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Existence of multiple solutions for the Brezis-Nirenberg-type problem with singular coefficients

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Abstract

By energy estimates and by establishing a local (PS) condition, we obtain the multiplicity of solutions to a class of Brezis-Nirenberg-type problem with singular coefficients via minimax methods and the Krasnoselskii genus theory.

Keywords: Brezis-Nirenberg-type problem; minimax method

1 Introduction and main results

This paper is concerned with multiple solutions for the semilinear Brezis-Nirenberg-type problem with singular coefficients

$$\begin{cases} -\operatorname{div}(\frac{Du}{|x|^{2a}}) = \lambda \frac{|u|^{2_*-2}u}{|x|^{2_*b}} + \beta \frac{|u|^{q-2}u}{|x|^{\alpha}}, & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ is a bounded smooth domain, and $0 \in \Omega$, $-\infty < a < \frac{n-2}{2}$, $a \le b < a+1$, $2_* = \frac{2n}{n-2d}$, $d = a+1-b \in (0,1]$, $1 < q < 2$, $\alpha < (1+a)q + n(1-\frac{q}{2})$. $\beta > 0$, $\lambda > 0$ are two real parameters.

The starting point of the variational approach to the problem is the Caffarelli-Kohn-Nirenberg inequality (see [1]): There is a constant $C_{a,b} > 0$ such that

$$\left(\int_{R^n} |x|^{-2_*b}|u|^{2_*}\,dx\right)^{2/2_*} \le C_{a,b}\int_{R^n} |x|^{-2a}|Du|^2\,dx, \tag{2}$$

for all $u \in C_0^\infty(R^n)$, where

$$-\infty < a < \frac{n-2}{2}, \quad a \le b < a+1, \qquad 2_* = \frac{2n}{n-2d}, \quad d = a+1-b.$$

Let $D_a^{1,2}(\Omega)$ be the completion of $C_0^\infty(R^n)$ with respect to the weighted norm $\|\cdot\|$ defined by

$$\|u\| = \left(\int_\Omega |x|^{-2a}|Du|^2\,dx\right)^{1/2}.$$

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From the boundedness of Ω and the standard approximation arguments, it is easy to see that (2) holds for any $u \in D_a^{1,2}(\Omega)$ in the sense:

$$\left(\int_\Omega |x|^{-\alpha}|u|^r\,dx\right)^{2/r} \le C\int_\Omega |x|^{-2a}|Du|^2\,dx \tag{3}$$

for $1 \le r \le 2^* = \frac{2n}{n-2}$, $\frac{\alpha}{r} \le (1+a) + n(\frac{1}{r} - \frac{1}{2})$, that is, the embedding $D_a^{1,2}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous, where $L^r(\Omega, |x|^{-\alpha})$ is the weighted L^r space with the norm

$$\|u\|_{r,\alpha} := \|u\|_{L^r(\Omega,|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u|^r\,dx\right)^{1/r}.$$

On $D_a^{1,2}(\Omega)$, we can define the energy functional

$$J(u) = \frac{1}{2}\int_\Omega |x|^{-2a}|Du|^2\,dx - \frac{\lambda}{2_*}\int_\Omega |x|^{-2_*b}|u|^{2_*}\,dx - \frac{\beta}{q}\int_\Omega |x|^{-\alpha}|u|^q\,dx. \tag{4}$$

From (4), J is well defined in $D_a^{1,2}(\Omega)$, and $J \in C^1(D_a^{1,2}(\Omega), R)$. Furthermore, the critical points of J are weak solutions of problem (1).

Breiz-Nirenberg-type problems have been generalized to many situations such as

$$\begin{cases} -\operatorname{div}(\frac{Du}{|x|^{2a}}) - \mu\frac{u}{|x|^{2(a+1)}} = \frac{|u|^{2_*-2}u}{|x|^{2_*b}} + \lambda\frac{u}{|x|^{2(a+1)-c}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

Xuan *et al.* [2] derived the explicit formula for the extremal functions of the best embedding constant by applying the Bliss lemma [3]. They got a nontrivial solution for problem (5) including the resonant and nonresonant cases by variational methods. He and Zou [4] studied problem (5) and obtained the multiplicity of solutions with the aid of a pseudo-index theory. In [5], problem (5) has been extended to the p-Laplace case by Xuan.

