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Study on integro-differential equation with generalized p-Laplacian operator

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Abstract

We tackle the existence and uniqueness of the solution for a kind of integro-differential equations involving the generalized p-Laplacian operator with mixed boundary conditions. This is achieved by using some results on the ranges for maximal monotone operators and pseudo-monotone operators. The method used in this paper extends and complements some previous work.

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1 Introduction

Nonlinear boundary value problems (BVPs) involving the p-Laplacian operator $-\Delta_p$ arise from a variety of physical phenomena such as non-Newtonian fluids, reaction-diffusion problems, petroleum extraction, flow through porous media, *etc.* Thus, the study of such problems and their generalizations have attracted numerous attention in recent years. Some of the BVPs studied in the literature include the following:

$$\begin{cases} -\Delta_p u + g(x, u(x)) = f(x), & \text{a.e. in } \Omega, \\ -\frac{\partial u}{\partial n} = 0, & \text{a.e. on } \Gamma \end{cases} \tag{1.1}$$

whose existence results in $L^p(\Omega)$ (for various ranges of p) can be found in [1–4]; a related BVP

$$\begin{cases} -\Delta_p u + g(x, u(x)) = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, |\nabla u|^{p-2}\nabla u\rangle \in \beta_x(u(x)), & \text{a.e. on } \Gamma \end{cases} \tag{1.2}$$

was tackled in [5–7] and later generalized to one that contains a perturbation term $|u|^{p-2}u$ [8, 9]

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + g(x, u(x)) = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, |\nabla u|^{p-2}\nabla u\rangle \in \beta_x(u(x)), & \text{a.e. on } \Gamma. \end{cases} \tag{1.3}$$

Motivated by Tolksdorf's work [10] where the following Dirichlet BVP has been discussed:

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] = f(x), & \text{a.e. in } K(1,S), \\ u = g, & \text{a.e. in } \Sigma(1,S), \end{cases} \tag{1.4}$$

several generalizations have been investigated. These include [11–14]

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + |u|^{p-2}u + g(x,u(x)) = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, (C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle = 0, & \text{a.e. on } \Gamma, \end{cases} \tag{1.5}$$

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + |u|^{p-2}u + g(x,u(x)) = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, (C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \beta_x(u(x)), & \text{a.e. on } \Gamma \end{cases} \tag{1.6}$$

and

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + \varepsilon|u|^{q-2}u + g(x,u(x)) = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, (C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \beta_x(u(x)), & \text{a.e. on } \Gamma, \end{cases} \tag{1.7}$$

where $0 \le C(x) \in L^p(\Omega)$, ε is a nonnegative constant and ϑ denotes the exterior normal derivative of Γ.

Inspired by all this research, recently we have studied the following nonlinear parabolic equation with mixed boundary conditions [15]:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + \varepsilon|u|^{p-2}u = f(x,t), & (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \beta(u) - h(x,t), & (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), & \text{a.e. } x \in \Omega. \end{cases} \tag{1.8}$$

We tackle the existence of solutions for (1.8) via the study of existence of solutions for two BVPs: (i) the elliptic equation with Dirichlet boundary conditions

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + \varepsilon|u|^{q-2}u = f(x), & \text{a.e. in } \Omega, \\ \gamma u = w, & \text{a.e. on } \Gamma \end{cases} \tag{1.9}$$

and (ii) the elliptic equation with Neumann boundary conditions

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] + \varepsilon|u|^{q-2}u = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, (C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \beta(u) - h(x), & \text{a.e. in } \Gamma. \end{cases} \tag{1.10}$$

By setting up the relations between the auxiliary equations (1.9) and (1.10) and by employing some results on ranges for maximal monotone operators, we showed that (1.8) has a unique solution in $L^p(0,T;W^{1,p}(\Omega))$, where $2 \le p < +\infty$, $1 \le q < +\infty$ if $p \ge N$, and $1 \le q \le \frac{2N-p}{N-p}$ if $p < N$.

In this paper, we shall employ the technique used in (1.8), viz. using the results on ranges for nonlinear operators, to study the existence and uniqueness of the solution to a nonlinear *integro-differential equation* with the generalized p-Laplacian operator. We note that most of the existing methods in the literature used to investigate such problems are based on the finite element method, hence our technique is *new* in tackling integro-differential equations. We shall consider the following nonlinear integro-differential problem with mixed boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon |u|^{q-2}u + a\frac{\partial}{\partial t}\int_\Omega u\,dx \\ \quad = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\rangle \in \beta_x(u), \quad (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), \quad x \in \Omega. \end{cases} \tag{1.11}$$

Our discussion is based on some results on the ranges for maximal monotone operators and pseudo-monotone operators in [16–18]. Some new methods of constructing appropriate mappings to achieve our goal are employed. Moreover, we weaken the restrictions on p and q. The paper is outlined as follows. In Section 2 we shall state the definitions and results needed, and in Section 3 we shall establish the existence and uniqueness of the solution to (1.11).

2 Preliminaries

Let X be a real Banach space with a strictly convex dual space X^*. We use $(\cdot,\cdot)$ to denote the generalized duality pairing between X and X^*. For a subset C of X, we use $\operatorname{Int} C$ to denote the interior of C. We also use '$\to$' and 'w-lim' to denote strong and weak convergences, respectively.

Let X and Y be Banach spaces. We use $X \hookrightarrow Y$ to denote that X is embedded continuously in Y.

The function Φ is called a *proper convex function* on X [17] if Φ is defined from X to $(-\infty,+\infty]$, Φ is not identically $+\infty$ such that $\Phi((1-\lambda)x+\lambda y) \le (1-\lambda)\Phi(x)+\lambda\Phi(y)$, whenever $x,y \in X$ and $0 \le \lambda \le 1$.

The function $\Phi : X \to (-\infty,+\infty]$ is said to be *lower-semicontinuous* on X [17] if $\liminf_{y\to x}\Phi(y) \ge \Phi(x)$ for any $x \in X$.

