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Lagrangian actions on 3-body problems with two fixed centers

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Abstract

In this paper, we study the existence of figure "∞"-type periodic solution for 3-body problems with strong-force potentials and two fixed centers, and we also give some remarks in the case with Newtonian weak-force potentials.

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Keywords: 3-body problems with two fixed centers, "∞"-type solutions, Lagrangian actions

1 Introduction and Main Result

We assume two masses $m_1 = m_2 = \frac{1}{2}$ are fixed at $q_1 = \left(\frac{-1}{2}, 0\right)$ and $q_2 = -q_1 = \left(\frac{1}{2}, 0\right)$, the third mass m_3 is affected by m_1 and m_2 and moving according to the Newton's second law and the general gravitational law [1,2], then the position $q(t)$ for m_3 satisfies

$$m_3\ddot{q}(t) = \frac{m_1 m_3 \alpha(q_1 - q)}{|q_1 - q|^{\alpha+2}} + \frac{m_2 m_3 \alpha(q_2 - q)}{|q_2 - q|^{\alpha+2}} \tag{1.1}$$

Equivalently,

$$\ddot{q}(t) = \frac{\alpha}{2}\left[\frac{q_1 - q}{|q_1 - q|^{\alpha+2}} + \frac{q_2 - q}{|q_2 - q|^{\alpha+2}}\right] \tag{1.2}$$

$$\ddot{q}(t) = \frac{\partial U(q)}{\partial q} \tag{1.3}$$

$$\text{Where } \alpha > 0, U(q) = \frac{1/2}{|q - q_1|^{\alpha}} + \frac{1/2}{|q - q_2|^{\alpha}}. \tag{1.4}$$

For the case $\alpha = 1$, Euler [3-5] studied (1.1)-(1.3), but didn't use variational methods to study periodic solutions.

Here we want to use variational minimizing method to look for periodic solution for m_3 which winds around q_1 and q_2, let

$$f(q) = \int_0^1 \left[\frac{1}{2}|\dot{q}|^2 + \frac{1/2}{|q - q_1|^\alpha} + \frac{1/2}{|q - q_2|^\alpha} \right] dt, \tag{1.5}$$

$$q \in \Lambda = \left\{ \begin{array}{c} q \in W^{1,2}(\mathbb{R}/\mathbb{Z},\ \ \mathbb{R}^2), \quad q(t) \neq q_1,\ \ q_2, \\ q\left(t + \frac{1}{2}\right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} q(t), \quad q(-t) = -q(t), \\ \deg(q - q_1) = 1,\ \deg(q - q_2) = -1 \end{array} \right\} \tag{1.6}$$

Theorem 1.1 For $\alpha \geq 2$, the minimizer of $f(q)$ on $\overline{\Lambda}$ does exist and is non-collision "∞"-type periodic solution of (1.1)-(1.3).(See Figure 1)

2 The Proof of Theorem 1.1

Using Palais'S symmetrical Principle [6], it's easy to prove the following variational Lemma:

Lemma 2.1 The critical point of $f(q)$ in Λ is the noncollision periodic solution winding around q_1 counter-clockwise and q_2 clockwise one time during one period.

Lemma 2.2 [7] If $x \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$ and $\exists\ t_0 \in [0,1]$, *s.t.* $x(t_0) = 0$, if $\alpha \geq 2$ and $a > 0$, then

$$\int_0^1 \left[\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|^\alpha} \right] dt = +\infty \tag{2.1}$$

