Feng et al. *Boundary Value Problems* 2012, **2012**:146
http://www.boundaryvalueproblems.com/content/2012/1/146

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

# General decay for a system of nonlinear viscoelastic wave equations with weak damping

Baowei Feng[1*], Yuming Qin[2] and Ming Zhang[1]

*Correspondence:
fengbaowei@hotmail.com
[1]College of Information Science and Technology, Donghua University, Shanghai, 201620, P.R. China
Full list of author information is available at the end of the article

**Abstract**

In this paper, we are concerned with a system of nonlinear viscoelastic wave equations with initial and Dirichlet boundary conditions in $\mathbb{R}^n$ ($n = 1,2,3$). Under suitable assumptions, we establish a general decay result by multiplier techniques, which extends some existing results for a single equation to the case of a coupled system.

**MSC:** 35L05; 35L55; 35L70

**Keywords:** viscoelastic system; general decay; weak damping

## 1 Introduction

In this paper, we are concerned with a coupled system of nonlinear viscoelastic wave equations with weak damping

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + u_t = f_1(u,v), & \text{in } \Omega \times (0,+\infty), \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + v_t = f_2(u,v), & \text{in } \Omega \times (0,+\infty), \\ u = v = 0, \quad \text{on } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = u_0, \qquad u_t(\cdot,0) = u_1, \qquad v(\cdot,0) = v_0, \qquad v_t(\cdot,0) = v_1, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ ($n = 1,2,3$) is a bounded domain with smooth boundary $\partial\Omega$, $u$ and $v$ represent the transverse displacements of waves. The functions $g_1$ and $g_2$ denote the kernel of a memory, $f_1(u,v)$ and $f_2(u,v)$ are the nonlinearities.

In recent years, many mathematicians have paid their attention to the energy decay and dynamic systems of the nonlinear wave equations, hyperbolic systems and viscoelastic equations.

Firstly, we recall some results concerning single viscoelastic wave equation. Kafini and Tatar [1] considered the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\,ds = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

They established the polynomial decay of the first-order energy of solutions for compactly supported initial data and for a not necessarily decreasing relaxation function. Later Tatar

[2] studied the problem (1.2) with the Dirichlet boundary condition and showed that the decay of solutions was an arbitrary decay not necessarily at exponential or polynomial rate. Cavalcanti *et al.* [3] studied the following equation with Dirichlet boundary condition:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + g * \Delta u - \gamma \Delta u_t = 0.$$

The authors established a global existence result for $\gamma \geq 0$ and an exponential decay of energy for $\gamma > 0$. They studied the interaction within the $|u_t|^\rho u_{tt}$ and the memory term $g * \Delta u$. Later on, several other results were published based on [4–6]. For more results on a single viscoelastic equation, we can refer to [7–14].

For a coupled system, Agre and Rammaha [15] investigated the following system:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u,v), & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + |v_t|^{r-1} v_t = f_2(u,v), & \text{in } \Omega \times (0,T), \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ ($n = 1,2,3$) is a bounded domain with smooth boundary. They considered the following assumptions on $f_i$ ($i = 1,2$):

($A_1$) Let

$$F(u,v) = a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}, \qquad f_1(u,v) = \frac{\partial F}{\partial u}, \qquad f_2(u,v) = \frac{\partial F}{\partial v}$$

with $a,b > 0$, $p \geq 3$ if $n = 1,2$ and $p = 3$ if $n = 3$; $m,r \geq 1$.

($A_2$) There exist two positive constants $c_0$, $c_1$ such that for all $u,v \in \mathbb{R}^2$, $F(u,v)$ satisfies

$$c_0\bigl(|u|^{p+1} + |v|^{p+1}\bigr) \leq F(u,v) \leq c_1\bigl(|u|^{p+1} + |v|^{p+1}\bigr).$$

Under the assumptions ($A_1$)-($A_2$), they established the global existence of weak solutions and the global existence of small weak solutions with initial and Dirichlet boundary conditions. Moreover, they also obtained the blow up of weak solutions. Mustafa [16] studied the following system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + f_1(u,v) = 0, \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + f_2(u,v) = 0, \end{cases} \tag{1.3}$$

in $\Omega \times (0,+\infty)$ with initial and Dirichlet boundary conditions, proved the existence and uniqueness to the system by using the classical Faedo-Galerkin method and established a stability result by multiplier techniques. But the author considered the following different assumptions on $f_i$ ($i = 1,2$) from ($A_1$)-($A_2$):