The purpose of this paper is to study the multiplicity of solutions for the Breiz-Nirenberg-type problem (1) with the aid of a minimax method. We obtain multiple nontrivial solutions of (1) by proving the local (PS) condition and energy estimates.

Our main results are the following.

Theorem 1.1 *Suppose* $1 < q < 2$, *then*

(i) $\forall\beta > 0$, $\exists\lambda_0 > 0$ *such that if* $0 < \lambda < \lambda_0$, *problem* (1) *has a sequence of solutions* $\{u_m\}$ *with* $J(u_m) < 0$ *and* $J(u_m) \to 0$ *as* $m \to \infty$.

(ii) $\forall\lambda > 0$, $\exists\beta_0 > 0$ *such that if* $0 < \beta < \beta_0$, *problem* (1) *has a sequence of solutions* $\{u_m\}$ *with* $J(u_m) < 0$ *and* $J(u_m) \to 0$ *as* $m \to \infty$.

2 Preliminary results

Lemma 2.1 [5] *Suppose that* $\Omega \subset R^n$ *is an open bounded domain with* C^1 *boundary and* $0 \in \Omega$, $-\infty < a < (n-2)/2$. *The embedding* $D_a^{1,2}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ *is compact if* $1 \le r < 2^*$, $\alpha < (1+a)r + n(1 - \frac{r}{2})$.

Lemma 2.2 (Concentration compactness principle [5]) *Let* $-\infty < a < (n-2)/2$, $a \le b \le a+1$, $2_* = 2n/(n-2d)$, $d = 1 + a - b \in [0,1]$, *and* $M(R^n)$ *be the space of bounded measures* *on R^n. Suppose that $\{u_m\} \subset D_a^{1,2}(R^n)$ is a sequence such that*

$$
\begin{aligned}
&u_m \rightharpoonup u \quad \text{in } D_a^{1,2}(R^n),\\
&\mu_m := \left||x|^{-a} Du_m\right|^2 dx \rightharpoonup \mu \quad \text{in } M(R^n),\\
&\nu_m := \left||x|^{-b} u_m\right|^{2_*} dx \rightharpoonup \nu \quad \text{in } M(R^n),\\
&u_m \to u \quad \text{a.e. on } R^n.
\end{aligned}
$$

Then there are the following statements:

(1) *There exists some at most countable set I, a family $\{x^{(i)} : i \in I\}$ of distinct points in R^n, and a family $\{\nu^{(i)} : i \in I\}$ of positive numbers such that*

$$\nu = \left||x|^{-b} u\right|^{2_*} dx + \sum_{i \in I} \nu^{(i)} \delta_{x^{(i)}}, \tag{6}$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in R^n$.

(2) *The following inequality holds*

$$\mu \geq \left||x|^{-a} Du\right|^2 dx + \sum_{i \in I} \mu^{(i)} \delta_{x^{(i)}} \tag{7}$$

for some family $\{\mu^{(i)} > 0 : i \in I\}$ satisfying

$$S\left(\nu^{(i)}\right)^{2/2^*} \leq \mu^{(i)} \quad \text{for all } i \in I, \tag{8}$$

where $S := \inf_{u \in D_a^{1,2}(R^n)\setminus\{0\}} E_{a,b}(u)$ to be the best embedding constants, and

$$E_{a,b}(u) = \frac{\int_{R^n} |x|^{-2a} |Du|^2 \, dx}{(\int_{R^n} |x|^{-2_* b} |u|^{2_*} \, dx)^{2/2_*}}.$$

In particular, $\sum_{i \in I} (\nu^{(i)})^{2/2_*} < \infty$.

Lemma 2.3 *Assume $\{u_n\}$ is a $(PS)_c$ sequence with $c < 0$, $1 < q < 2$, then*

(1) $\forall \lambda > 0$, *there exists $\beta_1 > 0$ such that for any $0 < \beta < \beta_1$, $\{u_n\}$ has a convergent subsequence in $D_a^{1,2}(\Omega)$.*

(2) $\forall \beta > 0$, *there exists $\lambda_1 > 0$ such that for any $0 < \lambda < \lambda_1$, $\{u_n\}$ has a convergent subsequence in $D_a^{1,2}(\Omega)$.*

Proof (1) The boundedness of $(\mathrm{PS})_c$ sequence.