Given a proper convex function Φ on X and a point $x \in X$, we denote by $\partial\Phi(x)$ the set of all $x^* \in X^*$ such that $\Phi(x) \le \Phi(y) + (x-y, x^*)$ for every $y \in X$. Such elements x^* are called *subgradients* of Φ at x, and $\partial\Phi(x)$ is called the *subdifferential* of Φ at x [17].

A mapping $T : D(T) = X \to X^*$ is said to be *demi-continuous* on X if $w\text{-}\lim_{n\to\infty} Tx_n = Tx$ for any sequence $\{x_n\}$ strongly convergent to x in X. A mapping $T : D(T) = X \to X^*$ is said to be *hemi-continuous* on X if $w\text{-}\lim_{t\to 0} T(x+ty) = Tx$ for any $x,y \in X$ [17].

With each multi-valued mapping $A : X \to 2^X$, we associate the subset A^0 as follows [17]:

$$A^0x = \{y \in Ax : \|y\| = |Ax|\},$$

where $|Ax| := \inf\{\|z\| : z \in Ax\}$. If X^* is strictly convex, then $D(A) = D(A^0)$ and A^0 is single-valued, which in this case is called the *minimal section* of A.

A multi-valued mapping $B : X \to 2^{X^*}$ is said to be *monotone* [18] if its graph $G(B)$ is a monotone subset of $X \times X^*$ in the sense that $(u_1 - u_2, w_1 - w_2) \ge 0$ for any $[u_i, w_i] \in G(B)$, $i = 1,2$. The monotone operator B is said to be *maximal monotone* if $G(B)$ is not properly contained in any other monotone subsets of $X \times X^*$.

Definition 2.1 [18] Let C be a closed convex subset of X, and let $A : C \to 2^{X^*}$ be a multi-valued mapping. Then A is said to be a *pseudo-monotone* operator provided that

(i) for each $x \in C$, the image Ax is a nonempty closed and convex subset of X^*;

(ii) if $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and if $f_n \in Ax_n$ is such that $\limsup_{n\to\infty}(x_n - x, f_n) \leq 0$, then to each element $y \in C$, there corresponds an $f(y) \in Ax$ with the property that

$$\bigl(x - y, f(y)\bigr) \leq \liminf_{n\to\infty}(x_n - x, f_n);$$

(iii) for each finite-dimensional subspace F of X, the operator A is continuous from $C \cap F$ to X^* in the weak topology.

Lemma 2.1 [19] *Let Ω be a bounded conical domain in $\mathbb{R}^N$. If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $0 < mp \leq N$ and $q = \frac{Np}{N-mp}$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; if $mp = N$ and $p > 1$, then for $1 \leq q < +\infty$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.*

Lemma 2.2 [18] *If $B : X \to 2^{X^*}$ is an everywhere defined, monotone, and hemi-continuous operator, then B is maximal monotone. If $B : X \to 2^{X^*}$ is a maximal monotone operator such that $D(B) = X$, then B is pseudo-monotone.*

Lemma 2.3 [18] *If X is a Banach space and $\Phi : X \to (-\infty, +\infty]$ is a proper convex and lower-semicontinuous function, then $\partial\Phi$ is maximal monotone from X to X^*.*

Lemma 2.4 [18] *If B_1 and B_2 are two maximal monotone operators in X such that $\operatorname{int} D(B_1) \cap D(B_2) \neq \emptyset$, then $B_1 + B_2$ is maximal monotone.*

Lemma 2.5 [20] *Let X and its dual X^* be strictly convex Banach spaces. Suppose $S : D(S) \subset X \to X^*$ is a closed linear operator and S^* is the conjugate operator of S. If $(u, Su) \geq 0$ $\forall u \in D(S)$ and $(v, S^*v) \geq 0$ $\forall v \in D(S^*)$, then S is a maximal monotone operator possessing a dense domain.*

Lemma 2.6 [18] *Any hemi-continuous mapping $T : X \to X^*$ is demi-continuous on $\operatorname{Int} D(T)$.*

Theorem 2.1 [16] *Let X be a real reflexive Banach space with X^* being its dual space. Let C be a nonempty closed convex subset of X. Assume that*

(i) *the mapping $A : C \to 2^{X^*}$ is a maximal monotone operator*;

(ii) *the mapping $B : C \to X^*$ is pseudo-monotone, bounded, and demi-continuous*;

(iii) *if the subset C is unbounded, then the operator B is A-coercive with respect to the fixed element $b \in X^*$, i.e., there exists an element $u_0 \in C \cap D(A)$ and a number $r > 0$ such that $(u - u_0, Bu) > (u - u_0, b)$ for all $u \in C$ with $\|u\| > r$.*

Then the equation $b \in Au + Bu$ has a solution.

3 Existence and uniqueness of the solution to (1.11)

We begin by stating some notations and assumptions used in this paper. Throughout, we shall assume that

$$1 < q \leq p < +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Let $V = L^p(0,T;W^{1,p}(\Omega))$ and V^* be the dual space of V. The duality pairing between V and V^* will be denoted by $\langle\langle\cdot,\cdot\rangle\rangle_V$. The norm in V will be denoted by $\|\cdot\|_V$, which is defined by

$$\|u\|_V = \left(\int_0^T \|u(t)\|^p_{W^{1,p}(\Omega)}\,dt\right)^{\frac{1}{p}}, \quad \forall u(x,t)\in V.$$

Let $W = L^q(0,T;W^{1,p}(\Omega))$ and W^* be the dual space of W. The norm in W will be denoted by $\|\cdot\|_W$, which is defined by

$$\|v\|_W = \left(\int_0^T \|v(t)\|^q_{W^{1,p}(\Omega)}\,dt\right)^{\frac{1}{q}}, \quad \forall v(x,t)\in W.$$

In the integro-differential equation (1.11), Ω is a bounded conical domain of a Euclidean space $\mathbb{R}^N$ where $N\geq 1$, Γ is the boundary of Ω with $\Gamma\in C^1$ [5], ϑ denotes the exterior normal derivative to Γ. Here, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean norm and the inner-product in $\mathbb{R}^N$, respectively. Also, $0\leq C(x,t)\in L^p(0,T;W^{1,p}(\Omega)), f(x,t)\in V^*$ is a given function, T and a are positive constants, and ε is a nonnegative constant. Moreover, β_x is the subdifferential of φ_x, where $\varphi_x = \varphi(x,\cdot):\mathbb{R}\to\mathbb{R}$ for $x\in\Gamma$, and $\varphi:\Gamma\times\mathbb{R}\to\mathbb{R}$ is a given function.