($A_1'$) $f_i : \mathbb{R}^2 \to \mathbb{R}$ ($i = 1,2$) are $C^1$ functions and there exists a function $F$ such that

$$f_1(x,y) = \frac{\partial F}{\partial x}, \qquad f_2(x,y) = \frac{\partial F}{\partial y}, \quad F \geq 0, x f_1(x,y) + y f_2(x,y) \geq F(x,y),$$

($A_2'$)

$$\left|\frac{\partial f_i}{\partial x}(x,y)\right| + \left|\frac{\partial f_i}{\partial y}(x,y)\right| \leq d\bigl(1 + |x|^{\beta_{i1}-1} + |y|^{\beta_{i2}-1}\bigr),$$

for all $(x,y) \in \mathbb{R}^2$, where the constant $d > 0$ and $\beta_{ij} \geq 1$, $(n-2)\beta_{ij} \leq n$ for $i,j = 1,2$.

Han and Wang [17] considered the following coupled nonlinear viscoelastic wave equations with weak damping:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + |u_t|^{m-1}u_t = f_1(u,v), & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + |v_t|^{r-1}v_t = f_2(u,v), & \text{in } \Omega \times (0,T), \\ u = v = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, \qquad u_t(\cdot,0) = u_1, \qquad v(\cdot,0) = v_0, \qquad v_t(\cdot,0) = v_1, \quad \text{in } \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Under the assumptions $(A_1)$-$(A_2)$ on $f_i$ $(i = 1,2)$, the initial data and the parameters in the equations, they established the local existence, global existence uniqueness and finite time blow up properties. When the weak damping terms $|u_t|^{m-1}u_t$, $|v_t|^{r-1}v_t$ were replaced by the strong damping terms $-\Delta u_t$, $-\Delta v_t$, Liang and Gao [18] showed that under certain assumption on initial data in the stable set, the decay rate of the solution energy is exponential when they take

$$f_1(u,v) = \left[a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{p+2}\right],$$
$$f_2(u,v) = \left[a|u+v|^{2(p+1)}(u+v) + b|u|^{p+2} v|v|^p\right],$$

$a, b > 0$ and $p > -1$ if $n = 1,2$, $-1 < p \le 1$ if $n = 3$. Moreover, they obtained that the solutions with positive initial energy blow up in a finite time for certain initial data in the unstable set. For more results on coupled viscoelastic equations, we can refer to [19–21].

If we take $m = r = 1$ in (1.4), the system will be transformed into (1.1). To the best of our knowledge, there is no result on general energy decay for the viscoelastic problem (1.1). Motivated by [16, 17], in this paper, we shall establish the general energy decay for the problem (1.1) by multiplier techniques, which extends some existing results for a single equation to the case of a coupled system. The rest of our paper is organized as follows. In Section 2, we give some preparations for our consideration and our main result. The statement and the proof of our main result will be given in Section 3.

For the reader's convenience, we denote the norm and the scalar product in $L^2(\Omega)$ by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. $C_1$ denotes a general constant, which may be different in different estimates.

## 2 Preliminaries and main result

To state our main result, in addition to $(A_1)$-$(A_2)$, we need the following assumption.

$(A_3)$ $g_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1,2$, are differentiable functions such that

$$g_i(0) > 0, \quad 1 - \int_0^{+\infty} g_i(s)\,ds = l_i > 0,$$

and there exist nonincreasing functions $\xi_1, \xi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$g_i'(t) \le -\xi_i(t) g_i(t), \quad t \ge 0.$$

Now, we define the energy functional

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + \left(1 - \int_0^t g_1(s)\,ds\right)|\nabla u|^2\right)dx \\ + \frac{1}{2}(g_1 \circ \nabla u)(t) + \frac{1}{2}(g_2 \circ \nabla v)(t) \\ + \frac{1}{2}\int_\Omega \left(v_t^2 + \left(1 - \int_0^t g_2(s)\,ds\right)|\nabla v|^2\right)dx - \int_\Omega F(u,v)\,dx \tag{2.1}$$

and the functional

$$D(t) = \left(1 - \int_0^t g_1(s)\,ds\right)\|\nabla u(t)\|^2 + \left(1 - \int_0^t g_2(s)\,ds\right)\|\nabla v(t)\|^2 \\ + 2\big[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)\big] - 4\int_\Omega F\big(u(t), v(t)\big)\,dx, \tag{2.2}$$

where

$$(g \circ y)(t) = \int_0^t g(t-s)\|y(t) - y(s)\|^2\,ds.$$

The existence of a global solution to the system (1.1) is established in [17] as follows.

