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Existence and multiplicity of positive solutions for a class of *p*(*x*)-Kirchhoff type equations

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Abstract

In this article, we study the existence and multiplicity of positive solutions for the Neumann boundary value problems involving the $p(x)$-Kirchhoff of the form

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \lambda |u|^{p(x)})dx\right)(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - \lambda |u|^{p(x)-2}u) = f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the sub-supersolution method and the variational method, under appropriate assumptions on f and M, we prove that there exists $\lambda_* > 0$ such that the problem has at least two positive solutions if $\lambda > \lambda_*$, at least one positive solution if $\lambda = \lambda_*$ and no positive solution if $\lambda < \lambda_*$. To prove these results we establish a special strong comparison principle for the Neumann problem.

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Keywords: $p(x)$-Kirchhoff, positive solution, sub-supersolution method, comparison principle

1 Introduction

In this article we study the following problem

$$\begin{cases} -M(t)(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - \lambda |u|^{p(x)-2}u) = f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda^f)$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N \geq 1$, $\frac{\partial u}{\partial v}$ is the outer unit normal derivative, $\lambda \in \mathbb{R}$ is a parameter, $p = p(x) \in C^1(\overline{\Omega})$ with $1 < p^- := \inf_\Omega p(x) \leq p^+ := \sup_\Omega p(x) < +\infty$, $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $M(t)$ is a function with $\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \lambda |u|^{p(x)})dx$ and satisfies the following condition:

(M_0) $M(t)$: $[0, +\infty) \to (m_0, +\infty)$ is a continuous and increasing function with $m_0 > 0$.

The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) := -\Delta_{p(x)}u$ is said to be the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [1-3]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [4,5]. Another field of application of equations with variable exponent growth conditions is image processing [6]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [7-11] for an overview of and references on this subject, and to [12-16] for the study of the variable exponent equations and the corresponding variational problems.

The problem $\left(P_{\lambda_1}^f\right)$ is a generalization of the stationary problem of a model introduced by Kirchhoff [17]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of Equation (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average $\frac{1}{L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, and hence the equation is no longer a pointwise identity. The equation

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 dx)\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

is related to the stationary analogue of the Equation (1.2). Equation (1.3) received much attention only after Lions [18] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [19-22]. Moreover, nonlocal boundary value problems like (1.3) can be used for modeling several physical and biological systems where u describes a process which depends on the average of itself, such as the population density [23-26]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [27-29]) and $p(x)$-Laplacian (see [30-33]).

Many authors have studied the Neumann problems involving the p-Laplacian, see e.g., [34-36] and the references therein. In [34,35] the authors have studied the problem $\left(P_{\lambda_1}^f\right)$ in the cases of $p(x) \equiv p = 2$, $M(t) \equiv 1$ and of $p(x) \equiv p > 1$, $M(t) \equiv 1$, respectively. In [36], Fan and Deng studied the Neumann problems with $p(x)$-Laplacian, with the nonlinear potential $f(x, u)$ under appropriate assumptions. By using the sub-supersolution method and variation method, the authors get the multiplicity of positive solutions of $\left(P_{\lambda_1}^f\right)$ with $M(t) \equiv 1$. The aim of the present paper is to generalize the main results of [34-36] to the $p(x)$-Kirchhoff case. For simplicity we shall restrict to the 0-Neumann boundary value problems, but the methods used in this article are also suitable for the inhomogeneous Neumann boundary value problems.

In this article we use the following notations:

$$F(x,t) = \int_0^t f(x,s)\,ds,$$

$\Lambda = \{\lambda \in \mathbb{R}$: there exists at least a positive solution of $\left(P_\lambda^f\right)\}$,

$$\lambda_* = \inf \Lambda.$$

The main results of this article are the following theorems. Throughout the article we always suppose that the condition (M_0) holds.

Theorem 1.1. *Suppose that f satisfies the following conditions:*

$$f(x,t) \geq 0, \quad f(x,t) \not\equiv 0 \ \forall x \in \Omega, \quad \forall t \geq 0 \tag{1.4}$$

and

$$\text{for each } x \in \Omega, f(x,t) \text{ is nondecreasing with respect to } t \geq 0. \tag{1.5}$$

Then $\Lambda \neq \emptyset$, $\lambda_* \geq 0$ *and* $(\lambda_*, +\infty) \subset \Lambda$. *Moreover, for every* $\lambda > \lambda_*$ *problem* $\left(P_\lambda^f\right)$*has a minimal positive solution* u_λ *in* $[0,w_1]$, *where* w_1 *is the unique solution of* $\left(P_\lambda^0\right)$*and* $u_{\lambda_1} < u_{\lambda_2}$*if* $\lambda_* < \lambda_2 < \lambda_1$.

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *also suppose that there exist positive constants M,* c_1 *and* c_2 *such that*

$$f(x,t) \leq c_1 + c_2 t^{q(x)-1}, \quad \forall x \in \Omega, \quad \forall t \geq M, \tag{1.6}$$

where $q \in C(\overline{\Omega})$*and* $1 \leq q(x) < p^*(x)$ *for* $x \in \overline{\Omega}$, $\mu \in (0,1)$ *such that*

$$\widehat{M}(t) \geq (1-\mu)M(t)t, \tag{1.7}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$*and* $M_1 > 0$, $\theta > \dfrac{p^+}{1-\mu}$*such that*

$$0 < \theta F(x,t) \leq tf(x,t), \quad \forall x \in \Omega, \quad \forall t \geq M_1. \tag{1.8}$$

Then for each $\lambda \in (\lambda_*, +\infty)$, $\left(P_\lambda^f\right)$*has at least two positive solutions* u_λ *and* v_λ, *where* u_λ *is a local minimizer of the energy functional and* $u_\lambda \leq v_\lambda$.

Theorem 1.3. (1) *Suppose that f satisfies* (1.4),

$$f(x,0) \leq f(x,t) \quad \text{for } t > 0 \text{ and } x \in \Omega \tag{1.9}$$

and the following conditions:

$$f(x,t) \leq c_3 + c_4 t^{r(x)-1}, \quad \forall x \in \Omega, \quad \forall t \geq M_2, \tag{1.10}$$

where M_2, c_3 *and* c_4 *are positive constants,* $r \in C(\overline{\Omega})$*and* $1 \leq r(x) < p(x)$ *for* $x \in \overline{\Omega}$. *Then* $\lambda_* = 0$.

