Boundary Value Problems
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Partial vanishing viscosity limit for the 2D Boussinesq system with a slip boundary condition

Liangbing Jin[1], Jishan Fan[2], Gen Nakamura[3] and Yong Zhou[1*]

* Correspondence: yzhoumath@zjnu.edu.cn
[1]Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P. R. China
Full list of author information is available at the end of the article

Abstract

This article studies the partial vanishing viscosity limit of the 2D Boussinesq system in a bounded domain with a slip boundary condition. The result is proved globally in time by a logarithmic Sobolev inequality.

2010 MSC: 35Q30; 76D03; 76D05; 76D07.

Keywords: Boussinesq system, inviscid limit, slip boundary condition

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain with smooth boundary $\partial\Omega$, and n is the unit outward normal vector to $\partial\Omega$. We consider the Boussinesq system in $\Omega \times (0, \infty)$:

$$\partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = \theta e_2, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t \theta + u \cdot \nabla \theta = \varepsilon \Delta \theta, \tag{1.3}$$

$$u \cdot n = 0, \quad \operatorname{curl} u = 0, \quad \theta = 0, \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.4}$$

$$(u, \theta)(x, 0) = (u_0, \theta_0)(x), \quad x \in \Omega, \tag{1.5}$$

where u, π, and θ denote unknown velocity vector field, pressure scalar and temperature of the fluid. $\epsilon > 0$ is the heat conductivity coefficient and $e_2 := (0, 1)^t$. $\omega := \operatorname{curl} u := \partial_1 u_2 - \partial_2 u_1$ is the vorticity.

The aim of this article is to study the partial vanishing viscosity limit $\epsilon \to 0$. When $\Omega := \mathbb{R}^2$, the problem has been solved by Chae [1]. When $\theta = 0$, the Boussinesq system reduces to the well-known Navier-Stokes equations. The investigation of the inviscid limit of solutions of the Navier-Stokes equations is a classical issue. We refer to the articles [2-7] when Ω is a bounded domain. However, the methods in [1-6] could not be used here directly. We will use a well-known logarithmic Sobolev inequality in [8,9] to complete our proof. We will prove:

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Theorem 1.1. *Let* $u_0 \in H^3$, $\operatorname{div} u_0 = 0$ *in* Ω, $u_0 \cdot n = 0$, $\operatorname{curl} u_0 = 0$ *on* $\partial\Omega$ *and* $\theta_0 \in H_0^1 \cap H^2$. *Then there exists a positive constant* C *independent of* ϵ *such that*

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(0,T;H^3)\cap L^2(0,T;H^4)} \le C, \ \|\theta_\varepsilon\|_{L^\infty(0,T;H^2)} \le C, \\ &\|\partial_t u_\varepsilon\|_{L^2(0,T;L^2)} \le C, \qquad \|\partial_t \theta_\varepsilon\|_{L^2(0,T;L^2)} \le C \end{aligned} \tag{1.6}$$

for any $T > 0$, *which implies*

$$(u_\varepsilon, q_\varepsilon) \to (u, \theta) \text{ strongly in } L^2(0, T; H^1) \text{ when } \varepsilon \to 0. \tag{1.7}$$

Here (u, θ) *is the unique solution of the problem (1.1)-(1.5) with* $\epsilon = 0$.

2 Proof of Theorem 1.1

Since (1.7) follows easily from (1.6) by the Aubin-Lions compactness principle, we only need to prove the a priori estimates (1.6). From now on we will drop the subscript e and throughout this section C will be a constant independent of $\epsilon > 0$.

First, we recall the following two lemmas in [8-10].

Lemma 2.1. *([8,9]) There holds*

$$\|\nabla u\|_{L^\infty(\Omega)} \le C(1 + \|\operatorname{curl} u\|_{L^\infty(\Omega)} \log(e + \|u\|_{H^3(\Omega)}))$$

for any $u \in H^3(\Omega)$ *with* $\operatorname{div} u = 0$ *in* Ω *and* $u \cdot n = 0$ *on* $\partial\Omega$.

Lemma 2.2. *([10]) For any* $u \in W^{s,p}$ *with* $\operatorname{div} u = 0$ *in* Ω *and* $u \cdot n = 0$ *on* $\partial\Omega$, *there holds*

$$\|u\|_{W^{s,p}} \le C\left(\|u\|_{L^p} + \|\operatorname{curl} u\|_{W^{s-1,p}}\right)$$

for any $s > 1$ *and* $p \in (1, \infty)$.