**Proposition** [17] *Let* $(A_1)$-$(A_3)$ *hold. Assume that* $D(0) = \|\nabla u_0\|^2 + \|\nabla v_0\|^2 - 4\int_\Omega F(u_0, v_0)\,dx > 0$, $\frac{2^p C_0}{l}(\frac{E(0)}{l})^{\frac{p-1}{2}} < 1$ *and that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *where* $C_0$ *is a computable constant and* $l = \min\{l_1, l_2\}$. *Then the problem* (1.1) *has a unique global solution* $(u(t), v(t))$ *satisfying*

$$\big(u(t), u_t(t)\big) \in C\big(\mathbb{R}^+; H_0^1(\Omega) \times L^2(\Omega)\big), \quad \big(v(t), v_t(t)\big) \in C\big(\mathbb{R}^+; H_0^1(\Omega) \times L^2(\Omega)\big).$$

We are now ready to state our main result.

**Theorem 2.1** *Let* $(A_1)$-$(A_3)$ *hold. Assume that* $D(0) = \|\nabla u_0\|^2 + \|\nabla v_0\|^2 - 4\int_\Omega F(u_0, v_0)\,dx > 0$, $\frac{2^p C_0}{l}(\frac{E(0)}{l})^{\frac{p-1}{2}} < 1$ *and that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *where* $C_0$ *is a computable constant and* $l = \min\{l_1, l_2\}$. *Then there exist constants* $C, \eta > 0$ *such that, for* $t$ *large, the solution of* (1.1) *satisfies*

$$E(t) \le Ce^{-\eta\int_0^t \xi(s)\,ds}, \tag{2.3}$$

*where*

$$\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, \quad t \ge 0. \tag{2.4}$$

## 3 Proof of Theorem 2.1

In this section, we carry out the proof of Theorem 2.1. Firstly, we will estimate several lemmas.

**Lemma 3.1** *Let $u(t)$, $v(t)$ be the solution of* (1.1). *Then the following energy estimate holds for any $t \geq 0$*:

$$\begin{aligned} E'(t) = {} & -\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{2}\left[\left(g_1' \circ \nabla u\right) + \left(g_2' \circ \nabla v\right)\right] \\ & - \frac{1}{2}\left[g_1(t)\left\|\nabla u(t)\right\|^2 + g_2(t)\left\|\nabla v(t)\right\|^2\right] \leq 0. \end{aligned} \tag{3.1}$$

*Proof* Multiplying the first equation of (1.1) by $u_t$ and the second equation by $v_t$, respectively, integrating the results over $\Omega$, performing integration by parts and noting that $F_t(u,v) = f_1(u,v)u_t + f_2(u,v)v_t$, we can easily get (3.1). The proof is complete. □

**Lemma 3.2** *Under the assumption* ($A_3$), *the following hold*:

$$\int_\Omega \left(\int_0^t g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right) d\tau\right)^2 dx \leq C_1(g \circ \nabla u), \tag{3.2}$$

$$\int_\Omega \left(\int_0^t -g'(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right) d\tau\right)^2 dx \leq -C_1\left(g' \circ \nabla u\right). \tag{3.3}$$

*Proof* Using Hölder's inequality, we get

$$\begin{aligned} & \int_\Omega \left(\int_0^t g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right) d\tau\right)^2 dx \\ & \leq \int_\Omega \left(\int_0^t g(\tau)\, d\tau\right)\left(\int_0^t g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)^2 d\tau\right) dx \\ & \leq \left(\int_0^t g(\tau)\, d\tau\right) \int_0^t g(t-\tau)\left(\int_\Omega \left(\nabla u(t) - \nabla u(\tau)\right)^2 dx\right) d\tau \\ & \leq C_1(g \circ \nabla u). \end{aligned}$$

On the other hand, we repeat the above proof with $-g'$, instead of $g$, we can get (3.3). The proof is now complete. □

**Lemma 3.3** *Let* ($A_1$)-($A_3$) *hold and $u(t)$, $v(t)$ be the solution of* (1.1). *Then the functional $I(t)$ defined by*

$$I(t) := \int_\Omega (u u_t + v v_t)\, dx$$

*satisfies*

$$\begin{aligned} I'(t) \leq {} & -\frac{l_1}{2}\left\|\nabla u(t)\right\|^2 - \frac{l_2}{2}\left\|\nabla v(t)\right\|^2 + \left(1 + \frac{1}{4\delta}\right)\left(\|u_t\|^2 + \|v_t\|^2\right) \\ & + \frac{C_1}{\delta}(g_1 \circ \nabla u) + \frac{C_1}{\delta}(g_2 \circ \nabla v) + C_1 \int_\Omega F(u,v)\, dx \end{aligned} \tag{3.4}$$

