langle J'_{\varepsilon,\lambda,\mu}(u, v), (u, v) \rangle = \|(u, v)\|_H^2 - \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz - \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz.$$

The Nehari manifold $\mathbf{M}_{\varepsilon,\lambda,\mu}$ contains all nontrivial weak solutions of $(E_{\varepsilon,\lambda,\mu})$.

Let

$$S_{\alpha,\beta} = \inf_{u,v \in H^1(\mathbb{R}^N) \setminus \{(0,0)\}} \frac{\|(u, v)\|_H^2}{(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dz)^{2/(\alpha+\beta)}}, \tag{1.2}$$

then by [2, Theorem 5], we have

$$S_{\alpha,\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \right] S_p,$$

where $p = \alpha + \beta$ and S_p is the best Sobolev constant defined by

$$S_p = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dz}{(\int_{\mathbb{R}^N} |u|^p dz)^{2/p}}.$$

For the semilinear elliptic systems $(\lambda = \mu = 0)$

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} f(\varepsilon z) |u|^{\alpha-2} |v|^\beta & \text{in } \mathbb{R}^N; \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} f(\varepsilon z) |u|^\alpha |v|^{\beta-2} & \text{in } \mathbb{R}^N; \\ (u, v) \in H, \end{cases} \tag{E_\varepsilon}$$

we define the energy functional $I_\varepsilon(u, v) = \frac{1}{2} \|(u, v)\|_H^2 - \frac{1}{\alpha+\beta} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz$, and

$$\mathbf{N}_\varepsilon = \{(u, v) \in H \setminus \{(0, 0)\} \mid \langle I'_\varepsilon(u, v), (u, v) \rangle = 0\}.$$

If $f \equiv \max_{z \in \mathbb{R}^N} f(z) (= 1)$, then we define $I_{\max}(u, v) = \frac{1}{2} \|(u, v)\|_H^2 - \frac{1}{\alpha+\beta} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dz$ and

$$\theta_{\max} = \inf_{(u,v) \in \mathbf{N}_{\max}} I_{\max}(u, v),$$

where $\mathbf{N}_{\max} = \{(u, v) \in H \setminus \{(0, 0)\} \mid \langle I'_{\max}(u, v), (u, v) \rangle = 0\}$.

It is well known that this problem

$$\begin{cases} -\Delta u + u = |u|^{p-2} u & \text{in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{E0}$$

has the unique, radially symmetric and positive ground state solution $w \in H^1(\mathbb{R}^N)$. Define $\bar{I}_{\max}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dz - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dz$ and $\bar{\theta}_{\max} = \inf_{u \in \bar{\mathbf{N}}_{\max}} \bar{I}_{\max}(u)$, where

$$\bar{\mathbf{N}}_{\max} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle \bar{I}'_{\max}(u), u \rangle = 0\}.$$

Moreover, we have that

$$\bar{\theta}_{\max} = \frac{p-2}{2p} S_p^{\frac{p}{p-2}} > 0. \quad (\text{See Wang [6, Theorems 4.12 and 4.13].})$$

This paper is organized as follows. First of all, we study the argument of the Nehari manifold $\mathbf{M}_{\varepsilon,\lambda,\mu}$. Next, we prove that the existence of a positive solution $(u_0, v_0) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ of $(E_{\varepsilon,\lambda,\mu})$. Finally, in Section 4, we show that the condition (A2) affects the number of positive solutions of $(E_{\varepsilon,\lambda,\mu})$; that is, there are at least k critical points $(u_i, v_i) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ of $J_{\varepsilon,\lambda,\mu}$ such that $J_{\varepsilon,\lambda,\mu}(u_i, v_i) = \beta_{\varepsilon,\lambda,\mu}^i$ ((PS)-value) for $1 \leq i \leq k$.

Theorem 1.1 $(E_{\varepsilon,\lambda,\mu})$ has at least one positive solution (u_0, v_0) , that is, $(\bar{E}_{\varepsilon,\lambda,\mu})$ admits at least one positive solution.

Theorem 1.2 There exist two positive numbers ε_0 and Λ^* such that $(E_{\varepsilon,\lambda,\mu})$ has at least k positive solutions for any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$, that is, $(\bar{E}_{\varepsilon,\lambda,\mu})$ admits at least k positive solutions.

2 Preliminaries

By studying the argument of Han [7, Lemma 2.1], we obtain the following lemma.

Lemma 2.1 Let $\Omega \subset \mathbb{R}^N$ (possibly unbounded) be a smooth domain. If $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$, and $u_n \rightarrow u$, $v_n \rightarrow v$ almost everywhere in Ω , then

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n - u|^\alpha |v_n - v|^\beta dz = \lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^\alpha |v_n|^\beta dz - \int_{\Omega} |u|^\alpha |v|^\beta dz.$$

Note that $J_{\varepsilon,\lambda,\mu}$ is not bounded from below in H . From the following lemma, we have that $J_{\varepsilon,\lambda,\mu}$ is bounded from below on $\mathbf{M}_{\varepsilon,\lambda,\mu}$.

Lemma 2.2 The energy functional $J_{\varepsilon,\lambda,\mu}$ is bounded from below on $\mathbf{M}_{\varepsilon,\lambda,\mu}$.

Proof For $(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$, by (1.1), we obtain that

$$J_{\varepsilon,\lambda,\mu}(u, v) = \left(\frac{1}{2} - \frac{1}{q}\right) \|(u, v)\|_H^2 + \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz > 0,$$

where $p = \alpha + \beta$. Hence, we have that $J_{\varepsilon,\lambda,\mu}$ is bounded from below on $\mathbf{M}_{\varepsilon,\lambda,\mu}$. □

We define

$$\theta_{\varepsilon,\lambda,\mu} = \inf_{(u,v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u, v).$$

Lemma 2.3 (i) There exist positive numbers σ and d_0 such that $J_{\varepsilon,\lambda,\mu}(u, v) \geq d_0$ for $\|(u, v)\|_H = \sigma$;

(ii) There exists $(\bar{u}, \bar{v}) \in H \setminus \{(0, 0)\}$ such that $\|(\bar{u}, \bar{v})\|_H > \sigma$ and $J_{\varepsilon,\lambda,\mu}(\bar{u}, \bar{v}) < 0$.

Proof (i) By (1.2), the Hölder inequality ($p_1 = \frac{p}{p-q}$, $p_2 = \frac{p}{q}$) and the Sobolev embedding theorem, we have that

$$\begin{aligned} J_{\varepsilon,\lambda,\mu}(u, v) &= \frac{1}{2} \|(u, v)\|_H^2 - \frac{1}{p} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz \\ &\quad - \frac{1}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz \\ &\geq \frac{1}{2} \|(u, v)\|_H^2 - \frac{1}{p} S_{\alpha,\beta}^{-p/2} \|(u, v)\|_H^p \\ &\quad - \frac{1}{q} \text{Max } S_p^{-q/2} (\lambda + \mu) \|(u, v)\|_H^q, \end{aligned}$$

where $p = \alpha + \beta$ and $\text{Max} = \max\{\|g\|_m, \|h\|_m\}$. Hence, there exist positive σ and d_0 such that $J_{\varepsilon,\lambda,\mu}(u, v) \geq d_0$ for $\|(u, v)\|_H = \sigma$.