(2) *If f satisfies* (1.4)-(1.8), *then* $\lambda_* \in \Lambda$.

Example 1.1. Let $M(t) = a + bt$, where a and b are positive constants. It is clear that

$$M(t) \geq a > 0.$$

Taking $\mu = \frac{1}{2}$, we have

$$\widehat{M}(t) = \int_0^t M(s)\,\mathrm{d}s = at + \frac{1}{2}bt^2 \geq \frac{1}{2}(a+bt)t = (1-\mu)M(t)t.$$

So the conditions (M_0) and (1.7) are satisfied.

The underlying idea for proving Theorems 1.1-1.3 is similar to the one of [36]. The special features of this class of problems considered in the present article are that they involve the nonlocal coefficient $M(t)$. To prove Theorems 1.1-1.3, we use the results of [37] on the global $C^{1,\alpha}$ regularity of the weak solutions for the $p(x)$-Laplacian equations. The main method used in this article is the sub-supersolution method for the Neumann problems involving the $p(x)$-Kirchhoff. A main difficulty for proving Theorem 1.1 is that a special strong comparison principle is required. It is well known that, when $p \neq 2$, the strong comparison principles for the p-Laplacian equations are very complicated (see e.g. [38-41]). In [13,42,43] the required strong comparison principles for the Dirichlet problems have be established, however, they cannot be applied to the Neumann problems. To prove Theorem 1.1, we establish a special strong comparison principle for the Neumann problem $\left(P_\lambda^f\right)$ (see Lemma 4.6 in Section 4), which is also valid for the inhomogeneous Neumann boundary value problems.

In Section 2, we give some preliminary knowledge. In Section 3, we establish a general principle of sub-supersolution method for the problem $\left(P_\lambda^f\right)$ based on the regularity results. In Section 4, we give the proof of Theorems 1.1-1.3.

2 Preliminaries

In order to discuss problem $\left(P_\lambda^f\right)$, we need some theories on $W^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly we state some basic properties of spaces $W^{1,p(x)}(\Omega)$ which will be used later (for details, see [17]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere.

Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), \quad h(x) > 1 \text{ for any } x \in \overline{\Omega}\}$$

and

$$L^{p(x)}(\Omega) = \left\{ u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are all separable Banach spaces. When $p^- > 1$ these spaces are reflexive.

Let $\lambda > 0$. Define for $u \in W^{1,p(x)}(\Omega)$,

$$\|u\|_\lambda = \inf\left\{\sigma > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\sigma}\right|^{p(x)} + \lambda\left|\frac{u}{\sigma}\right|^{p(x)}\right) dx \le 1\right\}.$$

Then $||u||_\lambda$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{W^{1,p(x)}(\Omega)}$.

By the definition of $||u||_\lambda$ we have the following

Proposition 2.1. [11,14]*Put* $\rho_\lambda(u) = \int_\Omega \left(|\nabla u|^{p(x)} + \lambda|u|^{p(x)}\right)dx$ *for* $\lambda > 0$ *and* $u \in W^{1,p(x)}(\Omega)$. *We have:*

(1) $\|u\|_\lambda \ge 1 \Rightarrow \|u\|_\lambda^{p_-} \le \rho_\lambda(u) \le \|u\|_\lambda^{p_+}$;

(2) $\|u\|_\lambda \le 1 \Rightarrow \|u\|_\lambda^{p_+} \le \rho_\lambda(u) \le \|u\|_\lambda^{p_-}$;

(3) $\lim_{k\to+\infty} \|u_k\|_\lambda = 0 \Leftrightarrow \lim_{k\to+\infty} \rho_\lambda(u_k) = 0$ *(as* $k \to +\infty$*)*;

(4) $\lim_{k\to+\infty} \|u_k\|_\lambda = +\infty \Leftrightarrow \lim_{k\to+\infty} \rho_\lambda(u_k) = +\infty$ *(as* $k \to +\infty$*)*.

Proposition 2.2. [14]*If* $u, u_k \in W^{1,p(x)}(\Omega)$, $k = 1,2,...$, *then the following statements are equivalent each other:*

(i) $\lim_{k\to+\infty} \|u_k - u\|_\lambda = 0$;

(ii) $\lim_{k\to+\infty} \rho_\lambda(u_k - u) = 0$;

(iii) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to+\infty} \rho_\lambda(u_k) = \rho(u)$.

Proposition 2.3. [14]*Let* $p \in C(\overline{\Omega})$. *If* $q \in C(\overline{\Omega})$ *satisfies the condition*

$$1 \le q(x) < p^*(x), \quad \forall x \in \overline{\Omega}, \tag{2.1}$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Proposition 2.4. [14]*The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have the following Hölder-type inequality*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)}.$$

Now, we discuss the properties of $p(x)$-Kirchhoff-Laplace operator

$$\Phi_K(u) := -M\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \lambda|u|^{p(x)})dx\right)(\mathrm{div}|\nabla u|^{p(x)-2}\nabla u - \lambda|u|^{p(x)-2}u),$$

where $\lambda > 0$ is a parameter. Denotes

$$\Phi(u) : \widehat{M}\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \lambda|u|^{p(x)})dx\right). \tag{2.2}$$

For simplicity we write $X = W^{1,p(x)}(\Omega)$, denote by $u_n \rightharpoonup u$ and $u_n \to u$ the weak convergence and strong convergence of sequence $\{u_n\}$ in X, respectively. It is obvious that the functional Φ is a Gâteaux differentiable whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\langle \Phi'(u), v\rangle = M\left(\int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \lambda|u|^{p(x)}\right)dx\right)\int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla v + \lambda|u|^{p(x)-2}uv)dx, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X and X^*. Therefore, the $p(x)$-Kirchhoff-Laplace operator is the derivative operator of Φ in the weak sense. We have the following properties about the derivative operator of Φ.