*for all $\delta > 0$.*

*Proof* By (1.1), a direct differentiation gives

$$\begin{aligned}
I'(t) &= \|u_t\|^2 - \|\nabla u\|^2 + \int_\Omega \nabla u \int_0^t g_1(t-\tau)\nabla u(\tau)\,d\tau\,dx - \int_\Omega u_t u\,dx + \int_\Omega f_1 u\,dx \\
&\quad + \|v_t\|^2 - \|\nabla v\|^2 + \int_\Omega \nabla v \int_0^t g_2(t-\tau)\nabla v(\tau)\,d\tau\,dx \\
&\quad - \int_\Omega v_t v\,dx + \int_\Omega f_2 v\,dx. \qquad (3.5)
\end{aligned}$$

From the assumptions $(A_1)$-$(A_2)$, we derive

$$\begin{aligned}
f_1(u,v) &= a(p+1)|u+v|^p + b(p+1)|u|^{\frac{p-1}{2}}|v|^{\frac{p+1}{2}}, \\
f_2(u,v) &= a(p+1)|u+v|^p + b(p+1)|u|^{\frac{p+1}{2}}|v|^{\frac{p-1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
f_1 u + f_2 v &= a(p+1)|u+v|^{p+1} + b(p+1)|uv|^{\frac{p+1}{2}} \\
&\le C_1 F(u,v). \qquad (3.6)
\end{aligned}$$

By Young's inequality and (3.2), we deduce for any $\delta > 0$

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \int_0^t g_1(t-\tau)\nabla u(\tau)\,d\tau\,dx \\
&\quad = \int_\Omega \nabla u \cdot \int_0^t g_1(t-\tau)\big(\nabla u(\tau) - \nabla u(t) + \nabla u(t)\big)\,d\tau\,dx \\
&\quad = \|\nabla u\|^2 \cdot \int_0^t g_1(\tau)\,d\tau + \int_\Omega \nabla u \cdot \int_0^t g_1(t-\tau)\big(\nabla u(\tau) - \nabla u(t)\big)\,d\tau\,dx \\
&\quad \le \|\nabla u\|^2 \cdot \int_0^t g_1(\tau)\,d\tau + \delta\|\nabla u\|^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^t g_1(t-\tau)\big|\nabla u(\tau) - u(t)\big|\,d\tau\right)^2 dx \\
&\quad \le \|\nabla u\|^2 \cdot \int_0^t g_1(\tau)\,d\tau + \delta\|\nabla u\|^2 + \frac{C_1}{4\delta}(g_1 \circ \nabla u). \qquad (3.7)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_\Omega \nabla v \cdot \int_0^t g_2(t-\tau)\nabla v(\tau)\,d\tau\,dx &\le \|\nabla v\|^2 \cdot \int_0^t g_2(\tau)\,d\tau + \delta\|\nabla v\|^2 \\
&\quad + \frac{C_1}{4\delta}(g_2 \circ \nabla v). \qquad (3.8)
\end{aligned}$$

Using Young's inequality and Poincaré's inequality, we obtain for any $\delta > 0$

$$\int_\Omega u u_t\,dx \le \delta\|u\|^2 + \frac{1}{4\delta}\|u_t\|^2 \le \delta\lambda^2\|\nabla u\|^2 + \frac{1}{4\delta}\|u_t\|^2, \qquad (3.9)$$

where $\lambda$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. Similarly,

$$\int_\Omega v v_t\,dx \le \delta\|v\|^2 + \frac{1}{4\delta}\|v_t\|^2 \le \delta\lambda^2\|\nabla v\|^2 + \frac{1}{4\delta}\|v_t\|^2,$$

which together with (3.5)-(3.9) gives

$$\begin{aligned}I'(t) \le{}& -\left(l_1 - \delta - \delta\lambda^2\right)\|\nabla u\|^2 - \left(l_2 - \delta - \delta\lambda^2\right)\|\nabla v\|^2 + \left(1 + \frac{1}{4\delta}\right)\left(\|u_t\|^2 + \|v_t\|^2\right) \\ &+ \frac{C_1}{4\delta}(g_1 \circ \nabla u) + \frac{C_1}{4\delta}(g_2 \circ \nabla v) + C_1 \int_\Omega F(u,v)\,dx. \end{aligned} \tag{3.10}$$