To tackle (1.11), we need the following assumptions which can be found in [5, 14].

Assumption 1 *Green's formula is available.*

Assumption 2 *For each $x\in\Gamma$, $\varphi_x = \varphi(x,\cdot):\mathbb{R}\to\mathbb{R}$ is a proper, convex, and lower-semicontinuous function and $\varphi_x(0)=0$.*

Assumption 3 $0\in\beta_x(0)$ *and for each $t\in\mathbb{R}$, the function $x\in\Gamma\to(I+\lambda\beta_x)^{-1}(t)\in\mathbb{R}$ is measurable for $\lambda>0$.*

We shall present a series of lemmas before we prove the main result.

Lemma 3.1 *Define the function $\Phi: V\to\mathbb{R}$ by*

$$\Phi(u) = \int_0^T\int_\Gamma \varphi_x\big(u|_\Gamma(x,t)\big)\,d\Gamma(x)\,dt, \quad \forall u\in V.$$

Then Φ is a proper, convex, and lower-semicontinuous mapping on V. Therefore, $\partial\Phi: V\to V^*$, the subdifferential of Φ, is maximal monotone.

Proof The proof of this lemma is analogous to that of Lemma 3.1 in [1]. We give the outline of the proof as follows.

Note that for each $s\in\mathbb{R}$, the function $x\in\Gamma\to\beta_x^0(s)\in\mathbb{R}$ is measurable, where $\beta_x^0(s)$ denotes the minimal section of β_x. Since for all $s_1,s_2\in\mathbb{R}$ we have

$$\big\{x\in\Gamma:\varphi_x(s_1)>s_2\big\} = \bigcup_n\left\{x\in\Gamma:\sum_{i=1}^n\frac{s_1}{n}\beta_x^0\left(\frac{is_1}{n}\right)>s_2\right\},$$

it implies that for $u\in V$, the function $\varphi_x(u|_\Gamma(x,t))$ is measurable on Γ. Then from the property of φ_x, we know that Φ is proper and convex on V.

To see that Φ is lower-semicontinuous on V, let $u_n \to u$ in V. We may assume that there exists a subsequence of u_n, for simplicity, we still denote it by u_n, such that $u_n|_\Gamma(x,t) \to u|_\Gamma(x,t)$ for $x \in \Gamma$ and $t \in (0,T)$ a.e. This yields

$$\varphi_x\big(u|_\Gamma(x,t)\big) \le \liminf_{n\to\infty} \varphi_x\big(u_n|_\Gamma(x,t)\big)$$

for all $x \in \Gamma$ and each $t \in (0,T)$ a.e. since φ_x is lower-semicontinuous for each $x \in \Gamma$. It then follows from Fatou's lemma that for each $t \in (0,T)$,

$$\begin{aligned}\int_\Gamma \varphi_x\big(u|_\Gamma(x,t)\big)\,d\Gamma(x) &\le \int_\Gamma \liminf_{n\to\infty} \varphi_x\big(u_n|_\Gamma(x,t)\big)\,d\Gamma(x) \\ &\le \liminf_{n\to\infty} \int_\Gamma \varphi_x\big(u_n|_\Gamma(x,t)\big)\,d\Gamma(x).\end{aligned}$$

So, $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$ whenever $u_n \to u$ in V. This completes the proof. □

Lemma 3.2 *Define* $S : D(S) = \{u \in V : \frac{\partial u}{\partial t} \in V^*, u(x,0) = u(x,T)\} \to V^*$ *by*

$$Su = \frac{\partial u}{\partial t} + a\frac{\partial}{\partial t}\int_\Omega u\,dx.$$

Then S is a linear maximal monotone operator possessing a dense domain in V.

Proof It is obvious that S is closed and linear.

For $u(x,t), w(x,t) \in D(S)$, integrating by parts gives

$$\begin{aligned}&\langle\langle w, Su\rangle\rangle_V + \left\langle\!\!\left\langle u, \frac{\partial w}{\partial t} + a\frac{\partial}{\partial t}\int_\Omega w\,dx\right\rangle\!\!\right\rangle_V \\ &\quad= \int_0^T\int_\Omega \frac{\partial u}{\partial t} w(x,t)\,dx\,dt + a\int_0^T\int_\Omega\left(\frac{\partial}{\partial t}\int_\Omega u\,dx\right) w(x,t)\,dx\,dt \\ &\qquad+ \int_0^T\int_\Omega u(x,t)\frac{\partial w}{\partial t}\,dx\,dt + a\int_0^T\int_\Omega\left(\frac{\partial}{\partial t}\int_\Omega w\,dx\right) u(x,t)\,dx\,dt \\ &\quad= \int_\Omega u(x,T)w(x,T)\,dx - \int_\Omega u(x,0)w(x,0)\,dx \\ &\qquad+ a\int_\Omega u(x,T)\,dx\int_\Omega w(x,T)\,dx - a\int_\Omega u(x,0)\,dx\int_\Omega w(x,0)\,dx = 0.\end{aligned}$$

Then $S^* w = -\frac{\partial w}{\partial t} - a\frac{\partial}{\partial t}\int_\Omega w\,dx$, where $D(S^*) = \{w \in V : \frac{\partial w}{\partial t} \in V^*, w(x,0) = w(x,T)\}$.