(ii) For any $(u, v) \in H \setminus \{(0, 0)\}$, since

$$\begin{aligned} J_{\varepsilon,\lambda,\mu}(tu, tv) &= \frac{t^2}{2} \|(u, v)\|_H^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz \\ &\quad - \frac{t^q}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz, \end{aligned}$$

then $\lim_{t \rightarrow \infty} J_{\varepsilon,\lambda,\mu}(tu, tv) = -\infty$. Fix some $(u, v) \in H \setminus \{(0, 0)\}$, there exists $\bar{t} > 0$ such that $\|(\bar{t}u, \bar{t}v)\|_H > \sigma$ and $J_{\varepsilon,\lambda,\mu}(\bar{t}u, \bar{t}v) < 0$. Let $(\bar{u}, \bar{v}) = (\bar{t}u, \bar{t}v)$. □

Define

$$\psi(u, v) = \langle J'_{\varepsilon,\lambda,\mu}(u, v), (u, v) \rangle.$$

Then for $(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$, we obtain that

$$\begin{aligned} \langle \psi'(u, v), (u, v) \rangle &= 2 \|(u, v)\|_H^2 - p \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz \\ &\quad - q \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz \\ &= (p - q) \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz - (p - 2) \|(u, v)\|_H^2 \end{aligned} \tag{2.1}$$

$$= (2 - q) \|(u, v)\|_H^2 + (q - p) \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz < 0. \tag{2.2}$$

Lemma 2.4 For each $(u, v) \in H \setminus \{(0, 0)\}$, there exists a unique positive number $t_{u,v}$ such that $(t_{u,v}u, t_{u,v}v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ and $J_{\varepsilon,\lambda,\mu}(t_{u,v}u, t_{u,v}v) = \sup_{t \geq 0} J_{\varepsilon,\lambda,\mu}(tu, tv)$.

Proof Fixed $(u, v) \in H \setminus \{(0, 0)\}$, we consider

$$\begin{aligned} R(t) &= J_{\varepsilon,\lambda,\mu}(tu, tv) \\ &= \frac{t^2}{2} \|(u, v)\|_H^2 - \frac{t^p}{p} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz - \frac{t^q}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz. \end{aligned}$$

Since $R(0) = 0$, $\lim_{t \rightarrow \infty} R(t) = -\infty$, by Lemma 2.3(i), then $\sup_{t \geq 0} R(t)$ is achieved at some $t_{u,v} > 0$. Moreover, we have that $R'(t_{u,v}) = 0$, that is, $(t_{u,v}u, t_{u,v}v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$. Next, we claim that $t_{u,v}$ is a unique positive number such that $R'(t_{u,v}) = 0$. Consider

$$r(t) = \|(u, v)\|_H^2 - t^{p-2} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz - t^{q-2} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz,$$

then $R'(t) = tr(t)$. Since $r(0) = \|(u, v)\|_H^2 > 0$,

$$\begin{aligned} r'(t) &= -(p-2)t^{p-3} \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz \\ &\quad - (q-2)t^{q-3} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz < 0, \end{aligned}$$

there exists a unique positive number $\bar{t}_{u,v}$ such that $r(\bar{t}_{u,v}) = 0$. It follows that $R'(\bar{t}_{u,v}) = 0$. Hence, $\bar{t}_{u,v} = t_{u,v}$. \square

Remark 2.5 By Lemma 2.3(i) and Lemma 2.4, then $\theta_{\varepsilon,\lambda,\mu} \geq d_0 > 0$ for some constant d_0 .

Lemma 2.6 Let $(u_0, v_0) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ satisfy

$$J_{\varepsilon,\lambda,\mu}(u_0, v_0) = \min_{(u,v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u, v) = \theta_{\varepsilon,\lambda,\mu},$$

then (u_0, v_0) is a solution of $(E_{\varepsilon,\lambda,\mu})$.

Proof By (2.2), $\langle \psi'(u, v), (u, v) \rangle < 0$ for $(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$. Since $J_{\varepsilon,\lambda,\mu}(u_0, v_0) = \min_{(u,v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u, v)$, by the Lagrange multiplier theorem, there is $\tau \in \mathbb{R}$ such that $J'_{\varepsilon,\lambda,\mu}(u_0, v_0) = \tau \psi'(u_0, v_0)$ in H^{-1} . Then we have

$$0 = \langle J'_{\varepsilon,\lambda,\mu}(u_0, v_0), (u_0, v_0) \rangle = \tau \langle \psi'(u_0, v_0), (u_0, v_0) \rangle.$$

It follows that $\tau = 0$ and $J'_{\varepsilon,\lambda,\mu}(u_0, v_0) = 0$ in H^{-1} . Therefore, (u_0, v_0) is a nontrivial solution of $(E_{\varepsilon,\lambda,\mu})$ and $J_{\varepsilon,\lambda,\mu}(u_0, v_0) = \theta_{\varepsilon,\lambda,\mu}$. \square

3 (PS) $_\gamma$ -condition in H for $J_{\varepsilon,\lambda,\mu}$

First of all, we define the Palais-Smale (denoted by (PS)) sequence and (PS)-condition in H for some functional J .

Definition 3.1 (i) For $\gamma \in \mathbb{R}$, a sequence $\{(u_n, v_n)\}$ is a (PS) $_\gamma$ -sequence in H for J if $J(u_n, v_n) = \gamma + o_n(1)$ and $J'(u_n, v_n) = o_n(1)$ strongly in H^{-1} as $n \rightarrow \infty$, where H^{-1} is the dual space of H ;

(ii) J satisfies the (PS) $_\gamma$ -condition in H if every (PS) $_\gamma$ -sequence in H for J contains a convergent subsequence.

Applying Ekeland's variational principle and using the same argument as in Cao-Zhou [8] or Tarantello [9], we have the following lemma.

Lemma 3.2 (i) There exists a (PS) $_{\theta_{\varepsilon,\lambda,\mu}}$ -sequence $\{(u_n, v_n)\}$ in $\mathbf{M}_{\varepsilon,\lambda,\mu}$ for $J_{\varepsilon,\lambda,\mu}$.

In order to prove the existence of positive solutions, we want to prove that $J_{\varepsilon,\lambda,\mu}$ satisfies the $(PS)_\gamma$ -condition in H for $\gamma \in (0, \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}})$.

Lemma 3.3 $J_{\varepsilon,\lambda,\mu}$ satisfies the $(PS)_\gamma$ -condition in H for $\gamma \in (0, \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}})$.