Proposition 2.5. *If* (M_0) *holds, then*

(i) $\Phi': X \to X^*$ *is a continuous, bounded and strictly monotone operator;*

(ii) Φ' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *and* $\overline{\lim_{n\to+\infty}}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* X;

(iii) $\Phi'(u): X \to X^*$ *is a homeomorphism;*

(iv) Φ *is weakly lower semicontinuous.*

Proof. Applying the similar method to prove [15, Theorem 2.1], with obvious changes, we can obtain the conclusions of this proposition.

3 Sub-supersolution principle

In this section we give a general principle of sub-supersolution method for the problem $\left(P_\lambda^f\right)$ based on the regularity results and the comparison principle.

Definition 3.1. $u \in X$ is called a weak solution of the problem $\left(P_\lambda^f\right)$ if for all $v \in X$,

$$M\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + \lambda|u|^{p(x)})dx\right)\int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla v + \lambda|u|^{p(x)-2}uv)dx = \int_\Omega f(x,u)v\,dx.$$

In this article, we need the global regularity results for the weak solution of $\left(P_\lambda^f\right)$. Applying Theorems 4.1 and 4.4 of [44] and Theorem 1.3 of [37], we can easily get the following results involving of the regularity of weak solutions of $\left(P_\lambda^f\right)$.

Proposition 3.1. (1) *If f satisfies* (1.6), *then* $u \in L^\infty(\Omega)$ *for every weak solution u of* $\left(P_\lambda^f\right)$.

(2) *Let* $u \in X \cap L^\infty(\Omega)$ *be a solution of* $\left(P_\lambda^f\right)$. *If the function p is log-Hölder continuous on* $\overline{\Omega}$, *i.e., there is a positive constant H such that*

$$\left|p(x) - p(y)\right| \le \frac{H}{-\log|x-y|} \quad \text{for } x, y \in \overline{\Omega} \text{ with } |x-y| \le \frac{1}{2}, \tag{3.2}$$

then $u \in C^{0,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$.

(3) *If in* (2), *the condition* (3.2) *is replaced by that p is Hölder continuous on* $\overline{\Omega}$, *then* $u \in C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$.

For u, $v \in \mathbf{S}(\Omega)$, we write $u \le v$ if $u(x) \le v(x)$ for a.e. $x \in \Omega$. In view of (M_0), applying Theorem 1.1 of [16], we have the following strong maximum principle.

Proposition 3.2. *Suppose that* $p(x) \in C_+(\overline{\Omega}) \cap C^1(\overline{\Omega})$, $u \in X$, $u \geq 0$ *and* $u \not\equiv 0$ *in* Ω. *If*

$$-M(t)(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - d(x)|u|^{p(x)-2}u) \geq 0,$$

where $t = \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}d(x)|u|^{p(x)}\right)dx$, $M(t) \geq m_0 > 0$, $0 \leq d(x) \in L^\infty(\Omega)$, $q(x) \in C(\overline{\Omega})$ *with* $p(x) \leq q(x) \leq p^*(x)$, *then* $u > 0$ *in* Ω.

Definition 3.2. $u \in X$ is called a subsolution (resp. supersolution) of $\left(P_\lambda^f\right)$ if for all $v \in X$ with $v \geq 0$, $u \leq 0$ (resp. $\geq$) on $\partial\Omega$ and

$$M\left(\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}\lambda|u|^{p(x)}\right)dx\right)\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + \lambda|u|^{p(x)-2}uv\right)dx \leq (\text{resp. } \geq) \int_\Omega f(x,u)v\,dx.$$

Theorem 3.1. *Let* $\lambda > 0$ *and* $q \in C(\overline{\Omega})$ *satisfies* (2.1). *Then for each* $h \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$, *the problem*

$$\begin{cases} -M\left(\int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}\lambda|u|^{p(x)}\right)dx\right)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - \lambda|u|^{p(x)-2}u\right) = h(x) \text{ in } \Omega \\ \frac{\partial u}{\partial v} = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{3.3'}$$

has a unique solution $u \in X$.

Proof. According to Propositions 2.3 and 2.4, $(f,v) := \int_\Omega f(x)v\,dx$ (for any $v \in X$) defines a continuous linear functional on X. Since Φ' is a homeomorphism, $\left(P_\lambda^f\right)$ has a unique solution.

Let $q \in C(\overline{\Omega})$ satisfy (2.1). For $h \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$, we denote by $K(h) = K_\lambda(h) = u$ the unique solution of (3.3_λ). $K = K_\lambda$ is called the solution operator for (3.3_λ). From the regularity results and the embedding theorems we can obtain the properties of the solution operator K as follows.

Proposition 3.3. (1) *The mapping* $K : L^{\frac{q(x)}{q(x)-1}}(\Omega) \to X$ *is continuous and bounded. Moreover, the mapping* $K : L^{\frac{q(x)}{q(x)-1}}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is completely continuous since the embedding* $X \boxtimes L^{q(x)}(\Omega)$ *is compact.*

(2) *If* p *is log-Hölder continuous on* $\overline{\Omega}$, *then the mapping* $K : L^\infty(\Omega) \to C^{0,\alpha}(\overline{\Omega})$ *is bounded, and hence the mapping* $K : L^\infty(\Omega) \to C(\overline{\Omega})$ *is completely continuous.*

(3) *If* p *is Hölder continuous on* $\overline{\Omega}$, *then the mapping* $K : L^\infty(\Omega) \to C^{1,\alpha}(\overline{\Omega})$ *is bounded, and hence the mapping* $K : L^\infty(\Omega) \to C^1(\overline{\Omega})$ *is completely continuous.*

Using the similar proof to [36], we have

Proposition 3.4. *If* $h \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$ *and* $h \geq 0$, *where* $q \in C(\overline{\Omega})$ *satisfies* (2.1), *then* $K(h) \geq 0$. *If* $p \in C^1(\Omega)$, $h \in L^\infty(\Omega)$ *and* $h \geq 0$, *then* $K(h) > 0$ *on* $\overline{\Omega}$.

Now we give a comparison principle as follows.