For $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence, then

$$J(u_n) = \frac{1}{2} \int_\Omega |x|^{-2a} |Du_n|^2 \, dx - \frac{\lambda}{2_*} \int_\Omega |x|^{-2_* b} |u_n|^{2_*} \, dx - \frac{\beta}{q} \int_\Omega |x|^{-\alpha} |u_n|^q \, dx, \tag{9}$$

$$\left\langle J'(u_n), u_n \right\rangle = \int_\Omega |x|^{-2a} |Du_n|^2 \, dx - \lambda \int_\Omega |x|^{-2_* b} |u_n|^{2_*} \, dx - \beta \int_\Omega |x|^{-\alpha} |u_n|^q \, dx. \tag{10}$$

So, we get

$$\begin{aligned} o(1)\big(1+\|u_n\|\big)+|c| &\geq J(u_n)-\frac{1}{2_*}\big\langle J'(u_n),u_n\big\rangle \\ &= \frac{d}{n}\|u_n\|^2-\left(\frac{1}{q}-\frac{1}{2_*}\right)\beta\int_\Omega \frac{|u_n|^q}{|x|^\alpha}\,dx \\ &\geq \frac{d}{n}\|u_n\|^2-\left(\frac{1}{q}-\frac{1}{2_*}\right)\beta C_\alpha\|u_n\|^q. \end{aligned}$$

We have the boundedness of $\{u_n\}$ for $1<q<2$, then there exists a subsequence, we still denote it by $\{u_n\}$, such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } D_a^{1,2}(\Omega), \\ &u_n \rightharpoonup u \quad \text{in } L^{2_*}\big(\Omega,|x|^{-2_*b}\big), \\ &u_n \to u \quad \text{in } L^r\big(\Omega,|x|^{-\alpha}\big), \forall 1\leq r<\frac{2n}{n-2}, \alpha<(1+a)r+n\left(1-\frac{r}{2}\right), \\ &u_n \to u \quad \text{a.e. on } \Omega. \end{aligned}$$

From the concentration compactness principle, there exist nonnegative measures μ, ν and a countable family $\{x_i\}\subset\Omega$ such that

$$\begin{aligned} &|x|^{-2_*b}|u_n|^{2_*}\,dx \rightharpoonup \nu = |x|^{-2_*b}|u|^{2_*}\,dx+\sum_{i\in I}\nu^{(i)}\delta_{x^{(i)}}, \\ &|x|^{-2a}|Du_n|^2\,dx \rightharpoonup \mu \geq |x|^{-2a}|Du|^2\,dx+S\sum_{i\in I}\big(\nu^{(i)}\big)^{2/2_*}\delta_{x^{(i)}}. \end{aligned}$$

(2) Up to a subsequence, $u_n\to u$ in $L^{2_*}(\Omega,|x|^{-2_*b})$.

Since $\{u_n\}$ is bounded in $D_a^{1,2}(\Omega)$, we may suppose, without loss of generality, that there exists $T\in(L^{2'}(\Omega,|x|^{-2a}))^n$ such that

$$Du_n \rightharpoonup T \quad \text{in } \big(L^{2'}\big(\Omega,|x|^{-2a}\big)\big)^n.$$

On the other hand, $|u_n|^{2_*-2}u_n$ is also bounded in $L^{2'_*}(\Omega,|x|^{-2_*b})$ and

$$|u_n|^{2_*-2}u_n \rightharpoonup |u|^{2_*-2}u \quad \text{in } L^{2'_*}\big(\Omega,|x|^{-2_*b}\big).$$

Note that

$$\begin{aligned} o(1)\|\varphi\| &= \big\langle J'(u_n),\varphi\big\rangle \\ &= \int_\Omega |x|^{-2a}Du_nD\varphi\,dx-\lambda\int_\Omega |x|^{-2_*b}|u_n|^{2_*-2}u_n\varphi\,dx \\ &\quad -\beta\int_\Omega |x|^{-\alpha}|u_n|^{q-2}u_n\varphi\,dx, \end{aligned} \tag{11}$$