Proof Let $\{(u_n, v_n)\}$ be a $(PS)_\gamma$ -sequence in H for $J_{\varepsilon,\lambda,\mu}$ such that $J_{\varepsilon,\lambda,\mu}(u_n, v_n) = \gamma + o_n(1)$ and $J'_{\varepsilon,\lambda,\mu}(u_n, v_n) = o_n(1)$ in H^{-1} . Then

$$\begin{aligned} \gamma + c_n + \frac{d_n \| (u_n, v_n) \|_H}{q} &\geq J_{\varepsilon,\lambda,\mu}(u_n, v_n) - \frac{1}{q} \langle J'_{\varepsilon,\lambda,\mu}(u_n, v_n), (u_n, v_n) \rangle \\ &= \left(\frac{1}{2} - \frac{1}{q} \right) \| (u_n, v_n) \|_H^2 + \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\mathbb{R}^N} f(\varepsilon z) |u_n|^\alpha |v_n|^\beta dz \\ &\geq \frac{q-2}{2q} \| (u_n, v_n) \|_H^2, \end{aligned}$$

where $c_n = o_n(1)$, $d_n = o_n(1)$ as $n \rightarrow \infty$. It follows that $\{(u_n, v_n)\}$ is bounded in H . Hence, there exist a subsequence $\{(u_n, v_n)\}$ and $(u, v) \in H$ such that

$$\begin{aligned} u_n &\rightharpoonup u, & v_n &\rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R}^N); \\ u_n &\rightarrow u, & v_n &\rightarrow v \quad \text{strongly in } L^s_{\text{loc}}(\mathbb{R}^N) \text{ for any } 1 \leq s < 2^*; \\ u_n &\rightarrow u, & v_n &\rightarrow v \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Moreover, we have that $J'_{\varepsilon,\lambda,\mu}(u, v) = 0$ in H^{-1} . We use the Brézis-Lieb lemma to obtain (3.1) and (3.2) as follows:

$$\int_{\mathbb{R}^N} g(\varepsilon z) |u_n - u|^q dz = \int_{\mathbb{R}^N} g(\varepsilon z) |u_n|^q dz - \int_{\mathbb{R}^N} g(\varepsilon z) |u|^q dz + o_n(1); \tag{3.1}$$

$$\int_{\mathbb{R}^N} h(\varepsilon z) |v_n - v|^q dz = \int_{\mathbb{R}^N} h(\varepsilon z) |v_n|^q dz - \int_{\mathbb{R}^N} h(\varepsilon z) |v|^q dz + o_n(1). \tag{3.2}$$

Next, we claim that

$$\int_{\mathbb{R}^N} g(\varepsilon z) |u_n - u|^q dz \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} h(\varepsilon z) |v_n - v|^q dz \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.4}$$

Since $g \in L^m(\mathbb{R}^N)$, where $m = p/(p - q)$, then for any $\sigma > 0$, there exists $r > 0$ such that $\int_{[B_r^N(0)]^c} g(\varepsilon z)^{\frac{p}{p-q}} dz < \sigma$. By the Hölder inequality and the Sobolev embedding theorem, we get

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} g(\varepsilon z) |u_n - u|^q dz \right| \\ &\leq \int_{B_r^N(0)} g(\varepsilon z) |u_n - u|^q dz \end{aligned}$$

$$\begin{aligned}
 & + \int_{[B_r^N(0)]^c} g(\varepsilon z) |u_n - u|^q dz \\
 & \leq \|g\|_m \left(\int_{B_r^N(0)} |u_n - u|^p dz \right)^{q/p} \\
 & \quad + S_p^{\frac{q}{2}} \left(\int_{[B_r^N(0)]^c} g(\varepsilon z)^{\frac{p}{p-q}} dz \right)^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 + |u_n - u|^2 dz \right)^{q/2} \\
 & \leq C' \sigma + o_n(1) \quad (\because \{u_n\} \text{ is bounded in } H^1(\mathbb{R}^N) \text{ and } u_n \rightarrow u \text{ in } L_{loc}^p(\mathbb{R}^N)).
 \end{aligned}$$

Similarly, $\int_{\mathbb{R}^N} h(\varepsilon z) |v_n - v|^q dz \rightarrow 0$ as $n \rightarrow \infty$. By (A1) and $u_n \rightarrow u, v_n \rightarrow v$ strongly in $L_{loc}^p(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} f(\varepsilon z) |u_n - u|^\alpha |v_n - v|^\beta dz = \int_{\mathbb{R}^N} f_\infty |u_n - u|^\alpha |v_n - v|^\beta dz = o_n(1). \tag{3.5}$$

Let $p_n = (u_n - u, v_n - v)$. By (3.1)-(3.4) and Lemma 2.1, we deduce that

$$\begin{aligned}
 \|p_n\|_H^2 & = (\|u_n\|_H^2 + \|v_n\|_H^2) - (\|u\|_H^2 + \|v\|_H^2) + o_n(1) \\
 & = \int_{\mathbb{R}^N} f(\varepsilon z) |u_n|^\alpha |v_n|^\beta dz + \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u_n|^q + \mu h(\varepsilon z) |v_n|^q) dz \\
 & \quad - \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz - \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz + o_n(1) \\
 & = \int_{\mathbb{R}^N} f(\varepsilon z) |u_n - u|^\alpha |v_n - v|^\beta dz + o_n(1),
 \end{aligned}$$

and

$$\frac{1}{2} \|p_n\|_H^2 - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} f(\varepsilon z) |u_n - u|^\alpha |v_n - v|^\beta dz = \gamma - J_{\varepsilon, \lambda, \mu}(u, v) + o_n(1). \tag{3.6}$$

We may assume that

$$\|p_n\|_H^2 \rightarrow l \quad \text{and} \quad \int_{\mathbb{R}^N} f(\varepsilon z) |u_n - u|^\alpha |v_n - v|^\beta dz \rightarrow l \quad \text{as } n \rightarrow \infty. \tag{3.7}$$

Recall that

$$S_{\alpha, \beta} = \inf_{u, v \in H^1(\mathbb{R}^N) \setminus \{(0)\}} \frac{\|(u, v)\|_H^2}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dz \right)^{2/p}}, \quad \text{where } p = \alpha + \beta.$$

If $l > 0$, by (3.5), then

$$\begin{aligned}
 S_{\alpha, \beta} l^{\frac{2}{p}} & = S_{\alpha, \beta} \left(\int_{\mathbb{R}^N} f(\varepsilon z) |u_n - u|^\alpha |v_n - v|^\beta dz \right)^{2/p} + o_n(1) \\
 & = S_{\alpha, \beta} \left(\int_{\mathbb{R}^N} f_\infty |u_n - u|^\alpha |v_n - v|^\beta dz \right)^{2/p} + o_n(1) \\
 & \leq (f_\infty)^{2/p} \|p_n\|_H^2 + o_n(1) = (f_\infty)^{2/p} l.
 \end{aligned}$$