Now, we choose $\delta > 0$ so small that

$$l_1 - \delta - \delta\lambda^2 \ge \frac{l_1}{2}, \qquad l_2 - \delta - \delta\lambda^2 \ge \frac{l_2}{2},$$

which together with (3.10) gives (3.4). The proof is complete. □

**Lemma 3.4** *Let* ($A_1$)-($A_3$) *hold and* $u(t)$, $v(t)$ *be the solution of* (1.1). *Then the functional* $J(t)$ *defined by*

$$J(t) = J_1(t) + J_2(t),$$

*with*

$$\begin{aligned}J_1(t) &:= -\int_\Omega u_t \int_0^t g_1(t-\tau)\big(u(t) - u(\tau)\big)\,d\tau\,dx, \\ J_2(t) &:= -\int_\Omega v_t \int_0^t g_2(t-\tau)\big(v(t) - v(\tau)\big)\,d\tau\,dx,\end{aligned}$$

*satisfies*

$$\begin{aligned}J'(t) \le{}& -\left(\int_0^t g_1(\tau) - 2\delta\right)\|u_t\|^2 + \delta C_1\|\nabla u\|^2 + \frac{C_1}{\delta}(g_1 \circ \nabla u) - \frac{C_1}{\delta}\left(g_1' \circ \nabla u\right) \\ &- \left(\int_0^t g_2(\tau) - 2\delta\right)\|v_t\|^2 + \delta C_1\|\nabla v\|^2 + \frac{C_1}{\delta}(g_2 \circ \nabla v) - \frac{C_1}{\delta}\left(g_2' \circ \nabla v\right). \end{aligned} \tag{3.11}$$

*Proof* A direct differentiation for $J_1(t)$ yields

$$\begin{aligned}J_1'(t) ={}& -\int_\Omega u_{tt} \cdot \int_0^t g_1(t-\tau)\big(u(t) - u(\tau)\big)\,d\tau - \int_\Omega u_t \cdot \int_0^t g_1'(t-\tau)\big(u(t) - u(\tau)\big)\,d\tau\,dx \\ &- \left(\int_0^t g_1(\tau)\,d\tau\right)\int_\Omega u_t^2\,dx. \end{aligned} \tag{3.12}$$

Using the first equation of (1.1) and integrating by parts, we obtain

$$\begin{aligned}J_1'(t) ={}& \left(1 - \int_0^t g_1(\tau)\,d\tau\right)\int_\Omega \nabla u \cdot \int_0^t g_1(t-\tau)\big(\nabla u(t) - \nabla u(\tau)\big)\,d\tau\,dx \\ &+ \int_\Omega \left(\int_0^t g_1(t-\tau)\big|\nabla u(t) - \nabla u(\tau)\big|\,d\tau\right)^2 dx \\ &+ \int_\Omega u_t \cdot \int_0^t g_1(t-\tau)\big(u(t) - u(\tau)\big)\,d\tau\,dx \end{aligned}$$

$$\begin{aligned}
&- \int_\Omega f_1(u,v) \int_0^t g_1(t-\tau)\big(u(t)-u(\tau)\big)\,d\tau\,dx \\
&- \int_\Omega u_t \cdot \int_0^t g_1'(t-\tau)\big(u(t)-u(\tau)\big)\,d\tau\,dx \\
&- \left(\int_0^t g_1(\tau)\,d\tau\right) \int_\Omega u_t^2\,dx.
\end{aligned} \tag{3.13}$$

From Young's inequality, Poincaré's inequality and Lemma 3.2, we derive

$$\begin{aligned}
&\left(1-\int_0^t g_1(\tau)\,d\tau\right) \int_\Omega \nabla u \cdot \int_0^t g_1(t-\tau)\big(\nabla u(t)-\nabla u(\tau)\big)\,d\tau\,dx \\
&\quad \le \delta\|\nabla u\|^2 + \frac{C_1}{\delta}(g_1 \circ \nabla u),
\end{aligned} \tag{3.14}$$

$$- \int_\Omega u_t \cdot \int_0^t g_1'(t-\tau)\big(u(t)-u(\tau)\big)\,d\tau\,dx \le \delta\|u_t\|^2 - \frac{C_1}{\delta}\big(g_1' \circ \nabla u\big), \tag{3.15}$$