taking $n\to\infty$ in (11), we have

$$\int_\Omega |x|^{-2a}TD\varphi\,dx=\lambda\int_\Omega |x|^{-2_*b}|u|^{2_*-2}u\varphi\,dx+\beta\int_\Omega |x|^{-\alpha}|u|^{q-2}u\varphi\,dx \tag{12}$$

for any $\varphi \in D_a^{1,2}(\Omega)$. Let $\varphi = \psi u_n$ in (12), where $\psi \in C(\overline{\Omega})$, then it follows that

$$\begin{aligned}
&\int_\Omega |x|^{-2a} Du_n u_n D\psi\,dx + \int_\Omega |x|^{-2a}|Du_n|^2\psi\,dx \\
&\quad = \lambda \int_\Omega |x|^{-2_*b}|u_n|^{2_*}\psi\,dx + \beta \int_\Omega |x|^{-\alpha}|u_n|^q\psi\,dx.
\end{aligned} \tag{13}$$

Taking $n \to \infty$ in (13), we have

$$\int_\Omega |x|^{-2a} uTD\psi\,dx + \int_\Omega \psi\,d\mu = \lambda \int_\Omega \psi\,d\nu + \beta \int_\Omega |x|^{-\alpha}|u|^q\psi\,dx. \tag{14}$$

Let $\varphi = \psi u$ in (12), then it follows that

$$\int_\Omega |x|^{-2a} T\psi u\,dx + \int_\Omega |x|^{-2a} Tu\,d\psi = \lambda \int_\Omega |x|^{-2_*b}|u|^{2_*}\psi\,dx + \beta \int_\Omega |x|^{-\alpha}|u|^q\psi\,dx. \tag{15}$$

Thus, it implies that

$$\int_\Omega \psi\,d\mu = \lambda \sum_{i\in I}\nu_i\psi(x_i) + \int_\Omega |x|^{-2a} TDu\psi\,dx, \tag{16}$$

which implies that

$$S(\nu_i)^{2/2_*} \le \mu_i = \lambda\nu_i.$$

Hence, $\nu_i \ge (\lambda^{-1}S)^{n/2d}$ if $\nu_i \ne 0$.

On the other hand,

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\left(J(u_n) - \frac{1}{2_*}\langle J'(u_n), u_n\rangle\right) \\
&= \lim_{n\to\infty}\left(\frac{d}{n}\|u_n\|^2 - \beta\left(\frac{1}{q} - \frac{1}{2_*}\right)\int_\Omega |x|^{-\alpha}|u_n|^q\,dx\right) \\
&\ge \frac{d}{n}\|u\|^2 - \beta C\|u\|^q,
\end{aligned}$$

then $\|u\|^q \le C\beta^{q/(2-q)}$, so that

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\left(J(u_n) - \frac{1}{2_*}\langle J'(u_n), u_n\rangle\right) \\
&= \lim_{n\to\infty}\left(\frac{d}{n}\|u_n\|^2 - \beta\left(\frac{1}{q} - \frac{1}{2_*}\right)\int_\Omega |x|^{-\alpha}|u_n|^q\,dx\right) \\
&\ge \frac{d}{n}\mu_i - \beta C\beta^{q/(2-q)} \\
&\ge \frac{d}{n}S^{\frac{n}{2d}}\left(\lambda^{-1}\right)^{\frac{n-2d}{2d}} - C\beta^{\frac{2}{2-q}}.
\end{aligned}$$

However, if $\beta > 0$ is given, we can choose $\lambda_1 > 0$ so small that for every $0 < \lambda < \lambda_1$, the last term on the right-hand side above is greater than 0, which is a contradiction. Similarly, if

$\lambda > 0$ is given, we can take $\beta_1 > 0$ so small that for every $0 < \beta < \beta_1$, the last term on the right-hand side above is greater than 0. Then $\nu_i = 0$ for each i.