Theorem 3.2. *Let* $u, v \in X$, $\varphi \in W_0^{1,p(x)}(\Omega)$. *If*

$$M(I_0(u))\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + \lambda|u|^{p(x)-2}u\varphi\right)dx \leq M(I_0(v))\int_\Omega \left(|\nabla v|^{p(x)-2}\nabla v\nabla\varphi + \lambda|v|^{p(x)-2}v\varphi\right)dx \tag{3.4}$$

with $\phi \geq 0$ *and* $u \leq v$ *on* $\partial\Omega$, $I_0(u) := \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{1}{p(x)}\lambda|u|^{p(x)}\right)dx$, *then* $u \leq v$ *in* Ω.

Proof. Taking $\phi = (u - v)^+$ as a test function in (3.4), we have

$$\begin{aligned}\langle\Phi'(u) - \Phi'(v), \varphi\rangle = M\left(\int_\Omega \left(\frac{|\nabla u|^{p(x)} + \lambda|u|^{p(x)}}{p(x)}\right)dx\right)\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\varphi + \lambda|u|^{p(x)-2}u\varphi\right)dx \\ - M\left(\int_\Omega \left(\frac{|\nabla v|^{p(x)} + \lambda|v|^{p(x)}}{p(x)}\right)dx\right)\int_\Omega \left(|\nabla v|^{p(x)-2}\nabla v\nabla\varphi + \lambda|v|^{p(x)-2}v\varphi\right)dx \\ \leq 0.\end{aligned}$$

Using the similar proof to Theorem 2.1 of [15] with obvious changes, we can show that

$$\begin{aligned}\langle\Phi'(u) - \Phi'(v), \varphi\rangle \geq m_0\left[\int_\Omega \frac{1}{2}(|\nabla u|^{p(x)-2} - |\nabla v|^{p(x)-2})(|\nabla u|^2 - |\nabla v|^2)dx\right] \\ + m_0\lambda\left[\int_\Omega \frac{1}{2}(|u|^{p(x)-2} - |v|^{p(x)-2})(|u|^2 - |v|^2)dx\right] \geq 0.\end{aligned}$$

Therefore, we get $\langle\Phi'(u) - \Phi'(v), \phi\rangle = 0$. Proposition 2.5 implies that $\phi \equiv 0$ or $u \equiv v$ in Ω. It follows that $u \leq v$ in Ω.

It follows from Theorem 3.2 that the solution operator K is increasing under the condition (M_0), that is, $K(u) \leq K(v)$ if $u \leq v$.

In this article we will use the following sub-supersolution principle, the proof of which is based on the well known fixed point theorem for the increasing operator on the order interval (see e.g., [45]) and is similar to that given in [12] for Dirichlet problems involving the $p(x)$-Laplacian.

Theorem 3.3. (A sub-supersolution principle) *Suppose that* u_0, $v^0 \in X \cap L^\infty(\Omega)$, u_0 *and* v^0 *are a subsolution and a supersolution of* $\left(P_\lambda^f\right)$ *respectively, and* $u_0 \leq v^0$. *If f satisfies the condition:*

$$f(x,t) \text{ is nondecreasingin } \ t \in [\inf u_0(x), \sup v^0(x)], \tag{3.5}$$

then $\left(P_\lambda^f\right)$ *has a minimal solution* u_* *and a maximal solution* v^* *in the order interval* $[u_0, v^0]$, *i.e.,* $u_0 \leq u_* \leq v^* \leq v^0$ *and if u is any solution of* $\left(P_\lambda^f\right)$ *such that* $u_0 \leq u \leq v^0$, *then* $u_* \leq u \leq v^*$.

The energy functional corresponding to $\left(P_\lambda^f\right)$ is

$$J_\lambda(u) = \Phi(u) - \int_\Omega F(x,u)dx, \quad \forall u \in X. \tag{3.6}$$

The critical points of J_λ are just the solutions of $\left(P_\lambda^f\right)$. Many authors, for example, Chang [46], Brezis and Nirenberg [47] and Ambrosetti et al. [48], have combined the sub-supersolution method with the variational method and studied successfully the semilinear elliptic problems, where a key lemma is that a local minimizer of the associated energy functional in the C^1-topology is also a local minimizer in the H^1-topology. Such lemma have been extended to the case of the p-Laplacian equations (see [43,49]) and also to the case of the $p(x)$-Laplacian equations (see [12, Theorem 3.1]). In [50], Fan extended the Brezis-Nirenberg type theorem to the case of the $p(x)$-Kirchhoff [50, Theorem 1.1]. The Theorem 1.1 of [50] concerns with the Dirichlet problems, but the method for proving the theorem is also valid for the Neumann problems. Thus we have the following

Theorem 3.4. *Let* $\lambda > 0$ *and* (1.6) *holds. If* $u \in C^1(\overline{\Omega})$ *is a local minimizer of* J_λ *in the* $C^1(\overline{\Omega})$*-topology, then u is also a local minimizer of* J_λ *in the X-topology.*

4 Proof of theorems

In this section we shall prove Theorems 1.1-1.3. Since only the positive solutions are considered, without loss of generality, we can assume that

$$f(x,t) = f(x,0) \quad \text{for } t < 0 \text{ and } x \in \overline{\Omega},$$

otherwise we may replace $f(x,t)$ by $f^{(+)}(x,t)$, where

$$f^{(+)}(x,t) = \begin{cases} f(x,t) & \text{if } t \geq 0, \\ f(x,0) & \text{if } t < 0. \end{cases}$$

The proof of Theorem 1.1 consists of the following several Lemmata 4.1-4.6.

Lemma 4.1. *Let* (1.4) *hold. Then* $\lambda > 0$ *if* $\lambda \in \Lambda$.

Proof. Let $\lambda \in \Lambda$ and u be a positive solution of $\left(P_\lambda^f\right)$. Taking $v \equiv 1$ as a test function in Definition 3.1. (1) yields

$$M\left(\int_\Omega \frac{\lambda}{p(x)}|u|^{p(x)}dx\right)\lambda\int_\Omega u^{p(x)-1}dx = \int_\Omega f(x,u)dx, \tag{4.1}$$

which implies $\lambda > 0$ because the value of the right side in (4.1) is positive.