It's easy to see

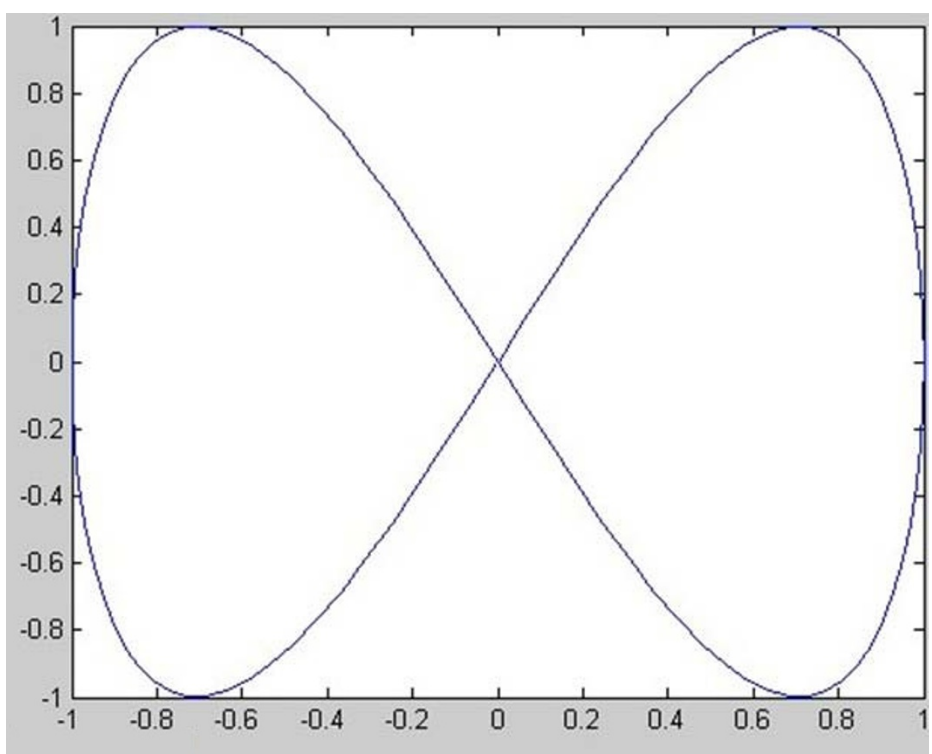

Figure 1 Figure-eight with 2-centres.

Lemma 2.3 $\overline{\Lambda}$ is a weakly closed subset of the Hilbert space $W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$.

Lemma 2.4 $f(q)$ is coercive and weakly lower-semicontinuous on the closure $\overline{\Lambda}$ of Λ.

Proof. By $q(-t) = -q(t)$ and $q(t) \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$, we have $\int_0^1 q(t)dt = 0$. By Wirtinger's inequality, we know $f(q)$ is coercive. By Sobolev's embedding Theorem and Fatou's Lemma, f is weakly lower-semi-continuous on the weakly closed set $\overline{\Lambda}$ of $W^{1,2}$.

Lemma 2.5 [8] Let X be a reflexive Banach space,$M \subset X$ be weakly closed subset,f : $M \to R$ be weakly lower semi-continous and coercive ($f(x) \to +\infty$ as $\|x\| \to +\infty$), then f attains its infimum on M.

According to *Lemmas* 2.1-2.5, we know that $f(q)$ attains its infimum on $\overline{\Lambda}$ and the minimizer of $f(q)$ on $\overline{\Lambda}$ is collision-free since if let $x_1 = q - q_1$, $x_2 = q - q_2$, *then*

$$\begin{aligned} f(q) &= \int_0^1 \left[\frac{1}{2}|\dot{q} - \dot{q}_1|^2 + \frac{1}{|q - q_1|^\alpha}\right] dt = \int_0^1 \left[\frac{1}{2}|\dot{q} - \dot{q}_2|^2 + \frac{1}{|q - q_2|^\alpha}\right] dt \\ &= \int_0^1 \left[\frac{1}{2}|\dot{x}_1|^2 + \frac{1}{|x_1|^\alpha}\right] dt = \int_0^1 \left[\frac{1}{2}|\dot{x}_2|^2 + \frac{1}{|x_2|^\alpha}\right] dt \end{aligned} \tag{2.2}$$

So if the minimizer of $f(q)$ on $\overline{\Lambda}$ has collision at some moment, then Gordon's Lemma tell us the minimum value is $+\infty$ which is a contradiction.