$$\int_\Omega u_t \cdot \int_0^t g_1(t-\tau)\big(u(t)-u(\tau)\big)\,d\tau\,dx \le \delta\|u_t\|^2 + \frac{C_1}{\delta}(g_1 \circ \nabla u), \tag{3.16}$$

$$\begin{aligned}
&\int_\Omega f_1(u,v) \int_0^t g_1(t-\tau)\big(u(t)-u(\tau)\big)\,d\tau\,dx \le \int_\Omega f_1^2(u,v)\,dx + \frac{C_1}{\delta}(g_1 \circ u) \\
&\quad \le \delta \int_\Omega f_1^2(u,v)\,dx + \frac{C_1}{\delta}(g_1 \circ \nabla u).
\end{aligned} \tag{3.17}$$

Now, we estimate the first term on the right-hand side of (3.17). Using the assumptions $(A_1)$-$(A_2)$ and Young's inequality, we arrive at

$$\begin{aligned}
&\int_\Omega f_1^2(u,v)\,dx \\
&\quad \le C_1 \int_\Omega |u+v|^{2p}\,dx + C_1 \int_\Omega |u|^{p-1}|v|^{p+1}\,dx \\
&\quad \le C_1\|u\|_{L^{2p}}^{2p} + C_1\|v\|_{L^{2p}}^{2p} + C_1\|u\|_{L^{3(p-1)}}^{2p-2} + C_1\|v\|_{L^{\frac{3(p+1)}{2}}}^{2p+2} \\
&\quad \le C_1\left(\frac{8E(0)}{l_1}\right)^{p-1}\|\nabla u\|^2 + C_1\left(\frac{8E(0)}{l_2}\right)^{p-1}\|\nabla v\|^2 \\
&\qquad + C_1\left(\frac{8E(0)}{l_1}\right)^{p-2}\|\nabla u\|^2 + C_1\left(\frac{8E(0)}{l_2}\right)^{p}\|\nabla v\|^2 \\
&\quad \le C_1\|\nabla u\|^2 + C_1\|\nabla v\|^2,
\end{aligned} \tag{3.18}$$

where we used the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ for $2 \le s \le 2n/(n-2)$ if $n = 3$ or $s \ge 2$ if $n = 1,2$ and the fact $\frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{1}{4}l_1\|\nabla u\|^2 + \frac{1}{4}l_2\|\nabla v\|^2 \le 2E(0)$ proved in Lemma 5.1 in [17]. Combining (3.13)-(3.18), we get

$$\begin{aligned}
J_1'(t) \le{}& -\left(\int_0^t g_1(\tau)\,d\tau - 2\delta\right)\|u_t\|^2 + \delta C_1\|\nabla u\|^2 + \delta C_1\|\nabla v\|^2 \\
&+ \frac{C_1}{\delta}(g_1 \circ \nabla u) - \frac{C_1}{\delta}\big(g_1' \circ \nabla u\big).
\end{aligned} \tag{3.19}$$

The same estimate to $J_2(t)$, we can derive

$$J_2'(t) \leq -\left(\int_0^t g_2(\tau)\,d\tau - 2\delta\right)\|v_t\|^2 + \delta C_1\|\nabla u\|^2 + \delta C_1\|\nabla v\|^2 + \frac{C_1}{\delta}(g_2 \circ \nabla v) - \frac{C_1}{\delta}(g_2' \circ \nabla v),$$

which together with (3.19) gives (3.11). The proof is now complete. □

*Proof of Theorem* 2.1 For $N_1, N_2 > 0$, we define the functional $\mathcal{K}$ by

$$\mathcal{K} := N_1 E(t) + N_2 J(t) + I(t),$$

and let

$$g_0 = \min\left\{\int_0^{t_0} g_1(s)\,ds, \int_0^{t_0} g_2(s)\,ds\right\}$$

for some fixed $t_0 > 0$.

Using Lemma 3.1 and Lemmas 3.3-3.4, a direct differentiation gives

$$\begin{aligned}\mathcal{K}'(t) \leq{}& -\left(\frac{l}{2} - N_2\delta C_1\right)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \left(\frac{C_1}{\delta} + N_2\frac{C_1}{\delta}\right)\left[(g_1 \circ \nabla u) + (g_2 \circ \nabla v)\right] \\ & - \left(N_1 + N_2 - 2\delta - 1 - \frac{1}{4\delta}\right)\left(\|u_t\|^2 + \|v_t\|^2\right) + C_1\int_\Omega F(u,v)\,dx \\ & + \left(\frac{N_1}{2} - \frac{N_2 C_1}{\delta}\right)\left[(g_1' \circ \nabla u) + (g_2' \circ \nabla v)\right],\end{aligned} \tag{3.20}$$

where $l = \min\{l_1, l_2\}$.