This implies $l \geq (S_{\alpha,\beta})^{p/(p-2)} / (f_\infty)^{2/(p-2)}$. By (3.6) and (3.7), we obtain that

$$\gamma = \left(\frac{1}{2} - \frac{1}{p} \right) l + J_{\varepsilon,\lambda,\mu}(u, v) \geq \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}},$$

which is a contradiction. Hence, $l = 0$, that is, $(u_n, v_n) \rightarrow (u, v)$ strongly in H . □

4 Existence of k solutions

Let $w \in H^1(\mathbb{R}^N)$ be the unique, radially symmetric and positive ground state solution of equation (E0) in \mathbb{R}^N . Recall the facts (or see Bahri-Li [10], Bahri-Lions [11], Gidas-Nirenberg [12] and Kwong [13]):

- (i) $w \in L^\infty(\mathbb{R}^N) \cap C_{\text{loc}}^{2,\theta}(\mathbb{R}^N)$ for some $0 < \theta < 1$ and $\lim_{|z| \rightarrow \infty} w(z) = 0$;
- (ii) for any $\varepsilon > 0$, there exist positive numbers C_1, C_2^ε and C_3^ε such that for all $z \in \mathbb{R}^N$

$$C_2^\varepsilon \exp(-(1 + \varepsilon)|z|) \leq w(z) \leq C_1 \exp(-|z|)$$

and

$$|\nabla w(z)| \leq C_3^\varepsilon \exp(-(1 - \varepsilon)|z|).$$

By Lien-Tzeng-Wang [14], then

$$S_p = \frac{\int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) dz}{\left(\int_{\mathbb{R}^N} w^p dz \right)^{2/p}}. \tag{4.1}$$

For $1 \leq i \leq k$, we define

$$w_\varepsilon^i(z) = w\left(z - \frac{a^i}{\varepsilon}\right), \quad \text{where } f(a^i) = \max_{z \in \mathbb{R}^N} f(z) = 1.$$

Clearly, $w_\varepsilon^i(z) \in H^1(\mathbb{R}^N)$.

First of all, we want to prove that

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{t \geq 0} J_{\varepsilon,\lambda,\mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) \leq \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

Lemma 4.1 For $\lambda > 0$ and $\mu > 0$, then

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{t \geq 0} J_{\varepsilon,\lambda,\mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) \leq \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

Moreover, we have that

$$0 < \theta_{\varepsilon,\lambda,\mu} \leq \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)}.$$

Proof Part I: Since $J_{\varepsilon,\lambda,\mu}$ is continuous in H , $J_{\varepsilon,\lambda,\mu}(0, 0) = 0$, and $\{(\sqrt{\alpha}w_\varepsilon^i, \sqrt{\beta}w_\varepsilon^i)\}$ is uniformly bounded in H for any $\varepsilon > 0$ and $1 \leq i \leq k$, then there exists $t_0 > 0$ such that for $0 \leq t < t_0$ and any $\varepsilon > 0$,

$$J_{\varepsilon,\lambda,\mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) < \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

From (A1), we have that $\inf_{z \in \mathbb{R}^N} f(z) > 0$. Then

$$\begin{aligned}
 J_{\varepsilon, \lambda, \mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) &\leq \frac{t^2}{2} \|(\sqrt{\alpha}w, \sqrt{\beta}w)\|_H^2 \\
 &\quad - \frac{t^{\alpha+\beta}}{\alpha+\beta} \left(\inf_{z \in \mathbb{R}^N} f(z) \right) \int_{\mathbb{R}^N} |\sqrt{\alpha}w|^\alpha |\sqrt{\beta}w|^\beta dz \\
 &\rightarrow -\infty \quad \text{as } t \rightarrow \infty.
 \end{aligned}$$

It follows that there exists $t_1 > 0$ such that for $t > t_1$ and any $\varepsilon > 0$,

$$J_{\varepsilon, \lambda, \mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) < \frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

From now on, we only need to show that

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{t_0 \leq t \leq t_1} J_{\varepsilon, \lambda, \mu}(tw_\varepsilon^i) \leq \frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

Since

$$\sup_{t \geq 0} \left(\frac{t^2}{2} a - \frac{t^{\alpha+\beta}}{\alpha+\beta} b \right) = \frac{\alpha+\beta-2}{2(\alpha+\beta)} \left(\frac{a}{b^{\frac{2}{\alpha+\beta}}} \right)^{\frac{\alpha+\beta}{\alpha+\beta-2}}, \quad \text{where } a, b > 0,$$

and by (4.1), then

$$\begin{aligned}
 &\sup_{t \geq 0} \left\{ \frac{t^2}{2} \|(\sqrt{\alpha}w_\varepsilon^i, \sqrt{\beta}w_\varepsilon^i)\|_H^2 - \frac{t^{\alpha+\beta}}{\alpha+\beta} \int_{\mathbb{R}^N} |\sqrt{\alpha}w_\varepsilon^i|^\alpha |\sqrt{\beta}w_\varepsilon^i|^\beta dz \right\} \\
 &= \frac{p-2}{2p} \left[\frac{(\alpha+\beta) \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) dz}{(\alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} \int_{\mathbb{R}^N} w^p dz)^{2/p}} \right]^{\frac{p}{p-2}} = \frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)}. \tag{4.2}
 \end{aligned}$$

For $t_0 \leq t \leq t_1$, by (4.2), we have that

$$\begin{aligned}
 J_{\varepsilon, \lambda, \mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) &= \frac{t^2}{2} \|(\sqrt{\alpha}w_\varepsilon^i, \sqrt{\beta}w_\varepsilon^i)\|_H^2 - \frac{t^{\alpha+\beta}}{\alpha+\beta} \int_{\mathbb{R}^N} f(\varepsilon z) |\sqrt{\alpha}w_\varepsilon^i|^\alpha |\sqrt{\beta}w_\varepsilon^i|^\beta dz \\
 &\quad - \frac{t^q}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |\sqrt{\alpha}w_\varepsilon^i|^q + \mu h(\varepsilon z) |\sqrt{\beta}w_\varepsilon^i|^q) dz \\
 &\leq \frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)} \\
 &\quad + \frac{t_1^p}{p} \int_{\mathbb{R}^N} (1-f(\varepsilon z)) |\sqrt{\alpha}w_\varepsilon^i|^\alpha |\sqrt{\beta}w_\varepsilon^i|^\beta dz.
 \end{aligned}$$