The most interesting case $\alpha = 1$ is the case for Newtonian potential, we try to prove the minimizer is collision-free, but it seems very difficult, here we give some remarks.

Lemma 2.6 [9] If $y(0) = 0$ and $2k$ is an even positive integer, then

$$\int_0^1 y^{2k} dx \le c \int_0^1 \dot{y}^{2k} dx, \tag{2.3}$$

where

$$c = \frac{1}{2k-1}\left(\frac{2k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right)^{2k}. \tag{2.4}$$

There is equality only for a certain hyperelliptic curve.

Now we estimate the lower bound of the Lagrangian action $f(q)$ on "∞"-type collisionorbits. Since $q_2 = -q_1 = \left(\frac{1}{2}, 0\right)$ and $q\left(t + \frac{1}{2}\right) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} q(t)$, so

$$\int_0^1 \frac{dt}{|q - q_1|} = \int_0^1 \frac{dt}{|q - q_2|}, \tag{2.5}$$

$$\begin{aligned} f(q) &= \int_0^1 \left[\frac{1}{2}|\dot{q}|^2 + \frac{1/2}{|q - q_1|} + \frac{1/2}{|q - q_2|}\right] dt \\ &= \int_0^1 \left[\frac{1}{2}|\dot{q}|^2 + \frac{1}{|q - q_1|}\right] dt \\ &= \int_0^1 \left[\frac{1}{2}|\dot{q} - \dot{q}_1|^2 + \frac{1}{|q - q_1|}\right] dt \end{aligned} \tag{2.6}$$

If $q(t)$ collides with q_1 at some moment $t_0 \in [0,1]$, without loss of generality, we assume $t_0 = 0$, then $q(0) - q_1 = 0$, we let $x(t) = q(t) - q_1$, $y(t) = |x(t)|$, then $x(0) = 0, y(0) = 0$. By Jensen's inequality and Hardy-Littlewood-pólya inequality [9], we have

$$
\begin{aligned}
f(q) &= \frac{1}{2}\int_0^1 \left(|\dot{x}|^2 dt + \int_0^1 \frac{dt}{|x|} \right) \\
&\geq \frac{1}{2}\int_0^1 \left[\frac{d}{dt}|x|\right]^2 dt + 1^{3/2}\left(\int_0^1 |x|^2 dt\right)^{-1/2} \\
&= \frac{1}{2}\int_0^1 \dot{y}^2 dt + \left(\int_0^1 y^2 dt\right)^{-1/2} \\
&\geq \frac{1}{2}\left(\frac{\pi}{2}\right)^2 \int_0^1 y^2 dt + \left(\int_0^1 y^2 dt\right)^{-1/2}
\end{aligned}
\tag{2.7}
$$

Let $\sqrt{\int_0^1 y^2 dt} = s \geq 0, \varphi(s) = \frac{\pi^2}{8}s^2 + s^{-1}$, then $\varphi''(s) = \frac{\pi^2}{4} + 2s^{-3} > 0$, that is ϕ is strictly convex.

Let $\phi'(s) = 0$, we solve it to get $s_0 = \left(\frac{\pi^2}{4}\right)^{-1/3}$ is the critical point for $\phi(s)$, and $\varphi(s_0) = \frac{3}{2}\left(\frac{\pi}{2}\right)^{\frac{2}{3}}$, which is the maximum value for $\phi(s)$ on $s > 0$ since ϕ is convex and $\phi(s) \to +\infty$ as $s \to 0^+$.

If we can find the test orbit $\tilde{q}(t) \in \Lambda$ such that

$$
f(\tilde{q}(t)) < \frac{3}{2}\left(\frac{\pi}{2}\right)^{2/3} \tag{2.8}
$$

then the minimizer of $f(q)$ on $\overline{\Lambda}$ is collision-free.

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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