Lemma 4.2. *Let* (1.4) *and* (1.5) *hold. Then* $\Lambda \neq \emptyset$.

Proof. By Theorem 3.1, Propositions 3.4 and 3.3. (3), the problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\right)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - |u|^{p(x)-2}u\right) = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial v} = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{4.2}$$

has a unique positive solution $w_1 \in C^1(\overline{\Omega})$ and $w_1(x) \geq \varepsilon > 0$ for $x \in \overline{\Omega}$. We can assume $\varepsilon \leq 1$. Put $d = \sup\{f(x, w_1(x)) : x \in \overline{\Omega}\}$, $M_3 = \dfrac{d}{m_0\varepsilon^{p_+-1}}$ and $\lambda_1 = 1 + M_3$. Then

$$\begin{aligned} &-M\left(\int_\Omega \frac{1}{p(x)}(|\nabla w_1|^{p(x)} + \lambda_1|w_1|^{p(x)})dx\right)\left(\Delta_{p(x)}w_1 - \lambda_1 w_1^{p(x)-1}\right) = \\ &\quad -M\left(\int_\Omega \frac{1}{p(x)}(|\nabla w_1|^{p(x)} + \lambda_1|w_1|^{p(x)})dx\right)\left(\Delta_{p(x)}w_1 - w_1^{p(x)-1}\right) \\ &\quad + M\left(\int_\Omega \frac{1}{p(x)}(|\nabla w_1|^{p(x)} + \lambda_1|w_1|^{p(x)})dx\right)M_3 w_1^{p(x)-1} \\ &\geq m_0 M_3 \varepsilon^{p_+-1} = d \geq f(x, w_1(x)). \end{aligned}$$

This shows that w_1 is a supersolution of the problem $\left(P^f_{\lambda_1}\right)$. Obviously 0 is a subsolution of $\left(P^f_{\lambda_1}\right)$. By Theorem 3.3, $\left(P^f_{\lambda_1}\right)$ has a solution u_{λ_1} such that $0 \le u_{\lambda_1} \le w_1$. By Proposition 3.4, $u_{\lambda_1} > 0$ on $\overline{\Omega}$. So $\lambda_1 \in \Lambda$ and $\Lambda \neq \emptyset$.

Lemma 4.3. *Let* (1.4) *and* (1.5) *hold. If* $\lambda_0 \in \Lambda$, *then* $\lambda \in \Lambda$ *for all* $\lambda > \lambda_0$.

Proof. Let $\lambda_0 \in \Lambda$ and $\lambda > \lambda_0$. Let u_{λ_0} be a positive solution of $\left(P^f_{\lambda_0}\right)$. Then, we have

$$\begin{aligned}-\Delta_{p(x)}u_{\lambda_0} + \lambda u_{\lambda_0}^{p(x)-1} &\ge -\Delta_{p(x)}u_{\lambda_0} + \lambda_0 u_{\lambda_0}^{p(x)-1}\\ &= \frac{f(x,u_{\lambda_0})}{M\left(\int_\Omega \frac{1}{p(x)}(\left|\nabla u_{\lambda_0}\right|^{p(x)} + \lambda_0\left|u_{\lambda_0}\right|^{p(x)})dx\right)}\\ &\ge \frac{f(x,u_{\lambda_0})}{M\left(\int_\Omega \frac{1}{p(x)}(\left|\nabla u_{\lambda_0}\right|^{p(x)} + \lambda\left|u_{\lambda_0}\right|^{p(x)})dx\right)}\end{aligned}$$

thanks to (M_0). This shows that u_{λ_0} is a supersolution of $\left(P^f_{\lambda}\right)$. We know that 0 is a subsolution of $\left(P^f_{\lambda}\right)$ By Theorem 3.3, $\left(P^f_{\lambda}\right)$ has a solution u_λ such that $0 \le u_\lambda \le u_{\lambda_0}$. By Proposition 3.4, $u_\lambda > 0$ on $\overline{\Omega}$. Thus $\lambda \in \Lambda$.

Lemma 4.4. *Let* (1.4) *and* (1.5) *hold. Then for every* $\lambda > \lambda_*$, *there exists a minimal positive solution* u_λ *of* $\left(P^f_{\lambda}\right)$*such that* $u_{\lambda_1} \le u_{\lambda_2}$*if* $\lambda_* < \lambda_2 < \lambda_1$.

Proof. The proof is similar to [36, Lemma 3.4], we omit it here.

Lemma 4.5. *Let* (1.4) *and* (1.5) *hold. Let* $\lambda_1, \lambda_2 \in \Lambda$ *and* $\lambda_2 < \lambda < \lambda_1$. *Suppose that* u_{λ_2}*and* u_{λ_2}*are the positive solutions of* $\left(P^f_{\lambda_1}\right)$*and* $\left(P^f_{\lambda_2}\right)$*respectively and* $u_{\lambda_1} \le u_{\lambda_2}$. *Then there exists a positive solution* v_λ *of* $\left(P^f_{\lambda}\right)$*such that* $u_{\lambda_1} \le v_\lambda \le u_{\lambda_2}$*and* v_λ *is a global minimizer of the restriction of* J_λ *to the order interval* $\left[u_{\lambda_1}, u_{\lambda_2}\right] \cap X$.