By the maximum principle, it follows from (1.2), (1.3), and (1.4) that

$$\|\theta\|_{L^\infty(0,T;L^\infty)} \le \|\theta_0\|_{L^\infty} \le C. \tag{2.1}$$

Testing (1.3) by θ, using (1.2), (1.3), and (1.4), we see that

$$\frac{1}{2}\frac{d}{dt}\int \theta^2 dx + \varepsilon \int |\nabla\theta|^2 dx = 0,$$

which gives

$$\sqrt{\varepsilon}\|\theta\|_{L^2(0,T;H^1)} \le C. \tag{2.2}$$

Testing (1.1) by u, using (1.2), (1.4), and (2.1), we find that

$$\frac{1}{2}\frac{d}{dt}\int u^2 dx + C\int |\nabla u|^2 dx = \int \theta e_2 u \le \|\theta\|_{L^2}\|u\|_{L^2} \le C\|u\|_{L^2},$$

which gives

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \le C. \tag{2.3}$$

Here we used the well-known inequality:

$$\|u\|_{H^1} \le C\|\operatorname{curl} u\|_{L^2}.$$

Applying curl to (1.1), using (1.2), we get

$$\partial_t \omega + u \cdot \nabla\omega - \Delta\omega = \operatorname{curl}(\theta e_2). \tag{2.4}$$

Testing (2.4) by $|\omega|^{p-2}\omega$ ($p > 2$), using (1.2), (1.4), and (2.1), we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int |\omega|^p dx + \frac{1}{2}\int |\omega|^{p-2}|\nabla\omega|^2 dx + 4\frac{p-2}{p^2}\int \left|\nabla|\omega|^{p/2}\right|^2 dx \\
&\quad = \int \operatorname{curl}(\theta e_2)|\omega|^{p-2}\omega dx \\
&\quad \le C\|\theta\|_{L^\infty}\int \left|\nabla\left(|\omega|^{p-2}\omega\right)\right| dx \\
&\quad \le \frac{1}{2}\left(\frac{1}{2}\int |\omega|^{p-2}|\nabla\omega|^2 dx + 4\frac{p-2}{p^2}\int \left|\nabla|\omega|^{p/2}\right|^2 dx\right) \\
&\qquad + C\int |\omega|^p dx + C,
\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0,T;W^{1,p})} \le C\|\omega\|_{L^\infty(0,T;L^p)} \le C. \tag{2.5}$$

(2.4) can be rewritten as

$$\begin{cases}
\partial_t\omega - \Delta\omega = \operatorname{div} f := \operatorname{curl}(\theta e_2) - \operatorname{div}(u\omega), \\
\omega = 0 \quad \text{on } \partial\Omega \times (0,\infty) \\
\omega(x,0) = \omega_0(x) \text{ in } \Omega
\end{cases}$$

with $f_1 := \theta - u_1\omega$, $f_2 := -u_2\omega$.

Using (2.1), (2.5) and the L^∞-estimate of the heat equation, we reach the key estimate

$$\|\omega\|_{L^\infty(0,T;L^\infty)} \le C\left(\|\omega_0\|_{L^\infty} + \|f\|_{L^\infty(0,T;L^p)} \le C\right). \tag{2.6}$$

Let τ be any unit tangential vector of $\partial\Omega$, using (1.4), we infer that

$$u\cdot\nabla\theta = ((u\cdot\tau)\tau + (u\cdot n)n)\cdot\nabla\theta = (u\cdot\tau)\tau\cdot\nabla\theta = (u\cdot\tau)\frac{\partial\theta}{\partial\tau} = 0 \tag{2.7}$$

on $\partial\Omega \times (0, \infty)$.

It follows from (1.3), (1.4), and (2.7) that

$$\Delta\theta = 0 \quad \text{on } \partial\Omega \times (0,\infty). \tag{2.8}$$

Applying Δ to (1.3), testing by $\Delta\theta$, using (1.2), (1.4), and (2.8), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta\theta|^2 dx + \varepsilon\int |\nabla\Delta\theta|^2 dx \\
&\quad = -\int (\Delta(u\cdot\nabla\theta) - u\nabla\Delta\theta)\Delta\theta dx \\
&\quad = -\int (\Delta u\cdot\nabla\theta + 2\sum_i \partial_i u\cdot\nabla\partial_i\theta)\Delta\theta dx \\
&\quad \le C\left(\|\Delta u\|_{L^4}\|\nabla\theta\|_{L^4} + \|\nabla u\|_{L^\infty}\|\Delta\theta\|_{L^2}\right)\|\Delta\theta\|_{L^2}.
\end{aligned} \tag{2.9}$$