Now, we choose $\delta = \frac{1}{4C_1N_2}$ and $N_1$, $N_2$ large enough so that

$$c_1 = \frac{l_1}{2} - N_2\delta C_1 = \frac{l}{2} - \frac{l}{4} = \frac{l}{4} > 0, \tag{3.21}$$

$$c_2 = N_1 + N_2 - \frac{l}{2C_1N_2} - 1 - \frac{C_1N_2}{l} > 0, \tag{3.22}$$

$$c_3 = \frac{N_1}{2} - \frac{4C_1^2N_2^2}{l} > 0. \tag{3.23}$$

Inserting (3.21)-(3.23) into (3.20), we have

$$\begin{aligned}\mathcal{K}'(t) \leq{}& -c_1\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - c_2\left(\|u_t\|^2 + \|v_t\|^2\right) + c_3\left[(g_1' \circ \nabla u) + (g_2' \circ \nabla v)\right] \\ & + \left(\frac{4C_1^2N_2}{l} + \frac{4C_1^2N_2}{l}\right)\left[(g_1 \circ \nabla u) + (g_2 \circ \nabla v)\right] + C_1\int_\Omega F(u,v)\,dx.\end{aligned} \tag{3.24}$$

Therefore, for two positive constants $\omega$ and $C$, we obtain

$$\mathcal{K}'(t) \leq -\omega E(t) + C\left[(g_1 \circ \nabla u) + (g_2 \circ \nabla v)\right], \quad \text{for all } t \geq t_0. \tag{3.25}$$

On the other hand, we choose $N_1$ even larger so that $\mathcal{K}(t)$ is equivalent to $E(t)$, *i.e.*,

$$\mathcal{K}(t) \sim E(t). \tag{3.26}$$

Multiplying (3.25) by $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ and using ($A_3$), we get

$$\begin{aligned}
\xi(t)\mathcal{K}'(t) \le{}& -\omega\xi(t)E(t) + C\int_\Omega\int_0^t \xi_1(t-\tau)g_1(t-\tau)\big|\nabla u(t) - \nabla u(\tau)\big|^2\,d\tau\,dx \\
&+ C\int_\Omega\int_0^t \xi_2(t-\tau)g_2(t-\tau)\big|\nabla v(t) - \nabla v(\tau)\big|^2\,d\tau\,dx \\
\le{}& -\omega\xi(t)E(t) - C\int_\Omega\int_0^t g_1'(t-\tau)\big|\nabla u(t) - \nabla u(\tau)\big|^2\,d\tau\,dx \\
&- C\int_\Omega\int_0^t g_2'(t-\tau)\big|\nabla v(t) - \nabla v(\tau)\big|^2\,d\tau\,dx \\
\le{}& -\omega\xi(t)E(t) - CE'(t), \quad \text{for all } t \ge t_0.
\end{aligned} \tag{3.27}$$

By virtue of ($A_3$) and $\xi(t) \le 0$, we have

$$\frac{d}{dt}\big(\xi(t)\mathcal{K}(t) + CE(t)\big) \le -\omega\xi(t)E(t), \quad \text{for all } t \ge t_0. \tag{3.28}$$

Using (3.26), we can easily get

$$\mathcal{L}(t) := \xi(t)\mathcal{K}(t) + CE(t) \sim E(t), \tag{3.29}$$

which together with (3.28) yields, for some positive constant $\eta$,

$$\mathcal{L}'(t) \le -\eta\xi(t)\mathcal{L}(t), \quad \text{for all } t \ge t_0. \tag{3.30}$$

Integrating (3.30) over $(t_0, t)$, we arrive at

$$\begin{aligned}
\mathcal{L}(t) &\le \mathcal{L}(t_0)e^{-\eta\int_t^{t_0}\xi(\tau)\,d\tau} \\
&\le Ce^{-\eta\int_t^{t_0}\xi(\tau)\,d\tau},
\end{aligned}$$

which together with (3.29) and the boundedness of $E$ and $\xi$ yields (2.3). The proof is now complete. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The paper is a joint work of all authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

**Author details**
[1]College of Information Science and Technology, Donghua University, Shanghai, 201620, P.R. China. [2]Department of Applied Mathematics, Donghua University, Shanghai, 201620, P.R. China.