Proof. Define $\widetilde{f} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(x,t) = \begin{cases} f(x,u_{\lambda_1}(x)), & \text{if } t < u_{\lambda_1}(x)\\ f(x,t), & \text{if } u_{\lambda_1}(x) \le t \le u_{\lambda_2}(x)\\ f(x,u_{\lambda_2}(x)), & \text{if } t > u_{\lambda_2}(x).\end{cases}$$

Define $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)ds$ and for all $u \in X$,

$$\widetilde{J}_\lambda(u) = \widehat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)} + \lambda|u|^{p(x)}}{p(x)}dx\right) - \int_\Omega \widetilde{F}(x,u)dx.$$

It is easy to see that the global minimum of $\widetilde{J}$ on X is achieved at some $v_\lambda \in X$. Thus v_λ is a solution of the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{|\nabla u|^{p(x)} + \lambda|u|^{p(x)}}{p(x)}dx\right)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - \lambda|u|^{p(x)-2}u\right) = \widetilde{f}(x,u) & \text{in } \Omega\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\end{cases} \tag{4.3}$$

and $v_\lambda \in C^1(\overline{\Omega})$. Noting that

$$f\left(x, u_{\lambda_1}\right) = \widetilde{f}\left(x, u_{\lambda_1}\right) \le \widetilde{f}(x, v_\lambda) \le \overline{f}(x, u_{\lambda_2}) = f(x, u_{\lambda_2})$$

and $\lambda_2 < \lambda < \lambda_1$, since K is increasing operator, we obtain that $u_{\lambda_1} \le v_\lambda \le u_{\lambda_2}$. So $\widetilde{f}(x, v_\lambda) = f(x, v_\lambda)$, and v_λ is a positive solution of $\left(P_\lambda^f\right)$. It is easy to see that there exists a constant c such that $J_\lambda(u)\widetilde{J}_\lambda(u) + c$ for $u \in \left[u_{\lambda_1}, u_{\lambda_2}\right] \cap X$. Hence v_λ is a global minimizer of $J_\lambda|_{[u_{\lambda_1}, u_{\lambda_2}]\cap X}$.

A key lemma of this paper is the following strong comparison principle.

Lemma 4.6 (A strong comparison principle). *Let (1.4) and (1.5) hold. Let* $\lambda_1, \lambda_2 \in \Lambda$ *and* $\lambda_2 < \lambda_1$. *Suppose that* u_{λ_1}*and* u_{λ_2}*are the positive solutions of* (1.1_{λ_1})*and* (1.1_{λ_2})*respectively. Then* $u_{\lambda_1} < u_{\lambda_2}$*on* $\overline{\Omega}$.

Proof. Since $u_{\lambda_1}, u_{\lambda_2} \in C^1(\overline{\Omega})$ and $u_{\lambda_1} > 0$ on $\overline{\Omega}$, in view of Lemma 4.4, there exist two positive constants $b_1 \le 1$ and b_2 such that

$$b_1 \le u_{\lambda_1} \le u_{\lambda_2} \le b_2 \text{ on } \overline{\Omega}.$$

For $\varepsilon \in \left(0, \frac{b_1}{2}\right)$, setting $v_\varepsilon = u_{\lambda_2} - \varepsilon$, then

$$\begin{aligned}
&-M\left(\int_\Omega \frac{|\nabla v_\varepsilon|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)}dx\right)(\mathrm{div}(|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon) - \lambda_1 v_\varepsilon^{p(x)-1})\\
&= -M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_2}\right|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)}dx\right)(\mathrm{div}(\left|\nabla u_{\lambda_2}\right|^{p(x)-2}\nabla u_{\lambda_2}) - \lambda_2 v_\varepsilon^{p(x)-1} + (\lambda_2 - \lambda_1)v_\varepsilon^{p(x)-1})\\
&= -M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_2}\right|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)}dx\right)(\mathrm{div}(\left|\nabla u_{\lambda_2}\right|^{p(x)-2}\nabla u_{\lambda_2}) - \lambda_2 u_{\lambda_2}^{p(x)-1})\\
&\quad + M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_2}\right|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)}dx\right)\left((\lambda_1 - \lambda_2)v_\varepsilon^{p(x)-1} - \lambda_2\left(u_{\lambda_2}^{p(x)-1} - v_\varepsilon^{p(x)-1}\right)\right)\\
&\ge f(x, u_{\lambda_2})\frac{M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_2}\right|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)}dx\right)}{M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_2}\right|^{p(x)} + \lambda_2 \left|u_{\lambda_2}\right|^{p(x)}}{p(x)}dx\right)}\\
&\quad + M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_2}\right|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)}dx\right)\left((\lambda_1 - \lambda_2)\left(\frac{b_1}{2}\right)^{p^+ - 1} - \lambda_2\left(u_{\lambda_2}^{p(x)-1} - v_\varepsilon^{p(x)-1}\right)\right).
\end{aligned}$$

Taking an $\varepsilon > 0$ sufficiently small such that

$$\lambda_2\left(u_{\lambda_2}^{p(x)-1} - v_\varepsilon^{p(x)-1}\right) < (\lambda_1 - \lambda_2)\left(\frac{b_1}{2}\right)^{p_+ - 1} \quad \text{for } x \in \overline{\Omega}$$

and

$$\lambda_1 \int_\Omega \frac{1}{p(x)}|v_\varepsilon|^{p(x)}dx \ge \lambda_2 \int_\Omega \frac{1}{p(x)}\left|u_{\lambda_2}\right|^{p(x)}dx,$$

then

$$-M\left(\int_\Omega \frac{|\nabla v_\varepsilon|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)}dx\right)(div(|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon) - \lambda_1 v_\varepsilon^{p(x)-1}) = g(x) \ge f(x, u_{\lambda_2}),$$

consequently, v_ε is a solution of the problem

$$\begin{cases} -M\left(\int_\Omega \dfrac{|\nabla v_\varepsilon|^{p(x)} + \lambda_1 |v_\varepsilon|^{p(x)}}{p(x)} dx\right)\left(\operatorname{div}(|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon) - \lambda_1 v_\varepsilon^{p(x)-1}\right) = g(x) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $g(x) \geq f(x, u_{\lambda_2})$. With other words, $v_\varepsilon = K_{\lambda_1}(g)$, where K_{λ_1} is the solution operator of (3.1_{λ_1}). Since $u_{\lambda_1} = K_{\lambda_1}(h)$, where $h(x) = f(x, u_{\lambda_1}) \leq f(x, u_{\lambda_2}) \leq g(x)$, noting that K_{λ_1} is increasing, we have $v_\varepsilon \geq u_{\lambda_1}$, that is, $u_{\lambda_2} - \varepsilon \geq u_{\lambda_1}$ on $\overline{\Omega}$.