Now using the Gagliardo-Nirenberg inequalities

$$\begin{aligned}
&\|\nabla\theta\|_{L^4}^2 \le C\|\theta\|_{L^\infty}\|\Delta\theta\|_{L^2}, \\
&\|\Delta u\|_{L^4}^2 \le C\|\nabla u\|_{L^\infty}\|u\|_{H^3},
\end{aligned} \tag{2.10}$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta\theta|^2 dx + \varepsilon \int |\nabla \Delta\theta|^2 dx \\
&\quad \le C\|\nabla u\|_{L^\infty} \|\Delta\theta\|_{L^2}^2 + C\|\Delta\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty} \|u\|_{H^3}^2 \\
&\quad \le C(1 + \|\nabla u\|_{L^\infty}) \left(\|u\|_{H^3}^2 + \|\Delta\theta\|_{L^2}^2\right) \\
&\quad \le C\left(1 + \|\omega\|_{L^\infty} \log(e + \|u\|_{H^3})\right)\left(1 + \|\Delta\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\right) \\
&\quad \le C\left(1 + \log(e + \|\Delta\omega\|_{L^2} + \|\Delta\theta\|_{L^2})\right)\left(1 + \|\Delta\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\right).
\end{aligned} \tag{2.11}$$

Similarly to (2.7) and (2.8), if follows from (2.4) and (1.4) that

$$u \cdot \nabla\omega = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{2.12}$$

$$\Delta\omega + \operatorname{curl}(\theta e_2) = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{2.13}$$

Applying Δ to (2.4), testing by $\Delta\omega$, using (1.2), (1.4), (2.13), (2.10), and Lemma 2.2, we reach

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta\omega|^2 dx + \int |\nabla\Delta\omega|^2 dx \\
&\quad = -\int (\Delta(u\cdot\nabla\omega) - u\nabla\Delta\omega)\Delta\omega dx - \int \nabla\operatorname{curl}(\theta e_2)\cdot\nabla\Delta\omega dx \\
&\quad \le C(\|\Delta u\|_{L^4}\|\nabla\omega\|_{L^4} + \|\nabla u\|_{L^\infty}\|\Delta\omega\|_{L^2})\|\Delta\omega\|_{L^2} + C\|\Delta\theta\|_{L^2}\|\nabla\Delta\omega\|_{L^2} \\
&\quad \le C\left(\|\Delta u\|_{L^4}^2 + \|\nabla u\|_{L^\infty}\|\Delta\omega\|_{L^2}\right)\|\Delta\omega\|_{L^2} + C\|\Delta\theta\|_{L^2}\|\nabla\Delta\omega\|_{L^2} \\
&\quad \le C\|\nabla u\|_{L^\infty}\|u\|_{H^3}\|\Delta\omega\|_{L^2} + C\|\Delta\theta\|_{L^2}\|\nabla\Delta\omega\|_{L^2} \\
&\quad \le C\|\nabla u\|_{L^\infty}(1 + \|\Delta\omega\|_{L^2})\|\Delta\omega\|_{L^2} + C\|\Delta\theta\|_{L^2}^2 + \frac{1}{2}\|\nabla\Delta\omega\|_{L^2}^2
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{d}{dt}\int |\Delta\omega|^2 dx + \int |\nabla\Delta\omega|^2 dx \\
&\quad \le C\|\nabla u\|_{L^\infty}(1 + \|\Delta\omega\|_{L^2})\|\Delta\omega\|_{L^2} + C\|\Delta\theta\|_{L^2}^2 \\
&\quad \le C\left(1 + \log(e + \|\Delta\omega\|_{L^2} + \|\Delta\theta\|_{L^2})\right)\left(1 + \|\Delta\omega\|_{L^2}^2 + \|\Delta\theta\|_{L^2}^2\right).
\end{aligned} \tag{2.14}$$

Combining (2.11) and (2.14), using the Gronwall inequality, we conclude that

$$\|\theta\|_{L^\infty(0,T;H^2)} + \sqrt{\varepsilon}\|\theta\|_{L^\infty(0,T;H^3)} \le C, \tag{2.15}$$

$$\|u\|_{L^\infty(0,T;H^3)} + \|u\|_{L^2(0,T;H^4)} \le C. \tag{2.16}$$

It follows from (1.1), (1.3), (2.15), and (2.16) that

$$\|\partial_t u\|_{L^2(0,T:L^2)} \le C, \quad \|\partial_t\theta\|_{L^2(0,T:L^2)} \le C.$$

This completes the proof.

Acknowledgements
This study was partially supported by the Zhejiang Innovation Project (Grant No. T200905), the ZJNSF (Grant No. R6090109), and the NSFC (Grant No. 11171154).

Author details
[1]Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P. R. China [2]Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, P.R. China [3]Department of Mathematics, Hokkaido University Sapporo 060-0810, Japan

Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 12 November 2011 Accepted: 15 February 2012 Published: 15 February 2012

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doi:10.1186/1687-2770-2012-20
Cite this article as: Jin *et al.*: Partial vanishing viscosity limit for the 2D Boussinesq system with a slip boundary condition. *Boundary Value Problems* 2012 **2012**:20.

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