**Acknowledgements**
Baowei Feng was supported by the Doctoral Innovational Fund of Donghua University with contract number BC201138, and Yuming Qin was supported by NNSF of China with contract numbers 11031003 and 11271066 and the grant of Shanghai Education Commission (No. 13ZZ048).

Received: 19 August 2012 Accepted: 26 November 2012 Published: 13 December 2012

**References**

1. Kafini, M, Tatar, N-e: A decay result to a viscoelastic in $\mathbb{R}^n$ with an oscillating kernel. J. Math. Phys. **51**(7), 073506 (2010)
2. Tatar, N-e: Arbitrary decay in linear viscoelastic. J. Math. Phys. **52**(1), 013502 (2011)
3. Cavalcanti, MM, Domingos Cavalcanti, VN, Ferreira, J: Existence and uniform decay for a non-linear viscoelastic equation with strong damping. Math. Methods Appl. Sci. **24**, 1043-1053 (2001)
4. Cavalcanti, MM, Domingos Cavalcanti, VN, Prates Filho, JS, Soriano, JA: Existence and uniform decay rates for viscoelastic problems with non-linear boundary damping. Differ. Integral Equ. **14**(1), 85-116 (2001)
5. Messaoudi, SA, Tatar, N-e: Global existence and uniform stability of solutions for a quasilinear viscoelastic problem. Math. Methods Appl. Sci. **30**, 665-680 (2007)
6. Messaoudi, SA, Tatar, N-e: Exponential and polynomial decay for a quasilinear viscoelastic problem. Math. Nachr. **282**, 1443-1450 (2009)
7. Fabrizio, M, Polidoro, S: Asymptotic decay for some differential systems with fading memory. Appl. Anal. **81**(6), 1245-1264 (2002)
8. Messaoudi, SA: General decay of solutions of a viscoelastic equation. J. Math. Anal. Appl. **341**, 1457-1467 (2008)
9. Messaoudi, SA: General decay of the solution energy in a viscoelastic equation with a nonlinear source. Nonlinear Anal. TMA **69**, 2589-2598 (2008)
10. Messaoudi, SA, Fareh, A: General decay for a porous thermoelastic system with memory: the case of equal speeds. Nonlinear Anal. TMA **74**, 6895-6906 (2011)
11. Messaoudi, SA, Tatar, N-e: Exponential and polynomial decay for a quasilinear viscoelastic equation. Nonlinear Anal. TMA **68**, 785-793 (2007)
12. Tatar, N-e: Exponential decay for a viscoelastic problem with a singular problem. Z. Angew. Math. Phys. **60**(4), 640-650 (2009)
13. Mediden, M, Tatar, N-e: Asymptotic behavior for a viscoelastic problem with not necessarily decreasing kernel. Appl. Math. Comput. **167**(2), 1221-1235 (2005)
14. Liu, W: General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms. J. Math. Phys. **50**(11), 113506 (2009)
15. Agre, K, Rammaha, MA: System of nonlinear wave equations with damping and source terms. Differ. Integral Equ. **19**, 1235-1270 (2006)
16. Mustafa, MI: Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations. Nonlinear Anal.: Real World Appl. **13**, 452-463 (2012)
17. Han, X, Wang, M: Global existence and blow up of solutions for a system of nonlinear viscoelastic wave equations with damping and source. Nonlinear Anal. TMA **71**, 5427-5450 (2009)
18. Liang, F, Gao, H: Exponential energy decay and blow up of solutions for a system of nonlinear viscoelastic wave equations with strong damping. Bound. Value Probl. (2011). doi:10.1186/1687-2770-2011-22
19. Liu, W: Uniform decay of solutions for a quasilinear system of viscoelastic equations. Nonlinear Anal. TMA **71**, 2257-2267 (2009)
20. Medeiros, LA, Miranda, MM: Weak solutions for a system of nonlinear Klein-Gordon equations. Ann. Mat. Pura Appl. **146**, 173-183 (1987)
21. Zhang, J: On the standing wave in coupled nonlinear Klein-Gordon equations. Math. Methods Appl. Sci. **26**, 11-25 (2003)

doi:10.1186/1687-2770-2012-146
**Cite this article as:** Feng et al.: **General decay for a system of nonlinear viscoelastic wave equations with weak damping.** *Boundary Value Problems* 2012 **2012**:146.