Up to now, we have shown that

$$\lim_{n\to\infty} \int_\Omega |x|^{-2_* b}|u_n|^{2_*}\,dx = \int_\Omega |x|^{-2_* b}|u|^{2_*}\,dx.$$

So, by the Breiz-Lieb lemma,

$$\begin{aligned} o(1)\|u_n\| &= \|u_n\|^2 - \lambda \int_\Omega |x|^{-2_* b}|u_n|^{2_*}\,dx - \beta \int_\Omega |x|^{-\alpha}|u_n|^q\,dx \\ &= \|u_n - u\|^2 - \|u\|^2 - \lambda \int_\Omega |x|^{-2_* b}|u|^{2_*}\,dx - \beta \int_\Omega |x|^{-\alpha}|u|^q\,dx \\ &= \|u_n - u\|^2 + o(1)\|u\| \end{aligned}$$

since $J'(u) = 0$. Thus, we prove that $\{u_n\}$ strongly converges to u in $D_a^{1,2}(\Omega)$. □

3 Existence of infinitely many solutions

In this section, we use the minimax procedure to prove the existence of infinitely many solutions. Let Σ be the class of subsets of $D_a^{1,2}(\Omega) \setminus \{0\}$, which are closed and symmetric with respect to the origin. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) = \min\bigl\{k \in N : \exists\phi \in C\bigl(A, R^k \setminus \{0\}\bigr), \phi(x) = -\phi(-x)\bigr\}.$$

Assume that $1 < q < 2$, then we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |x|^{-2a}|Du|^2\,dx - \frac{\lambda}{2_*}\int_\Omega |x|^{-2_* b}|u|^{2_*}\,dx - \frac{\lambda}{q}\int_\Omega |x|^{-\alpha}|u|^q\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{C_b\lambda}{2_*}\|u\|^{2_*} - \frac{\beta C_\alpha}{q}\|u\|^q. \end{aligned}$$

Define

$$h(t) = \frac{1}{2}t^2 - \lambda C_1 t^{2_*} - \beta C_2 t^q.$$

Then, given $\beta > 0$, there exists $\lambda_2 > 0$ so small that for every $0 < \lambda < \lambda_2$, there exists $0 < T_0 < T_1$ such that $h(t) < 0$ for $0 < t < T_0$, $h(t) > 0$ for $T_0 < t < T_1$, $h(t) < 0$ for $t > T_1$. Similarly, given $\lambda > 0$, we can choose $\beta_2 > 0$ with the property that T_0, T_1 as above exist for each $0 < \beta < \beta_2$. Clearly, $h(T_0) = h(T_1) = 0$. Following the same idea as in [6–8], we consider the truncated functional

$$\tilde{J}(u) = \frac{1}{2}\int_\Omega |x|^{-2a}|Du|^2\,dx - \frac{\lambda}{2_*}\psi(u)\int_\Omega |x|^{-2_* b}|u|^{2_*}\,dx - \frac{\lambda}{q}\int_\Omega |x|^{-\alpha}|u|^q\,dx,$$

where $\psi(u) = \tau(\|u\|)$, and $\tau : R^+ \to [0,1]$ is a nonincreasing C^∞ function such that $\tau(t) = 1$ if $t \leq T_0$ and $\tau(t) = 0$ if $t \geq T_1$. The main properties of $\tilde{J}$ are the following.

Lemma 3.1

(1) $\tilde{J} \in C^1$ *and* $\tilde{J}$ *is bounded below.*
(2) *If* $\tilde{J}(u) \leq 0$, *then* $\|u\| \leq T_0$ *and* $\tilde{J}(u) = J(u)$.
(3) *For any* $\lambda > 0$, *there exists* $\beta_0 = \min\{\beta_1, \beta_2\}$ *such that if* $0 < \beta < \beta_0$ *and* $c < 0$, *then* $\tilde{J}$ *satisfies* $(PS)_c$ *condition.*
(4) *for any* $\beta > 0$, *there exists* $\lambda_0 = \min\{\lambda_1, \lambda_2\}$ *such that if* $0 < \lambda < \lambda_0$ *and* $c < 0$, *then* $\tilde{J}$ *satisfies* $(PS)_c$ *condition.*

Proof (1) and (2) are immediate. To prove (3) and (4), observe that all $(\mathrm{PS})_c$ sequences for $\tilde{J}$ with $c < 0$ must be bounded. Similar to the proof of Lemma 2.3, there exists a convergent subsequence. □

Lemma 3.2 *Given* $m \in N$, *there is* $\varepsilon_m < 0$ *such that*

$$\gamma\left(\left\{u \in D_a^{1,2}(\Omega) : \tilde{J}(u) \leq \varepsilon_m\right\}\right) \geq m.$$