Since

$$\begin{aligned}
 &\int_{\mathbb{R}^N} (1-f(\varepsilon z)) |\sqrt{\alpha}w_\varepsilon^i|^\alpha |\sqrt{\beta}w_\varepsilon^i|^\beta dz \\
 &= \alpha^{\frac{\alpha}{2}} \beta^{\frac{\beta}{2}} \int_{\mathbb{R}^N} (1-f(\varepsilon z + a^i)) w^p dz = o(1) \quad \text{as } \varepsilon \rightarrow 0^+ \text{ uniformly in } i,
 \end{aligned}$$

then

$$\limsup_{\varepsilon \rightarrow 0^+} \sup_{t_0 \leq t \leq t_1} J_{\varepsilon, \lambda, \mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) \leq \frac{p-2}{2p}(S_{\alpha, \beta})^{p/(p-2)},$$

that is, for $\lambda > 0$ and $\mu > 0$,

$$\limsup_{\varepsilon \rightarrow 0^+} \sup_{t \geq 0} J_{\varepsilon, \lambda, \mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) \leq \frac{p-2}{2p}(S_{\alpha, \beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

Part II: By Lemma 2.4, there is a number $t_\varepsilon^i > 0$ such that $(t_\varepsilon^i \sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i \sqrt{\beta}w_\varepsilon^i) \in \mathbf{M}_{\varepsilon, \lambda, \mu}$, where $1 \leq i \leq k$. Hence, from the result of Part I, we have that for $\lambda > 0$ and $\mu > 0$,

$$0 < \theta_{\varepsilon, \lambda, \mu} \leq \limsup_{\varepsilon \rightarrow 0^+} \sup_{t \geq 0} J_{\varepsilon, \lambda, \mu}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) \leq \frac{p-2}{2p}(S_{\alpha, \beta})^{p/(p-2)}. \quad \square$$

Proof of Theorem 1.1 By Lemma 3.2, there exists a $(\text{PS})_{\theta_{\varepsilon, \lambda, \mu}}$ -sequence $\{(u_n, v_n)\}$ in $\mathbf{M}_{\varepsilon, \lambda, \mu}$ for $J_{\varepsilon, \lambda, \mu}$. Since $0 < \theta_{\varepsilon, \lambda, \mu} \leq \frac{p-2}{2p}(S_{\alpha, \beta})^{p/(p-2)} < \frac{p-2}{2p} \frac{(S_{\alpha, \beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}$ for $\lambda > 0$ and $\mu > 0$, by Lemma 3.3, there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0, v_0) \in H$ such that $(u_n, v_n) \rightarrow (u_0, v_0)$ strongly in H . It is easy to check that (u_0, v_0) is a nontrivial solution of $(E_{\varepsilon, \lambda, \mu})$ and $J_{\varepsilon, \lambda, \mu}(u_0, v_0) = \theta_{\varepsilon, \lambda, \mu}$. Since $J_{\varepsilon, \lambda, \mu}(u_0, v_0) = J_{\lambda, \mu}(|u_0|, |v_0|)$ and $(|u_0|, |v_0|) \in \mathbf{M}_{\varepsilon, \lambda, \mu}$, by Lemma 2.6, we may assume that $u_0 \geq 0, v_0 \geq 0$. Applying the maximum principle, $u_0 > 0$ and $v_0 > 0$ in Ω . \square

Choosing $0 < \rho_0 < 1$ such that

$$\overline{B_{\rho_0}^N(a^i)} \cap \overline{B_{\rho_0}^N(a^j)} = \emptyset \quad \text{for } i \neq j \text{ and } 1 \leq i, j \leq k,$$

where $\overline{B_{\rho_0}^N(a^i)} = \{z \in \mathbb{R}^N \mid |z - a^i| \leq \rho_0\}$ and $f(a^i) = \max_{z \in \mathbb{R}^N} f(z) = 1$, define $\mathbf{K} = \{a^i \mid 1 \leq i \leq k\}$ and $\mathbf{K}_{\rho_0/2} = \bigcup_{i=1}^k \overline{B_{\rho_0/2}^N(a^i)}$. Suppose $\bigcup_{i=1}^k \overline{B_{\rho_0}^N(a^i)} \subset B_{r_0}^N(0)$ for some $r_0 > 0$. Let Q_ε be given by

$$Q_\varepsilon(u, v) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z) |u|^\alpha |v|^\beta dz}{\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dz},$$

where $\chi : \mathbb{R}^N \rightarrow \mathbb{R}^N, \chi(z) = z$ for $|z| \leq r_0$ and $\chi(z) = r_0 z/|z|$ for $|z| > r_0$.

For each $1 \leq i \leq k$, we define

$$\begin{aligned} O_{\varepsilon, \lambda, \mu}^i &= \{(u, v) \in \mathbf{M}_{\varepsilon, \lambda, \mu} \mid |Q_\varepsilon(u, v) - a^i| < \rho_0\}, \\ \partial O_{\varepsilon, \lambda, \mu}^i &= \{(u, v) \in \mathbf{M}_{\varepsilon, \lambda, \mu} \mid |Q_\varepsilon(u, v) - a^i| = \rho_0\}, \\ \beta_{\varepsilon, \lambda, \mu}^i &= \inf_{(u, v) \in O_{\varepsilon, \lambda, \mu}^i} J_{\varepsilon, \lambda, \mu}(u, v) \quad \text{and} \quad \tilde{\beta}_{\varepsilon, \lambda, \mu}^i = \inf_{(u, v) \in \partial O_{\varepsilon, \lambda, \mu}^i} J_{\varepsilon, \lambda, \mu}(u, v). \end{aligned}$$

By Lemma 2.4, there exists $t_\varepsilon^i > 0$ such that $(t_\varepsilon^i \sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i \sqrt{\beta}w_\varepsilon^i) \in \mathbf{M}_{\varepsilon, \lambda, \mu}$ for each $1 \leq i \leq k$. Then we have the following result.

Lemma 4.2 *There exists $\varepsilon_1 > 0$ such that if $\varepsilon \in (0, \varepsilon_1)$, then $Q_\varepsilon(t_\varepsilon^i \sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i \sqrt{\beta}w_\varepsilon^i) \in \mathbf{K}_{\rho_0/2}$ for each $1 \leq i \leq k$.*

Proof Since

$$\begin{aligned} Q_\varepsilon(t_\varepsilon^i \sqrt{\alpha} w_\varepsilon^i, t_\varepsilon^i \sqrt{\beta} w_\varepsilon^i) &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z) |w(z - \frac{a^i}{\varepsilon})|^p dz}{\int_{\mathbb{R}^N} |w(z - \frac{a^i}{\varepsilon})|^p dz} \\ &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z + a^i) |w(z)|^p dz}{\int_{\mathbb{R}^N} |w(z)|^p dz} \\ &\rightarrow a^i \quad \text{as } \varepsilon \rightarrow 0^+, \end{aligned}$$

there exists $\varepsilon_1 > 0$ such that

$$Q_\varepsilon(t_\varepsilon^i \sqrt{\alpha} w_\varepsilon^i, t_\varepsilon^i \sqrt{\beta} w_\varepsilon^i) \in \mathbf{K}_{\rho_0/2} \quad \text{for any } \varepsilon \in (0, \varepsilon_1) \text{ and each } 1 \leq i \leq k. \quad \square$$

We need the following lemmas to prove that $\beta_{\lambda, \mu}^i < \tilde{\beta}_{\lambda, \mu}^i$ for sufficiently small $\varepsilon, \lambda, \mu$.