For $u(x,t) \in D(S)$, we find

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial u}{\partial t}u(x,t)\,dx\,dt &= \int_\Omega \big|u(x,T)\big|^2\,dx - \int_\Omega\big|u(x,0)\big|^2\,dx - \int_0^T\int_\Omega \frac{\partial u}{\partial t}u(x,t)\,dx\,dt \\ &= -\int_0^T\int_\Omega \frac{\partial u}{\partial t}u(x,t)\,dx\,dt,\end{aligned}$$

which implies that

$$\int_0^T\int_\Omega \frac{\partial u}{\partial t}u(x,t)\,dx\,dt = 0.$$

Similarly, for $u(x,t) \in D(S)$,

$$a\int_0^T \int_\Omega u(x,t)\left(\frac{\partial}{\partial t}\int_\Omega u\,dx\right)dx\,dt$$
$$= a\left(\int_\Omega u(x,T)\,dx\right)^2 - a\left(\int_\Omega u(x,0)\,dx\right)^2 - a\int_0^T \int_\Omega u(x,t)\left(\frac{\partial}{\partial t}\int_\Omega u\,dx\right)dx\,dt,$$

which implies that

$$a\int_0^T \int_\Omega u(x,t)\left(\frac{\partial}{\partial t}\int_\Omega u\,dx\right)dx\,dt = 0.$$

Thus,

$$\langle\langle u, Su\rangle\rangle_V = \int_0^T \int_\Omega \frac{\partial u}{\partial t} u(x,t)\,dx\,dt + a\int_0^T \int_\Omega u(x,t)\left(\frac{\partial}{\partial t}\int_\Omega u\,dx\right)dx\,dt = 0.$$

In the same manner, we have $\langle\langle w, S^* w\rangle\rangle_V = 0$ for $w \in D(S^*)$. Therefore, noting Lemma 2.5 the result follows. □

In view of Lemmas 2.3 and 2.4, we have the following result.

Lemma 3.3 *$S + \partial\Phi : V \to V^*$ is maximal monotone.*

Lemma 3.4 [14] *Define the mapping $B_{p,q} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ as follows:*

$$(\overline{v}, B_{p,q}\overline{u}) = \int_\Omega \left\langle \left(C(x,t) + |\nabla\overline{u}|^2\right)^{\frac{p-2}{2}} \nabla\overline{u}, \nabla\overline{v}\right\rangle dx + \varepsilon\int_\Omega |\overline{u}|^{q-2}\overline{u}\overline{v}\,dx, \quad \forall \overline{u}, \overline{v} \in W^{1,p}(\Omega).$$

Then $B_{p,q}$ is maximal monotone.

Lemma 3.5 [14] *Let X_0 denote the closed subspace of all constant functions in $W^{1,p}(\Omega)$. Let X be the quotient space $\frac{W^{1,p}(\Omega)}{X_0}$. For $\overline{u} \in W^{1,p}(\Omega)$, define the mapping $P : W^{1,p}(\Omega) \to X_0$ by*

$$P\overline{u} = \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega \overline{u}\,dx.$$

Then, there is a constant $C > 0$ such that for every $\overline{u} \in W^{1,p}(\Omega)$,

$$\|\overline{u} - P\overline{u}\|_{L^p(\Omega)} \le C\|\nabla\overline{u}\|_{(L^p(\Omega))^N}.$$

Here $\mathrm{meas}(\Omega)$ *denotes the measure of* Ω.

Definition 3.1 Define $A : V \to V^*$ as follows:

$$\langle\langle v, Au\rangle\rangle_V = \int_0^T (v, B_{p,q}u)\,dt - \int_0^T \int_\Omega f(x,t)v(x,t)\,dx\,dt, \quad \forall u, v \in V.$$

Lemma 3.6 *The mapping $A : V \to V^*$ is everywhere defined, bounded, monotone, and hemi-continuous. Therefore, Lemma 2.2 implies that it is also pseudo-monotone.*

Proof From Lemma 2.1, we know that $W^{1,p}(\Omega) \hookrightarrow C_B(\Omega)$ when $p > N$, and $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ when $p = N$. If $p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-p}}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^q(\Omega)$ since $1 < q \le p < +\infty$. Thus, for all $\overline{w} \in W^{1,p}(\Omega)$, $\|\overline{w}\|_{L^q(\Omega)} \le k\|\overline{w}\|_{W^{1,p}(\Omega)}$, where $k > 0$ is a constant. Therefore, for $u, v \in V$, we have

$$\int_0^T \|u\|^q_{L^q(\Omega)}\,dt \le \text{const} \cdot \int_0^T \|u\|^q_{W^{1,p}(\Omega)}\,dt = \text{const} \cdot \|u\|^q_W$$

and

$$\int_0^T \|v\|^q_{L^q(\Omega)}\,dt \le \text{const} \cdot \int_0^T \|v\|^q_{W^{1,p}(\Omega)}\,dt = \text{const} \cdot \|v\|^q_W.$$

Moreover, since $1 < q \le p < +\infty$, then $L^p(0,T;W^{1,p}(\Omega)) \hookrightarrow L^q(0,T;W^{1,p}(\Omega))$, which implies that $\|u\|_W \le \|u\|_V$ and $\|v\|_W \le \|v\|_V$ for $u, v \in V$.