The proof of Theorem 1.1 is complete. Let us now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let (1.4)-(1.8) hold. Let $\lambda > \lambda_*$. Take $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_2 < \lambda < \lambda_1$ and let $u_{\lambda_1} \leq u_\lambda \leq u_{\lambda_2}$ be as in Lemma 4.5.

We claim that u_λ is a local minimizer of J_λ in the X-topology.

Indeed, Lemma 4.6 implies that $u_{\lambda_1} < u_\lambda < u_{\lambda_2}$ on $\overline{\Omega}$. It follows that there is a C^0-neighborhood U of u_λ such that $U \subset [u_{\lambda_1}, u_{\lambda_2}]$, consequently u_λ is a local minimizer of J_λ in the C^0-topology, and of course, also in the C^1-topology. By Theorem 3.4, u_λ is also a local minimizer of J_λ in the X-topology.

Define

$$\widetilde{f}_\lambda(x,t) = \begin{cases} f(x,t), & \text{if } t > u_\lambda(x), \\ f(x, u_\lambda(x)), & \text{if } t \leq u_\lambda(x), \end{cases}$$

and $\widetilde{F}_\lambda(x,t) = \int_0^t \widetilde{f}_\lambda(x,s)ds$. Consider the problem

$$\begin{cases} -M\left(\int_\Omega \dfrac{|\nabla u|^{p(x)} + \lambda |u|^{p(x)}}{p(x)} dx\right)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - \lambda |u|^{p(x)-2}u\right) = \widetilde{f}_\lambda(x,u) \text{ in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \end{cases}$$

and denote by $\widetilde{J}_\lambda$ the energy functional corresponding to (4.4_λ). By the definition of $\widetilde{f}_\lambda$, we have $\widetilde{f}_\lambda(x, u(x)) \geq f(x, u_\lambda(x))$ for every $u \in X$. Hence, for each solution u of (4.4_λ), we have that $u \geq u_\lambda$, consequently $\widetilde{f}_\lambda(x,u) = f(x,u)$ and u is also a solution of $\left(P_\lambda^f\right)$. It is easy to see that u_{λ_1} and u_{λ_2} are a subsolution and a supersolution of (4.4_λ) respectively. By Theorems 3.3 and 1.2, there exists $u_\lambda^* \in [u_{\lambda_1}, u_{\lambda_2}] \cap C^1(\overline{\Omega})$ such that u_λ^* is a solution of (4.4_λ) and is a local minimizer of $\widetilde{J}_\lambda$ in the C^1-topology. As was noted above, we know that $u_\lambda^* \geq u_\lambda$ and u_λ^* is also a solution of $\left(P_\lambda^f\right)$. If $u_\lambda^* \neq u_\lambda$, then the assertion of Theorem 1.2 already holds, hence we can assume that $u_\lambda^* = u_\lambda$. Now u_λ is a local minimizer of $\widetilde{J}_\lambda$ in the C^1-topology, and so also in the X-topology. We can assume that u_λ is a strictly local minimizer of $\widetilde{J}_\lambda$ in the X-topology, otherwise we have obtained the assertion of Theorem 1.2. It is easy to verify that, under the assumptions of Theorem 1.3, $\widetilde{J}_\lambda \in C^1(X, \mathbb{R})$ and $\widetilde{J}_\lambda$ satisfies the (P.S.) condition (see e.g., [30]). It follows from the condition (1.7) and (1.8) that $\{\widetilde{J}_\lambda(u) : u \in X\} = -\infty$ (see e.g., [30]). Using the mountain pass lemma (see [51]), we know that (4.4_λ) has a solution v_λ such that $v_\lambda \neq u_\lambda$. v_λ, as a solution of (4.4_λ), must satisfy $v_\lambda \geq u_\lambda$, and v_λ is also a solution of $\left(P_\lambda^f\right)$. The proof of Theorem 1.2 is complete.

Proof of Theorem 1.3. (1) Let f satisfy (1.4), (1.9), and (1.10). For given any $\lambda > 0$, consider the energy functional J_λ defined by (3.3). By (1.10) and noting that $r(x) < p(x)$ for $x \in \overline{\Omega}$, there is a positive constant M_4 such that

$$\left|F(x,t)\right| \le \frac{\lambda m_0}{2p^+}|t|^{p(x)}, \quad \forall x \in \Omega, \quad \forall\, |t| \ge M_4. \tag{4.5}$$

For $u \in X$ with $||u||_\lambda \ge 1$, we have that

$$\begin{aligned} J_\lambda(u) &\ge \frac{m_0}{p^+}\int_\Omega \left(|\nabla|^{p(x)} + \lambda|u|^{p(x)}\right)dx - \frac{\lambda m_0}{2p^+}\int_\Omega |u|^{p(x)}dx - c_5 \\ &\ge \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{\lambda m_0}{2p^+}\int_\Omega |u|^{p(x)}dx - c_5 \\ &\ge \frac{m_0}{2p^+}\,\|u\|_\lambda^{p_-} - c_5, \end{aligned}$$

where c_5 is a positive constant. This shows that $J_\lambda(u) \to +\infty$ as $||u||_\lambda \to +\infty$, that is, J_λ is coercive. In view of Proposition 2.5. (iv), the condition (1.10) also implies that J_λ is weakly sequentially lower semi-continuous. Thus J_λ has a global minimizer u_0. Put $v_0(x) = |u_0(x)|$ for $x \in \overline{\Omega}$. It is easy to see that $J_\lambda(v_0) \le J_\lambda(u_0)$, consequently, v_0 is a global minimizer of J_λ and is a positive solution of $\left(P_\lambda^f\right)$. This shows that $\lambda \in \Lambda$ for all $\lambda > 0$. Hence $\lambda_* = 0$ and the statement (1) is proved.

To prove Theorem 1.3. (2) we give the following lemma.

Lemma 4.7. *Let* (1.4) *and* (1.5) *hold. Then for each* $\lambda > \lambda_*$, $\left(P_\lambda^f\right)$ *has a positive solution* u_λ *such that* $J_\lambda(u_\lambda) \le 0$.