Lemma 4.3 $\theta_{\max} = \frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)}$.

Proof From Part I of Lemma 4.1, we obtain $\sup_{t \geq 0} I_{\max}(t\sqrt{\alpha} w_\varepsilon^i, t\sqrt{\beta} w_\varepsilon^i) = \frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)}$ uniformly in i . Similarly to Lemma 2.4, there is a sequence $\{s_{\max}^i\} \subset \mathbb{R}^+$ such that $(s_{\max}^i \sqrt{\alpha} w_\varepsilon^i, s_{\max}^i \sqrt{\beta} w_\varepsilon^i) \in \mathbf{N}_{\max}$ and

$$\theta_{\max} \leq I_{\max}(s_{\max}^i \sqrt{\alpha} w_\varepsilon^i, s_{\max}^i \sqrt{\beta} w_\varepsilon^i) = \sup_{t \geq 0} J_{\max}(t\sqrt{\alpha} u_\varepsilon^i, t\sqrt{\beta} u_\varepsilon^i) = \frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)}.$$

Let $\{(u_n, v_n)\} \subset \mathbf{N}_{\max}$ be a minimizing sequence of θ_{\max} for I_{\max} . It follows that $\|(u_n, v_n)\|_H^2 = \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dz$ and

$$\begin{aligned} \theta_{\max} &= \frac{1}{2} \|(u_n, v_n)\|_H^2 - \frac{1}{p} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dz + o_n(1) \\ &= \frac{p-2}{2p} \|(u_n, v_n)\|_H^2 + o_n(1). \end{aligned}$$

We may assume that $\|(u_n, v_n)\|_H^2 \rightarrow l$ and $\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dz \rightarrow l$ as $n \rightarrow \infty$, where $l = \frac{2p}{p-2} \theta_{\max} > 0$. By the definition of $S_{\alpha, \beta}$, then $S_{\alpha, \beta} l^{\frac{2}{p}} \leq l$. We can deduce that $S_{\alpha, \beta} \leq l^{\frac{p-2}{p}} = (\frac{2p}{p-2} \theta_{\max})^{\frac{p-2}{p}}$, that is, $\frac{p-2}{2p} (S_{\alpha, \beta})^{p/(p-2)} \leq \theta_{\max}$. \square

Lemma 4.4 *There exists a number $\delta_0 > 0$ such that if $(u, v) \in \mathbf{N}_\varepsilon$ and $I_\varepsilon(u, v) \leq \theta_{\max} + \delta_0$, then $Q_\varepsilon(u, v) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \varepsilon_1$.*

Proof On the contrary, there exist the sequences $\{\varepsilon_n\} \subset \mathbb{R}^+$ and $\{(u_n, v_n)\} \subset \mathbf{N}_{\varepsilon_n}$ such that $\varepsilon_n \rightarrow 0$, $I_{\varepsilon_n}(u_n, v_n) = \theta_{\max} (> 0) + o_n(1)$ as $n \rightarrow \infty$ and $Q_{\varepsilon_n}(u_n, v_n) \notin \mathbf{K}_{\rho_0/2}$ for all $n \in \mathbb{N}$. It is easy to check that $\{(u_n, v_n)\}$ is bounded in H . Suppose that $\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dz \rightarrow 0$ as $n \rightarrow \infty$. Since

$$\|(u_n, v_n)\|_H^2 = \int_{\mathbb{R}^N} f(\varepsilon_n z) |u_n|^\alpha |v_n|^\beta dz \quad \text{for each } n \in \mathbb{N},$$

then

$$\theta_{\max} + o_n(1) = I_{\varepsilon_n}(u_n, v_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f(\varepsilon_n z) |u_n|^\alpha |v_n|^\beta dz \leq o_n(1),$$

which is a contradiction. Thus, $\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta dz \rightarrow 0$ as $n \rightarrow \infty$. Similarly to the concentration-compactness principle (see Lions [15, 16] or Wang [6, Lemma 2.16]), then there exist a constant $c_0 > 0$ and a sequence $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that

$$\int_{B^N(\tilde{z}_n, 1)} |u_n|^{\frac{\alpha l}{p}} |v_n|^{\frac{\beta l}{p}} dz \geq c_0 > 0, \tag{4.3}$$

where $2 < l < p = \alpha + \beta < 2^*$ and $p = l(1 - t) + 2^*t$ for some $t \in ((N - 2)/N, 1)$. Let $(\tilde{u}_n(z), \tilde{v}_n(z)) = (u_n(z + \tilde{z}_n), v_n(z + \tilde{z}_n))$. Then there are a subsequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ and $(\tilde{u}, \tilde{v}) \in H$ such that $\tilde{u}_n \rightharpoonup \tilde{u}$ and $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in $H^1(\mathbb{R}^N)$. Using the similar computation of Lemma 2.4, there is a sequence $\{s_{\max}^n\} \subset \mathbb{R}^+$ such that $(s_{\max}^n \tilde{u}_n, s_{\max}^n \tilde{v}_n) \in \mathbf{N}_{\max}$ and

$$\begin{aligned} 0 < \theta_{\max} &\leq I_{\max}(s_{\max}^n \tilde{u}_n, s_{\max}^n \tilde{v}_n) = I_{\max}(s_{\max}^n u_n, s_{\max}^n v_n) \\ &\leq I_{\varepsilon_n}(s_{\max}^n u_n, s_{\max}^n v_n) \leq I_{\varepsilon_n}(u_n, v_n) = \theta_{\max} + o_n(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We deduce that a subsequence $\{s_{\max}^n\}$ satisfies $s_{\max}^n \rightarrow s_0 > 0$. Then there are a subsequence $\{(s_{\max}^n \tilde{u}_n, s_{\max}^n \tilde{v}_n)\}$ and $(s_0 \tilde{u}, s_0 \tilde{v}) \in H$ such that $s_{\max}^n \tilde{u}_n \rightharpoonup s_0 \tilde{u}$ and $s_{\max}^n \tilde{v}_n \rightharpoonup s_0 \tilde{v}$ weakly in $H^1(\mathbb{R}^N)$. By (4.3), then $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$. Applying Ekeland's variational principle, there exists a $(PS)_{\theta_{\max}}$ -sequence $\{(U_n, V_n)\}$ for I_{\max} and $\|(U_n - s_{\max}^n \tilde{u}_n, V_n - s_{\max}^n \tilde{v}_n)\|_H = o_n(1)$. Similarly to the proof of Lemma 3.3, there exist a subsequence $\{(U_n, V_n)\}$ and $(U_0, V_0) \in H$ such that $U_n \rightarrow U_0, V_n \rightarrow V_0$ strongly in $H^1(\mathbb{R}^N)$ and $I_{\max}(U_0, V_0) = \theta_{\max}$. Now, we want to show that there exists a subsequence $\{z_n\} = \{\varepsilon_n \tilde{z}_n\}$ such that $z_n \rightarrow z_0 \in \mathbf{K}$.