If $p \ge 2$, then for $u, v \in V$, we have

$$\begin{aligned}
&\bigl|\langle\langle v, Au\rangle\rangle_V\bigr| \\
&\quad\le \int_0^T \int_\Omega \bigl|C(x,t) + |\nabla u|^2\bigr|^{\frac{p-2}{2}} |\nabla u| \cdot |\nabla v|\,dx\,dt \\
&\qquad + \varepsilon \int_0^T \int_\Omega |u|^{q-1}|v|\,dx\,dt + \int_0^T \int_\Omega |f| \cdot |v|\,dx\,dt \\
&\quad\le \int_0^T \int_\Omega \bigl|2\max\bigl(C(x,t), |\nabla u|^2\bigr)\bigr|^{\frac{p-2}{2}} |\nabla u| \cdot |\nabla v|\,dx\,dt \\
&\qquad + \text{const} \cdot \varepsilon \|v\|_W \|u\|_W^{\frac{q}{q'}} + \|f\|_{V^*}\|v\|_V \\
&\quad\le 2^{\frac{p-2}{2}} \int_0^T \int_\Omega C(x,t)^{\frac{p-2}{2}} |\nabla u| \cdot |\nabla v|\,dx\,dt + 2^{\frac{p-2}{2}} \|u\|_V^{\frac{p}{p'}} \|v\|_V \\
&\qquad + \text{const} \cdot \varepsilon \|v\|_W \|u\|_W^{\frac{q}{q'}} + \|f\|_{V^*}\|v\|_V \\
&\quad\le 2^{\frac{p-2}{2}} \left(\int_0^T \int_\Omega C(x,t)^{\frac{p-2}{2}p'} |\nabla v|^{p'}\,dx\,dt\right)^{\frac{1}{p'}} \|u\|_V + 2^{\frac{p-2}{2}} \|u\|_V^{\frac{p}{p'}} \|v\|_V \\
&\qquad + \text{const} \cdot \varepsilon \|v\|_W \|u\|_W^{\frac{q}{q'}} + \|f\|_{V^*}\|v\|_V \\
&\quad\le 2^{\frac{p-2}{2}} \bigl\|C(x,t)\bigr\|_V^{p-2} \|u\|_V \|v\|_V + 2^{\frac{p-2}{2}} \|u\|_V^{\frac{p}{p'}} \|v\|_V + \text{const} \cdot \varepsilon \|v\|_V \|u\|_V^{\frac{q}{q'}} + \|f\|_{V^*}\|v\|_V,
\end{aligned}$$

which implies that A is everywhere defined and bounded.

If $1 < p < 2$, then for $u, v \in V$, we have

$$\begin{aligned}
&\bigl|\langle\langle v, Au\rangle\rangle_V\bigr| \\
&\quad\le \int_0^T \int_\Omega \bigl|C(x,t) + |\nabla u|^2\bigr|^{\frac{p-2}{2}} |\nabla u| \cdot |\nabla v|\,dx\,dt \\
&\qquad + \varepsilon \int_0^T \int_\Omega |u|^{q-1}|v|\,dx\,dt + \int_0^T \int_\Omega |f| \cdot |v|\,dx\,dt \\
&\quad= \int_0^T \int_\Omega \frac{|\nabla u| \cdot |\nabla v|}{|C(x,t) + |\nabla u|^2|^{\frac{2-p}{2}}}\,dx\,dt + \text{const} \cdot \varepsilon \|v\|_W \|u\|_W^{\frac{q}{q'}} + \|f\|_{V^*}\|v\|_V
\end{aligned}$$

$$\begin{aligned}&\le \int_0^T \int_\Omega \frac{|\nabla u| \cdot |\nabla v|}{|\nabla u|^{2-p}}\,dx\,dt + \mathrm{const} \cdot \varepsilon \|v\|_W \|u\|_W^{\frac{q}{q'}} + \|f\|_{V^*}\|v\|_V \\ &\le \|u\|_V^{\frac{p}{p'}} \|v\|_V + \mathrm{const} \cdot \varepsilon \|v\|_V \|u\|_V^{\frac{q}{q'}} + \|f\|_{V^*}\|v\|_V,\end{aligned}$$

which also implies that A is everywhere defined and bounded.

Since $B_{p,q}$ is monotone, we can easily see that for $u, v \in V$,

$$\langle\langle u - v, Au - Av\rangle\rangle_V = \int_0^T (u - v, B_{p,q}u - B_{p,q}v)\,dt \ge 0,$$

which implies that A is monotone.

To show that A is hemi-continuous, it suffices to show that for any $u, v, w \in V$ and $k \in [0,1]$, $\langle\langle w, A(u + kv) - Au\rangle\rangle_V \to 0$, as $k \to 0$. Noting the fact that $B_{p,q}$ is hemi-continuous and using the Lebesgue's dominated convergence theorem, we have

$$0 \le \lim_{k\to 0}\bigl|\langle\langle w, A(u + kv) - Au\rangle\rangle_V\bigr| \le \int_0^T \lim_{k\to 0}\bigl|\bigl(w, B_{p,q}(u + kv) - B_{p,q}u\bigr)\bigr|\,dt = 0.$$

Hence, A is hemi-continuous.

This completes the proof. □

Lemma 3.7 *The mapping $A : V \to V^*$ satisfies that for $u \in D(S)$,*

$$\frac{\langle\langle u - u_0, Au\rangle\rangle_V}{\|u\|_V} \to +\infty, \tag{3.1}$$

as $\|u\|_V \to +\infty$ in V.

Proof First, we shall show that for $u \in V$,

$$\|u\|_V \to +\infty$$

is equivalent to

$$\left\| u - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega u\,dx \right\|_V \to +\infty.$$

In fact, from Lemma 3.5, we know that for $u \in V$,

$$\left\| u - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega u\,dx \right\|_{L^p(\Omega)} \le C\|\nabla u\|_{(L^p(\Omega))^N},$$

where C is a positive constant. Thus,

$$\begin{aligned}&\left\| u - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega u\,dx \right\|_{W^{1,p}(\Omega)}^p \\ &\quad = \left\| u - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega u\,dx \right\|_{L^p(\Omega)}^p + \left\| \nabla\left(u - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega u\,dx \right)\right\|_{(L^p(\Omega))^N}^p \\ &\quad \le \bigl(C^p + 1\bigr)\|\nabla u\|_{(L^p(\Omega))^N}^p,\end{aligned}$$

which implies that

$$\left\| u - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\,dx \right\|_V \le \left[\left(C^p + 1\right) \int_0^T \|\nabla u\|^p_{(L^p(\Omega))^N}\,dt \right]^{\frac{1}{p}}$$
$$\le \left(C^p + 1\right)^{\frac{1}{p}} \|u\|_V. \tag{3.2}$$