Proof. Let $\lambda > \lambda_*$. Take $\lambda_2 \in (\lambda_*, \lambda)$ and let u_{λ_2} be a positive solution of $\left(P_{\lambda_2}^f\right)$. then u_{λ_2} is a supersolution of $\left(P_\lambda^f\right)$. We know that 0 is a subsolution of $\left(P_\lambda^f\right)$. Analogous to the proof of Lemma 4.5, we can prove that $\left(P_\lambda^f\right)$ has a positive solution $u_\lambda \in [0, u_{\lambda_2}]$ such that $J_\lambda(u_\lambda) = \inf\{J_\lambda(u) : u \in [0, u_{\lambda_2}]\}$. So $J_\lambda(u_\lambda) \le J_\lambda(0) = 0$.

Proof of Theorem 1.3. (2). Let (1.4)-(1.8) hold. Let $\lambda_n > \lambda_*$ and $\lambda_n \to \lambda_*$ as $n \to +\infty$. By Lemma 4.7, for each n, $\left(P_{\lambda_n}^f\right)$ has a positive solution u_{λ_n} such that $J_{\lambda_n}(u_{\lambda_n}) \le 0$, that is

$$\widehat{M}\left(\int_\Omega \frac{\left|\nabla u_{\lambda_n}\right|^{p(x)} + \lambda_n\left|u_{\lambda_n}\right|^{p(x)}}{p(x)}dx\right) \le \int_\Omega F(x, u_{\lambda_n})dx.$$

Since u_{λ_n} is a solution of $\left(P_{\lambda_n}^f\right)$, we have that

$$M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_n}\right|^{p(x)} + \lambda_n\left|u_{\lambda_n}\right|^{p(x)}}{p(x)}dx\right)\int_\Omega \left(\left|\nabla u_{\lambda_n}\right|^{p(x)} + \lambda_n\left|u_{\lambda_n}\right|^{p(x)}\right)dx = \int_\Omega f(x, u_{\lambda_n})u_{\lambda_n}dx.$$

It follows from (1.8) that there exists a positive constant c_6 such that

$$\int_\Omega F(x, u_{\lambda_n})dx \le c_6 + \frac{1}{\theta}\int_\Omega f(x, u_{\lambda_n})u_{\lambda_n}dx.$$

Thus, using condition (1.7), we have that

$$\frac{1-\mu}{p_+}M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_n}\right|^{p(x)}+\lambda_n\left|u_{\lambda_n}\right|^{p(x)}}{p(x)}dx\right)\int_\Omega \left(\left|\nabla u_{\lambda_n}\right|^{p(x)}+\lambda_n\left|u_{\lambda_n}\right|^{p(x)}\right)dx$$
$$\le c_6+\frac{1}{\theta}\int_\Omega f(x,u_{\lambda_n})u_{\lambda_n}dx$$
$$\le c_6+\frac{1}{\theta}M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_n}\right|^{p(x)}+\lambda_n\left|u_{\lambda_n}\right|^{p(x)}}{p(x)}dx\right)\int_\Omega \left(\left|\nabla u_{\lambda_n}\right|^{p(x)}+\lambda_n\left|u_{\lambda_n}\right|^{p(x)}\right)dx,$$

and consequently,

$$m_0\left(\frac{1-\mu}{p_+}-\frac{1}{\theta}\right)\left\|u_{\lambda_n}\right\|_{\lambda_n}^{p_-}\le c_7,$$

where the positive constant c_7 is independent of n. This shows that $\{\|u_{\lambda_n}\|_{\lambda_n}\}$ is bounded. Noting that $\lambda_n \to \lambda_* > 0$, we have that $\{\|u_{\lambda_n}\|\}$ is bounded. Without loss of generality, we can assume that $u_{\lambda_n} \rightharpoonup u_*$ in X and $u_{\lambda_n}(x) \to u_*(x)$ for a.e. $x \in \Omega$. By (1.6) and the $L^\infty(\Omega)$-regularity results of [44], the boundedness of $\{\|u_{\lambda_n}\|\}$ implies the boundedness of $\left\{|u_{\lambda_n}|_{L^\infty(\Omega)}\right\}$. By the $C^{1,\alpha}(\overline{\Omega})$-regularity results of [37], the boundedness of $\left\{|u_{\lambda_n}|_{L^\infty(\Omega)}\right\}$ implies the boundedness of $\left\{\|u_{\lambda_n}\|_{C^{1,\alpha}(\overline{\Omega})}\right\}$, where $\alpha \in (0, 1)$ is a constant. Thus we have $u_{\lambda_n} \rightharpoonup u_*$ in $C^1(\overline{\Omega})$. For every $v \in X$, since u_{λ_n} is a solution of $\left(P^f_{\lambda_n}\right)$, we have that, for each n,

$$M\left(\int_\Omega \frac{\left|\nabla u_{\lambda_n}\right|^{p(x)}+\lambda_n\left|u_{\lambda_n}\right|^{p(x)}}{p(x)}dx\right)\int_\Omega \left(\left|\nabla u_{\lambda_n}\right|^{p(x)-2}\nabla u_{\lambda_n}\nabla v+\lambda_n\left|u_{\lambda_n}\right|^{p(x)-2}u_{\lambda_n}v\right)dx=\int_\Omega f(x,u_{\lambda_n})v\,dx$$

Passing the limit of above equality as $n \to +\infty$, yields

$$M\left(\int_\Omega \frac{|\nabla u_*|^{p(x)}+\lambda_*|u_*|^{p(x)}}{p(x)}dx\right)\int_\Omega \left(|\nabla u_*|^{p(x)-2}\nabla u_*\nabla v+\lambda_*|u_*|^{p(x)-2}u_*v\right)dx=\int_\Omega f(x,u_*)v\,dx,$$

which shows that u_* is a solution of $\left(P^f_{\lambda_*}\right)$. Obviously $u_* \ge 0$ and $u_* \not\equiv 0$. Hence u_* is a positive solution of $\left(P^f_{\lambda_*}\right)$ and $\lambda_* \in \Lambda$.

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Authors' contributions
GD conceived of the study, and participated in its design and coordination and helped to draft the manuscript. RM participated in the design of the study. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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