(i) Claim that the sequence $\{z_n\}$ is bounded in \mathbb{R}^N . On the contrary, assume that $|z_n| \rightarrow \infty$, then

$$\begin{aligned} \theta_{\max} &= I_{\max}(U_0, V_0) < \frac{1}{2} \|(U_0, V_0)\|_H^2 - \frac{1}{p} \int_{\mathbb{R}^N} f_\infty |U_0|^\alpha |V_0|^\beta dz \\ &\leq \liminf_{n \rightarrow \infty} \left[\frac{(s_{\max}^n)^2}{2} \|(\tilde{u}_n, \tilde{v}_n)\|_H^2 - \frac{(s_{\max}^n)^p}{p} \int_{\mathbb{R}^N} f(\varepsilon_n z + z_n) |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dz \right] \\ &= \liminf_{n \rightarrow \infty} \left[\frac{(s_{\max}^n)^2}{2} \|(u_n, v_n)\|_H^2 - \frac{(s_{\max}^n)^p}{p} \int_{\mathbb{R}^N} f(\varepsilon_n z) |u_n|^\alpha |v_n|^\beta dz \right] \\ &= \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(s_{\max}^n u_n, s_{\max}^n v_n) \leq \liminf_{n \rightarrow \infty} I_{\varepsilon_n}(u_n, v_n) = \theta_{\max}, \end{aligned}$$

which is a contradiction.

(ii) Claim that $z_0 \in \mathbf{K}$. On the contrary, assume that $z_0 \notin \mathbf{K}$, that is, $f(z_0) < 1 = \max_{z \in \mathbb{R}^N} f(z)$. Then use the argument of (i) to obtain that

$$\begin{aligned} \theta_{\max} &= I_{\max}(U_0, V_0) \leq I_{\max}(s_0 U_0, s_0 V_0) \\ &< \frac{(s_0)^2}{2} \|(U_0, V_0)\|_H^2 - \frac{(s_0)^p}{p} \int_{\mathbb{R}^N} f(z_0) |U_0|^\alpha |V_0|^\beta dz \\ &\leq \liminf_{n \rightarrow \infty} \left[\frac{(s_{\max}^n)^2}{2} \|(\tilde{u}_n, \tilde{v}_n)\|_H^2 - \frac{(s_{\max}^n)^p}{p} \int_{\mathbb{R}^N} f(\varepsilon_n z + z_n) |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta dz \right] \\ &\leq \theta_{\max}, \end{aligned}$$

which is a contradiction.

Since $\|(U_n - s_{\max}^n \tilde{u}_n, V_n - s_{\max}^n \tilde{v}_n)\|_H = o_n(1)$ and $U_n \rightarrow U_0, V_n \rightarrow V_0$ strongly in $H^1(\mathbb{R}^N)$, we have that

$$\begin{aligned} Q_{\varepsilon_n}(u_n, v_n) &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n z) |\tilde{u}_n(z - \tilde{z}_n)|^\alpha |\tilde{v}_n(z - \tilde{z}_n)|^\beta dz}{\int_{\mathbb{R}^N} |\tilde{u}_n(z - \tilde{z}_n)|^\alpha |\tilde{v}_n(z - \tilde{z}_n)|^\beta dz} \\ &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n z + \varepsilon_n \tilde{z}_n) |U_0|^\alpha |V_0|^\beta dz}{\int_{\mathbb{R}^N} |U_0|^\alpha |V_0|^\beta dz} \rightarrow z_0 \in \mathbf{K}_{\rho_0/2} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which is a contradiction.

Hence, there exists $\delta_0 > 0$ such that if $(u, v) \in \mathbf{N}_\varepsilon$ and $I_\varepsilon(u, v) \leq \theta_{\max} + \delta_0$, then $Q_\varepsilon(u, v) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \varepsilon_1$. \square

Lemma 4.5 *If $(u, v) \in \mathbf{M}_{\varepsilon, \lambda, \mu}$ and $J_{\varepsilon, \lambda, \mu}(u, v) \leq \theta_{\max} + \delta_0/2$, then there exists a number $\Lambda^* > 0$ such that $Q_\varepsilon(u, v) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \varepsilon_1$ and $0 < \lambda + \mu < \Lambda^*$.*

Proof Using the similar computation in Lemma 2.4, we obtain that there is the unique positive number

$$s_\varepsilon = \left(\frac{\|(u, v)\|_H^2}{\int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz} \right)^{1/(p-2)}$$

such that $(s_\varepsilon u, s_\varepsilon v) \in \mathbf{N}_\varepsilon$. We want to show that there exists $\Lambda_0 > 0$ such that if $0 < \lambda + \mu < \Lambda_0$, then $s_\varepsilon < c$ for some constant $c > 0$ (independent of u and v). First, for $(u, v) \in \mathbf{M}_{\varepsilon, \lambda, \mu}$,

$$0 < d_0 \leq \theta_{\varepsilon, \lambda, \mu} \leq J_{\varepsilon, \lambda, \mu}(u, v) \leq \theta_{\max} + \delta_0/2.$$

Since $\langle J'_{\varepsilon, \lambda, \mu}(u, v), (u, v) \rangle = 0$, then

$$\begin{aligned} \theta_{\max} + \delta_0/2 &\geq J_{\varepsilon, \lambda, \mu}(u, v) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \|(u, v)\|_H^2 + \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz \\ &\geq \frac{q-2}{2q} \|(u, v)\|_H^2, \quad \text{that is, } \|(u, v)\|_H^2 \leq c_1 = \frac{2q}{q-2} (\theta_{\max} + \delta_0/2), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} d_0 &\leq J_{\varepsilon, \lambda, \mu}(u, v) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \|(u, v)\|_H^2 - \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\Omega} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz \\ &\leq \frac{p-2}{2p} \|(u, v)\|_H^2, \quad \text{that is, } \|(u, v)\|_H^2 \geq c_2 = \frac{2p}{p-2} d_0. \end{aligned} \tag{4.5}$$

Moreover, we have that

$$\begin{aligned} \int_{\Omega} f(\varepsilon z) |u|^\alpha |v|^\beta dz &= \|(u, v)\|_H^2 - \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz \\ &\geq c_2 - \text{Max } S_p^{-\frac{q}{2}} (\lambda + \mu) c_1^{q/2}, \end{aligned}$$

where $\text{Max} = \max\{\|g\|_m, \|h\|_m\}$. It follows that there exists $\Lambda_0 > 0$ such that for $0 < \lambda + \mu < \Lambda_0$

$$\int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz \geq c_2 - \text{Max} S_p^{-\frac{q}{2}} (\lambda + \mu) (c_1)^{q/2} > 0. \tag{4.6}$$