On the other hand, we have

$$\left\| u - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\,dx \right\|_{W^{1,p}(\Omega)} \ge \|u\|_{W^{1,p}(\Omega)} - \left\| \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\,dx \right\|_{W^{1,p}(\Omega)},$$

which implies that

$$\|u\|_{W^{1,p}(\Omega)} \le \left\| u - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\,dx \right\|_{W^{1,p}(\Omega)} + \text{const.}$$

Hence,

$$\|u\|_V \le \left\| u - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\,dx \right\|_V + \text{const.} \tag{3.3}$$

In view of (3.2) and (3.3), we have shown that for $u \in V$, $\|u\|_V \to +\infty$ is equivalent to $\|u - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\,dx\|_V \to +\infty$.

Next, we shall show that A satisfies (3.1). In fact, we have

$$\begin{aligned}
&\frac{\langle\langle u - u_0, Au\rangle\rangle_V}{\|u\|_V} \\
&= \frac{\int_0^T \int_\Omega \langle (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, \nabla u\rangle\,dx\,dt}{\|u\|_V} \\
&\quad + \varepsilon \frac{\int_0^T \int_\Omega |u|^q\,dx\,dt}{\|u\|_V} - \frac{\int_0^T \int_\Omega f(x,t)(u - u_0)\,dx\,dt}{\|u\|_V} \\
&\quad - \frac{\int_0^T \int_\Omega \langle (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, \nabla u_0\rangle\,dx\,dt}{\|u\|_V} - \varepsilon \frac{\int_0^T \int_\Omega |u|^{q-2} u u_0\,dx\,dt}{\|u\|_V}.
\end{aligned} \tag{3.4}$$

If $1 < p < 2$, then

$$\begin{aligned}
&\frac{\int_0^T \int_\Omega \langle (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, \nabla u\rangle\,dx\,dt}{\|u\|_V} + \varepsilon \frac{\int_0^T \int_\Omega |u|^q\,dx\,dt}{\|u\|_V} \\
&= \frac{1}{\|u\|_V} \left[\int_0^T \int_\Omega \left(C(x,t) + |\nabla u|^2\right)^{\frac{p}{2}}\,dx\,dt \right. \\
&\quad \left. - \int_0^T \int_\Omega \frac{C(x,t)}{(C(x,t) + |\nabla u|^2)^{\frac{2-p}{2}}}\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^q\,dx\,dt \right] \\
&\ge \frac{1}{\|u\|_V} \left[\int_0^T \int_\Omega |\nabla u|^p\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^q\,dx\,dt \right] \\
&\quad - \frac{1}{\|u\|_V} \int_0^T \int_\Omega \frac{C(x,t)}{(C(x,t) + |\nabla u|^2)^{\frac{2-p}{2}}}\,dx\,dt
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{\|u\|_V}\left[\int_0^T\int_\Omega |\nabla u|^p\,dx\,dt + \varepsilon\int_0^T\int_\Omega |u|^q\,dx\,dt\right] \\
&\quad - \frac{1}{\|u\|_V}\int_0^T\int_\Omega \frac{C(x,t)}{C(x,t)^{\frac{2-p}{2}}}\,dx\,dt \\
&\geq \frac{1}{\|u\|_V}\int_0^T\int_\Omega |\nabla u|^p\,dx\,dt - \frac{1}{\|u\|_V}\int_0^T\int_\Omega C(x,t)^{\frac{p}{2}}\,dx\,dt.
\end{aligned} \tag{3.5}$$

From (3.2) and (3.3), we know that

$$\int_0^T\int_\Omega |\nabla u|^p\,dx\,dt \geq \frac{1}{C^p+1}\left\|u - \frac{1}{\operatorname{meas}(\Omega)}\int_\Omega u\,dx\right\|_V^p \geq \frac{1}{C^p+1}\|u\|_V^p + \text{const.}$$

Also,

$$\int_0^T\int_\Omega C(x,t)^{\frac{p}{2}}\,dx\,dt \leq \left\|C(x,t)\right\|_V^p < +\infty.$$

It follows from (3.5) that

$$\frac{\int_0^T\int_\Omega \langle (C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u, \nabla u\rangle\,dx\,dt}{\|u\|_V} + \varepsilon\frac{\int_0^T\int_\Omega |u|^q\,dx\,dt}{\|u\|_V} \to +\infty,$$

as $\|u\|_V \to +\infty$.

Moreover, we have

$$\begin{aligned}
&\left|\frac{\int_0^T\int_\Omega \langle (C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u, \nabla u_0\rangle\,dx\,dt}{\|u\|_V}\right. \\
&\qquad\left. + \varepsilon\frac{\int_0^T\int_\Omega |u|^{q-2}uu_0\,dx\,dt}{\|u\|_V} + \frac{\int_0^T\int_\Omega f(x,t)(u-u_0)\,dx\,dt}{\|u\|_V}\right| \\
&\leq \frac{\int_0^T\int_\Omega (C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u|\cdot|\nabla u_0|\,dx\,dt}{\|u\|_V} \\
&\quad + \varepsilon\frac{\int_0^T\int_\Omega |u|^{q-1}|u_0|\,dx\,dt}{\|u\|_V} + \frac{\int_0^T\int_\Omega |f|\cdot|u-u_0|\,dx\,dt}{\|u\|_V} \\
&\leq \frac{1}{\|u\|_V}\int_0^T\int_\Omega \frac{|\nabla u|\cdot|\nabla u_0|}{(C(x,t)+|\nabla u|^2)^{\frac{2-p}{2}}}\,dx\,dt \\
&\quad + \frac{\varepsilon}{\|u\|_V}\int_0^T\int_\Omega |u|^{q-1}|u_0|\,dx\,dt + \frac{\|f\|_{V^*}\|u-u_0\|_V}{\|u\|_V} \\
&\leq \frac{1}{\|u\|_V}\int_0^T\int_\Omega |\nabla u|^{p-1}|\nabla u_0|\,dx\,dt + \text{const}\cdot\frac{\varepsilon\|u\|_V^{\frac{q}{q'}}\|u_0\|_V}{\|u\|_V} + \frac{\|f\|_{V^*}\|u-u_0\|_V}{\|u\|_V} \\
&\leq \frac{1}{\|u\|_V}\left[\|u\|_V^{\frac{p}{p'}}\|u_0\|_V + \text{const}\cdot\varepsilon\|u\|_V^{\frac{q}{q'}}\|u_0\|_V + \|f\|_{V^*}\|u_0\|_V\right] + \|f\|_{V^*} \\
&\leq \text{const.}
\end{aligned} \tag{3.6}$$