Proof Fix m and let H_m be an m-dimensional subspace of $D_a^{1,2}(\Omega)$. Take $u \in H_m$, $u \neq 0$, write $u = r_m v$ with $v \in H_m$, $\|v\| = 1$ and $r_m = \|u\|$. Thus, for $0 < r_m < T_0$, since all the norms are equivalent, we have

$$\begin{aligned}
\tilde{J}(u) &= J(u) = \frac{1}{2}\int_\Omega |x|^{-2a}|Du|^2\,dx - \frac{\lambda}{2_*}\int_\Omega |x|^{-2_*b}|u|^{2_*}\,dx - \frac{\lambda}{q}\int_\Omega |x|^{-\alpha}|u|^q\,dx \\
&\leq \frac{1}{2}\|u\|^2 - \frac{\lambda C_1}{2_*}\|u\|^{2_*} - \frac{\lambda C_2}{q}\|u\|^q \\
&= \frac{1}{2}r_m^2 - \frac{\lambda C_1}{2_*}r_m^{2_*} - \frac{\lambda C_2}{q}r_m^q := \varepsilon_m.
\end{aligned}$$

Therefore, we can choose $r_m \in (0, T_0)$ so small that $\tilde{J}(u) \leq \varepsilon_m < 0$. Let $S_{r_m} = \{u \in D_a^{1,2}(\Omega) : \|u\| = r_m\}$, then $S_{r_m} \cap H_m \subset \tilde{J}^{\varepsilon_m}$. Hence, $\gamma(\tilde{J}^{\varepsilon_m}) \geq \gamma(S_{r_m} \cap H_m) = m$. Denote $\Gamma_m = \{A \in \Sigma : \gamma(A) \geq m\}$ and let

$$c_m = \inf_{A \in \Gamma_m} \sup_{u \in A} \tilde{J}(u).$$

Then $-\infty < c_m \leq \varepsilon_m < 0$ because $\tilde{J}^{\varepsilon_m} \in \Gamma_m$ and $\tilde{J}$ is bounded from below. □

Lemma 3.3 *Let* λ, β *be as in* (3) *or* (4) *of Lemma* 3.1. *Then all* c_m *are critical values of* $\tilde{J}$ *as* $c_m \to 0$.

Proof It is clear that $c_m \leq c_{m+1}$, $c_m < 0$. Hence, $c_m \to \bar{c} \leq 0$. Moreover, since all c_m are critical values of $\tilde{J}$, we claim that $\bar{c} = 0$. If $\bar{c} < 0$, because $K_{\bar{c}}$ is compact and $K_{\bar{c}} \in \Sigma$, it follows that $\gamma(K_{\bar{c}}) = N_0 < +\infty$ and there exists $\delta > 0$ such that $\gamma(K_{\bar{c}}) = \gamma(N_\delta(K_{\bar{c}})) = N_0$. By the deformation lemma there exist $\varepsilon > 0$ ($\bar{c} + \varepsilon < 0$) and an odd homeomorphism η such that

$$\eta\left(\tilde{J}^{\bar{c}+\varepsilon} \setminus N_\delta(K_{\bar{c}})\right) \subset \tilde{J}^{\bar{c}-\varepsilon}.$$

Since c_m is increasing and converges to $\overline{c}$, there exists $m \in N$ such that $c_m > \overline{c} - \varepsilon$ and $c_{m+N_0} \leq \overline{c}$ and there exists $A \in \Gamma_{m+N_0}$ such that $\sup_{u\in A} \tilde{J}(u) < \overline{c} + \varepsilon$. By the properties of γ, we have

$$\gamma\left(\overline{A \setminus N_\delta(K_{\overline{c}})}\right) \geq \gamma(A) - \gamma\left(N_\delta(K_{\overline{c}})\right) \geq m, \quad \gamma\left(\overline{A \setminus N_\delta(K_{\overline{c}})}\right) \geq m.$$

Therefore,

$$\eta\left(\overline{A \setminus N_\delta(K_{\overline{c}})}\right) \in \Gamma_m.$$

Consequently,

$$\sup_{u\in\eta(\overline{A\setminus N_\delta(K_{\overline{c}})})} \tilde{J}(u) \geq c_m > \overline{c} - \varepsilon,$$

a contradiction, hence $c_m \to 0$. □

With Lemma 3.1 to Lemma 3.3, we have proved Theorem 1.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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