Hence, by (4.4), (4.5) and (4.6), $s_\varepsilon < c$ for some constant $c > 0$ (independent of u and v) for $0 < \lambda + \mu < \Lambda_0$. Now, we obtain that

$$\begin{aligned} \theta_{\max} + \delta_0/2 &\geq J_{\varepsilon,\lambda,\mu}(u, v) = \sup_{t \geq 0} J_{\varepsilon,\lambda,\mu}(tu, tv) \geq J_{\varepsilon,\lambda,\mu}(s_\varepsilon u, s_\varepsilon v) \\ &= \frac{1}{2} \|(s_\varepsilon u, s_\varepsilon v)\|_H^2 - \frac{1}{p} \int_{\mathbb{R}^N} f(\varepsilon z) |s_\varepsilon u|^\alpha |s_\varepsilon v|^\beta dz \\ &\quad - \frac{1}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |s_\varepsilon u|^q + \mu h(\varepsilon z) |s_\varepsilon v|^q) dz \\ &\geq I_\varepsilon(s_\varepsilon u, s_\varepsilon v) - \frac{1}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |s_\varepsilon u|^q + \mu h(\varepsilon z) |s_\varepsilon v|^q) dz. \end{aligned}$$

From the above inequality, we deduce that for any $0 < \varepsilon < \varepsilon_1$ and $0 < \lambda + \mu < \Lambda_0$,

$$\begin{aligned} I_\varepsilon(s_\varepsilon u, s_\varepsilon v) &\leq \theta_{\max} + \delta_0/2 + \frac{1}{q} \int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |s_\varepsilon u|^q + \mu h(\varepsilon z) |s_\varepsilon v|^q) dz \\ &\leq \theta_{\max} + \delta_0/2 + \text{Max}(\lambda + \mu) S_p^{-\frac{q}{2}} \|(s_\varepsilon u, s_\varepsilon v)\|_H^q \\ &< \theta_{\max} + \delta_0/2 + \text{Max} S_p^{-\frac{q}{2}} (\lambda + \mu) c^q (c_1)^{q/2}. \end{aligned}$$

Hence, there exists $\Lambda^* \in (0, \Lambda_0)$ such that for $0 < \lambda + \mu < \Lambda^*$,

$$I_\varepsilon(s_\varepsilon u, s_\varepsilon v) \leq \theta_{\max} + \delta_0, \quad \text{where } (s_\varepsilon u, s_\varepsilon v) \in \mathbf{N}_\varepsilon.$$

By Lemma 4.4, we obtain

$$Q_\varepsilon(s_\varepsilon u, s_\varepsilon v) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z) |s_\varepsilon u|^\alpha |s_\varepsilon v|^\beta dz}{\int_{\mathbb{R}^N} |s_\varepsilon u|^\alpha |s_\varepsilon v|^\beta dz} \in \mathbf{K}_{\rho_0/2},$$

or $Q_\varepsilon(u, v) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$. □

Since $f_\infty < 1$, then by Lemma 4.3,

$$\theta_{\max} = \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)} < \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}. \tag{4.7}$$

By Lemmas 4.1, 4.2 and (4.7), for any $0 < \varepsilon < \varepsilon_0$ ($< \varepsilon_1$) and $0 < \lambda + \mu < \Lambda^*$,

$$\beta_{\varepsilon,\lambda,\mu}^i \leq J_{\varepsilon,\lambda,\mu}(t_\varepsilon^i \sqrt{\alpha} w_\varepsilon^i, t_\varepsilon^i \sqrt{\beta} w_\varepsilon^i) < \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}. \tag{4.8}$$

Applying above Lemma 4.5, we get that

$$\tilde{\beta}_{\varepsilon,\lambda,\mu}^i \geq \theta_{\max} + \delta_0/2 \quad \text{for any } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \lambda + \mu < \Lambda^*. \tag{4.9}$$

For each $1 \leq i \leq k$, by (4.8) and (4.9), we obtain that

$$\beta_{\varepsilon,\lambda,\mu}^i < \tilde{\beta}_{\varepsilon,\lambda,\mu}^i \quad \text{for any } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \lambda + \mu < \Lambda^*.$$

It follows that

$$\beta_{\varepsilon,\lambda,\mu}^i = \inf_{(u,v) \in O_{\varepsilon,\lambda,\mu}^i \cup \partial O_{\varepsilon,\lambda,\mu}^i} J_{\varepsilon,\lambda,\mu}(u,v) \quad \text{for any } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \lambda + \mu < \Lambda^*.$$

Then applying Ekeland's variational principle and using the standard computation, we have the following lemma.

Lemma 4.6 *For each $1 \leq i \leq k$, there is a $(PS)_{\beta_{\varepsilon,\lambda,\mu}^i}$ -sequence $\{(u_n, v_n)\} \subset O_{\varepsilon,\lambda,\mu}^i$ in H for $J_{\varepsilon,\lambda,\mu}$.*

Proof See Cao-Zhou [8]. □

Proof of Theorem 1.2 For any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$, by Lemma 4.6, there is a $(PS)_{\beta_{\varepsilon,\lambda,\mu}^i}$ -sequence $\{(u_n, v_n)\} \subset O_{\varepsilon,\lambda,\mu}^i$ for $J_{\varepsilon,\lambda,\mu}$ where $1 \leq i \leq k$. By (4.8), we obtain that

$$\beta_{\varepsilon,\lambda,\mu}^i < \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}.$$

Since $J_{\varepsilon,\lambda,\mu}$ satisfies the $(PS)_\gamma$ -condition for $\gamma \in (-\infty, \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}})$, then $J_{\varepsilon,\lambda,\mu}$ has at least k critical points in $M_{\varepsilon,\lambda,\mu}$ for any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$. Set $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. Replace the terms $\int_{\mathbb{R}^N} f(\varepsilon z) |u|^\alpha |v|^\beta dz$ and $\int_{\mathbb{R}^N} (\lambda g(\varepsilon z) |u|^q + \mu h(\varepsilon z) |v|^q) dz$ of the functional $J_{\varepsilon,\lambda,\mu}$ by $\int_{\mathbb{R}^N} f(\varepsilon z) u_+^\alpha v_+^\beta dz$ and $\int_{\mathbb{R}^N} (\lambda g(\varepsilon z) u_+^q + \mu h(\varepsilon z) v_+^q) dz$, respectively. It follows that $(E_{\varepsilon,\lambda,\mu})$ has k nonnegative solutions. Applying the maximum principle, $(E_{\varepsilon,\lambda,\mu})$ admits at least k positive solutions. □

Competing interests

The author declares that he has no competing interests.

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