Therefore, it follows from (3.4), (3.5), and (3.6) that A satisfies (3.1) when $1 < p < 2$.

If $p \geq 2$, then

$$\begin{aligned}
&\frac{\langle\langle u-u_0, Au\rangle\rangle_V}{\|u\|_V} \\
&\quad\geq \frac{\int_0^T\int_\Omega\langle(C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u,\nabla u\rangle\,dx\,dt}{\|u\|_V} \\
&\qquad + \varepsilon\frac{\int_0^T\int_\Omega|u|^q\,dx\,dt}{\|u\|_V} - \frac{\int_0^T\int_\Omega|f|\cdot|u-u_0|\,dx\,dt}{\|u\|_V} \\
&\qquad - \frac{\int_0^T\int_\Omega(C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}|\nabla u|\cdot|\nabla u_0|\,dx\,dt}{\|u\|_V} - \varepsilon\frac{\int_0^T\int_\Omega|u|^{q-1}|u_0|\,dx\,dt}{\|u\|_V} \\
&\quad\geq \frac{\int_0^T\int_\Omega\langle(C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u,\nabla u\rangle\,dx\,dt}{\|u\|_V} + \varepsilon\frac{\int_0^T\int_\Omega|u|^q\,dx\,dt}{\|u\|_V} - \frac{\|f\|_{V^*}\|u-u_0\|_V}{\|u\|_V} \\
&\qquad - \frac{2^{\frac{p-2}{2}}\int_0^T\int_\Omega C(x,t)^{\frac{p-2}{2}}|\nabla u|\cdot|\nabla u_0|\,dx\,dt}{\|u\|_V} - \frac{2^{\frac{p-2}{2}}\int_0^T\int_\Omega|\nabla u|^{p-1}|\nabla u_0|\,dx\,dt}{\|u\|_V} \\
&\qquad - \varepsilon\frac{\int_0^T\int_\Omega|u|^{q-1}|u_0|\,dx\,dt}{\|u\|_V} \\
&\quad\geq \frac{\int_0^T\int_\Omega|\nabla u|^p\,dx\,dt}{\|u\|_V} - \frac{2^{\frac{p-2}{2}}(\int_0^T\int_\Omega|\nabla u|^p\,dx\,dt)^{\frac{1}{p'}}(\int_0^T\int_\Omega|\nabla u_0|^p\,dx\,dt)^{\frac{1}{p}}}{\|u\|_V} \\
&\qquad - \frac{\|f\|_{V^*}\|u-u_0\|_V}{\|u\|_V} + \varepsilon\frac{\int_0^T\int_\Omega|u|^q\,dx\,dt}{\|u\|_V} - \varepsilon\frac{\|u_0\|_V(\int_0^T\int_\Omega|u|^q\,dx\,dt)^{\frac{1}{q'}}}{\|u\|_V} \\
&\qquad - 2^{\frac{p-2}{2}}\big\|C(x,t)\big\|_V^{p-2}\|u_0\|_V \\
&\quad\geq \frac{M(\|u-\frac{1}{|\Omega|}\int_\Omega u\,dx\|_V^p - \|u_0\|_V\|u-\frac{1}{|\Omega|}\int_\Omega u\,dx\|_V^{\frac{p}{p'}})}{\|u\|_V} - \frac{\|f\|_{V^*}\|u-u_0\|_V}{\|u\|_V} \\
&\qquad + \frac{\varepsilon(\int_0^T\int_\Omega|u|^q\,dx\,dt)^{\frac{1}{q'}}[(\int_0^T\int_\Omega|u|^q\,dx\,dt)^{1-\frac{1}{q'}} - \|u_0\|_V]}{\|u\|_V} \\
&\qquad - 2^{\frac{p-2}{2}}\big\|C(x,t)\big\|_V^{p-2}\|u_0\|_V,
\end{aligned} \tag{3.7}$$

where M is a positive constant. We can easily see that

$$\frac{\|u-\frac{1}{|\Omega|}\int_\Omega u\,dx\|_V^p - \|u_0\|_V\|u-\frac{1}{|\Omega|}\int_\Omega u\,dx\|_V^{\frac{p}{p'}}}{\|u\|_V} \to +\infty,$$

as $\|u\|_V \to +\infty$. Moreover, if $\int_0^T\int_\Omega|u|^q\,dx\,dt < +\infty$, then

$$\frac{\varepsilon(\int_0^T\int_\Omega|u|^q\,dx\,dt)^{\frac{1}{q'}}[(\int_0^T\int_\Omega|u|^q\,dx\,dt)^{1-\frac{1}{q'}} - \|u_0\|_V]}{\|u\|_V} \to 0,$$

as $\|u\|_V \to +\infty$; while if $\int_0^T\int_\Omega|u|^q\,dx\,dt \to +\infty$,

$$\frac{\varepsilon(\int_0^T\int_\Omega|u|^q\,dx\,dt)^{\frac{1}{q'}}[(\int_0^T\int_\Omega|u|^q\,dx\,dt)^{1-\frac{1}{q'}} - \|u_0\|_V]}{\|u\|_V} > 0.$$

Hence, the right side of (3.7) tends to $+\infty$ as $\|u\|_V \to +\infty$, which implies that A satisfies (3.1).

This completes the proof. □

Lemma 3.8 *If $w(x,t) \in \partial\Phi(u)$, then $w(x,t) = \widetilde{w}(x,t) \in \partial\beta_x(u)$ a.e. on $\Gamma \times (0,T)$.*

Proof If $w(x,t) \in \partial\Phi(u)$, then from the definition of subdifferential, we have

$$\int_0^T \int_\Gamma \varphi_x\bigl(u|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt \le \int_0^T \int_\Gamma \varphi_x\bigl(w|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt + \int_0^T \int_\Gamma w(x,t)(u-w)\,d\Gamma(x)\,dt,$$

which implies that the result is true. □

We are now ready to prove the main result.

Theorem 3.1 *The integro-differential equation* (1.11) *has a unique solution in V for $f(x,t) \in V^*$.*

Proof First, we shall show the existence of a solution. Noting Lemmas 2.6, 3.6, 3.7 and 3.3, and by using Theorem 2.1, we know that there exists $u(x,t) \in D(S) \subset V$ such that

$$0 = Su + Au + \partial\Phi(u). \tag{3.8}$$

Then we have for all $w \in V$,

$$\langle\langle u-w, Su\rangle\rangle_V + \langle\langle u-w, Au\rangle\rangle_V + \bigl\langle\!\bigl\langle u-w, \partial\Phi(u)\bigr\rangle\!\bigr\rangle_V = 0.$$

The definition of subdifferential implies that

$$\left\langle\!\!\left\langle u-w, \frac{\partial u}{\partial t}\right\rangle\!\!\right\rangle_V + \left\langle\!\!\left\langle u-w, a\frac{\partial}{\partial t}\int_\Omega u\,dx\right\rangle\!\!\right\rangle_V + \langle\langle u-w, Au\rangle\rangle_V + \Phi(u) - \Phi(w) \le 0.$$

From the definition of S, we have

$$u(x,0) = u(x,T). \tag{3.9}$$

Moreover,

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u}{\partial t}(u-w)\,dx\,dt + \int_0^T \int_\Omega \left(a\frac{\partial}{\partial t}\int_\Omega u\,dx\right)(u-w)\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \bigl\langle \bigl(C(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u, \nabla(u-w)\bigr\rangle\,dx\,dt \\
&\quad + \varepsilon \int_0^T \int_\Omega |u|^{q-2}u(u-w)\,dx\,dt \\
&\quad\quad - \int_0^T \int_\Omega f(x,t)(u-w)\,dx\,dt + \Phi(u) - \Phi(w) \le 0.
\end{aligned} \tag{3.10}$$

Let $w = u \pm \psi$, where $\psi \in C_0^\infty(\Omega \times (0,T))$. Then we have

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u}{\partial t}\psi\,dx\,dt + \int_0^T \int_\Omega \left(a\frac{\partial}{\partial t}\int_\Omega u\,dx\right)\psi\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \left\langle \left(C(x,t) + |\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u, \nabla\psi\right\rangle dx\,dt \\
&\quad + \varepsilon \int_0^T \int_\Omega |u|^{q-2}u\psi\,dx\,dt = \int_0^T \int_\Omega f(x,t)\psi\,dx\,dt.
\end{aligned}$$

From the properties of a generalized function, we get

$$\begin{aligned}
&\frac{\partial u}{\partial t} + a\frac{\partial}{\partial t}\int_\Omega u\,dx - \operatorname{div}\left[\left(C(x,t) + |\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right] + \varepsilon|u|^{q-2}u \\
&\quad = f(x,t), \quad \text{a.e. in } \Omega \times (0,T).
\end{aligned} \tag{3.11}$$

Noting (3.10) again, by using Green's formula, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u}{\partial t}(w-u)\,dx\,dt + \int_0^T \int_\Omega \left(a\frac{\partial}{\partial t}\int_\Omega u\,dx\right)(w-u)\,dx\,dt \\
&\qquad - \int_0^T \int_\Omega \operatorname{div}\left[\left(C(x,t) + |\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right](w-u)\,dx\,dt \\
&\qquad + \int_0^T \int_\Gamma \left\langle \vartheta, \left(C(x,t) + |\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right\rangle (w-u)|_\Gamma\,d\Gamma(x)\,dt \\
&\qquad + \varepsilon \int_0^T \int_\Omega |u|^{q-2}u(w-u)\,dx\,dt + \Phi(w) - \Phi(u) \\
&\quad \geq \int_0^T \int_\Omega f(x,t)(w-u)\,dx\,dt.
\end{aligned}$$

Then using (3.10), we obtain

$$\Phi(w) - \Phi(u) \geq -\int_0^T \int_\Gamma \left\langle \vartheta, \left(C(x,t) + |\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right\rangle (w-u)|_\Gamma\,d\Gamma(x)\,dt.$$

Thus, $-\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \partial\Phi(u)$.

In view of Lemma 3.8, we have $-\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle \in \beta_x(u)$ a.e. on $\Gamma \times (0,T)$. Combining it with (3.8) and (3.11), we know that (1.11) has a solution in V.

Next, we shall prove the uniqueness of the solution. Let $u(x,t)$ and $v(x,t)$ be two solutions of (1.11). By (3.8), we have

$$\big\langle\!\big\langle u - v, (A + \partial\Phi)u - (A + \partial\Phi)v \big\rangle\!\big\rangle_V = -\langle\!\langle u - v, Su - Sv\rangle\!\rangle_V \leq 0$$

since S is monotone. But $A + \partial\Phi$ is monotone too, so $\langle\!\langle u - v, Su - Sv\rangle\!\rangle_V = 0$, which implies that $u(x,t) = v(x,t)$.

The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors approve the final manuscript.

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