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Positive solutions for nonlocal fourth-order boundary value problems with all order derivatives

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Abstract

In this article, by the fixed point theorem in a cone and the nonlocal fourth-order BVP's Green function, the existence of at least one positive solution for the nonlocal fourth-order boundary value problem with all order derivatives

$$\begin{cases} u^{(4)}(t) + Au''(t) = \lambda f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds \end{cases}$$

is considered, where f is a nonnegative continuous function, $\lambda > 0$, $0 < A < \pi^2$, $p, q \in L[0, 1]$, $p(s) \geq 0$, $q(s) \geq 0$. The emphasis here is that f depends on all order derivatives.

Keywords: fourth-order boundary value problem, fixed point theorem, Green's function, positive solution

1 Introduction

The deformation of an elastic beam in equilibrium state, whose two ends are simply supported, can be described by a fourth-order ordinary equation boundary value problem. Owing to its significance in physics, the existence of positive solutions for the fourth-order boundary value problem has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point index theory, the Krasnosel'skii's fixed point theorem and the method of upper and lower solutions, in reference [1-10].

In recent years, there has been much attention on the question of positive solutions of the fourth-order differential equations with one or two parameters. By the Krasnosel'skii's fixed point theorem in cone [11], Bai [5] investigated the following fourth-order boundary value problem with one parameter

$$\begin{cases} u^{(4)}(t) + \beta u''(t) = \lambda f(t, u(t), u''(t)), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds, \end{cases}$$

where $\lambda > 0$, $0 < \beta < \pi^2$, f: $C([0, 1] \times [0, \infty) \times (-\infty, 0], [0, \infty))$ is continuous, $p, q \in L[0, 1]$, $p(s) \geq 0$, $q(s) \geq 0$, $\int_0^1 p(s)ds < 1$, $\int_0^1 q(s)\sin\sqrt{\beta}s ds + \int_0^1 q(s)\sin\sqrt{\beta}(1-s)ds < \sin\sqrt{\beta}$.

By the fixed point index in cone, Ma [7] proved the existence of symmetric positive solutions for the nonlocal fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = h(t)f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds. \end{cases}$$

All the above works were done under the assumption that all order derivatives u', u'', u''' are not involved explicitly in the nonlinear term f. In this article, we are concerned with the existence of positive solutions for the nonlocal fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) + Au''(t) = \lambda f(t, u(t), u'(t), u''(t), u'''(t)), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds. \end{cases} \tag{1.1}$$

Throughout, we assume

(H_1) $\lambda > 0$, $0 < A < \pi^2$;

(H_2) f: $[0, 1] \times R^4 \to R^+$ is continuous, $p, q \in L[0, 1]$, $p(s) \geq 0$, $q(s) \geq 0$, $\int_0^1 p(s)ds < 1$, $\int_0^1 q(s)\sin\sqrt{A}s ds + \int_0^1 q(s)\sin\sqrt{A}(1-s)ds < \sin\sqrt{A}$.

We will impose all order derivatives in f and make use of two continuous convex functionals which will ensure the existence of at least one positive solution to (1.1). Bai [5] applied Krasnoselskii's fixed point theorem. Ma [8] used fixed point index in cone and Leray-Schauder degree. In this article, to show the existence of positive solutions to (1.1), we define two positive continuous convex functionals. Then, using the new fixed point theorem [12] in a cone and the nonlocal fourth-order BVP's Green function, we give some new criteria for the existence of positive solutions to (1.1).

2 The preliminary lemmas

Let $Y = C[0, 1]$ be the Banach space equipped with the norm

$$||u(t)||_0 = \max_{t \in [0,1]} |u(t)|.$$

Set λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 + A\lambda$, namely $\lambda_1 = 0$, $\lambda_2 = -A$. By (H_1), it is obviously that $-\pi^2 < \lambda_2 < 0$.

Let $Q_1(t, s)$, $Q_2(t, s)$ be, respectively the Green's functions of the following problems

$$\begin{cases} -u''(t) + \lambda_1 u(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \end{cases} \qquad \begin{cases} -u''(t) + \lambda_2 u(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 q(s)u(s)ds. \end{cases}$$

Then, carefully calculation yield

$$Q_1(t, s) = G_1(t, s) + \frac{\int_0^1 G_1(s, x)p(x)dx}{1 - \int_0^1 p(x)dx},$$

$$Q_2(t, s) = G_2(t, s) + \frac{\left[\sin\sqrt{A}t + \sin\sqrt{A}(1-t)\right]\int_0^1 G_2(s, x)q(x)dx}{\sin\sqrt{A} - \int_0^1 q(x)\sin\sqrt{A}x dx - \int_0^1 q(x)\sin\sqrt{A}(1-x)dx},$$

$$G_1(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{\sin\sqrt{A}s \sin\sqrt{A}(1-t)}{\sqrt{A}\sin\sqrt{A}}, & 0 \le s \le t \le 1, \\ \dfrac{\sin\sqrt{A}t \sin\sqrt{A}(1-s)}{\sqrt{A}\sin\sqrt{A}}, & 0 \le t \le s \le 1. \end{cases}$$

Denote

$$\omega_1 = \frac{1}{1-\int_0^1 p(x)dx},$$
$$\omega_2(t) = \frac{\sin\sqrt{A}t + \sin\sqrt{A}(1-t)}{\sin\sqrt{A} - \int_0^1 q(x)\sin\sqrt{A}x dx - \int_0^1 q(x)\sin\sqrt{A}(1-x)dx}.$$

Lemma 2.1. [5] Suppose that (H_1) and (H_2) hold. Then for any $y(t) \in C[0, 1]$, the problem

$$\begin{cases} u^{(4)}(t) + Au''(t) = y(t), & 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds. \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 \int_0^1 Q_1(t,s)Q_2(s,\tau)y(\tau)d\tau ds, \tag{2.2}$$

where

$$Q_1(t,s) = G_1(t,s) + \omega_1 \int_0^1 G_1(s,x)p(x)dx,$$
$$Q_2(s,\tau) = G_2(s,\tau) + \omega_2(s) \int_0^1 G_2(\tau,x)q(x)dx.$$

By (2.2), we get

$$u'(t) = \int_0^1 \int_0^1 Q_2(s,\tau)y(\tau)d\tau ds - \int_0^1 \int_0^1 sQ_2(s,\tau)y(\tau)d\tau ds; \tag{2.3}$$

$$u''(t) = -\int_0^1 Q_2(t,\tau)y(\tau)d\tau, \tag{2.4}$$

$$u'''(t) = -\int_0^1 \frac{\partial Q_2(t,\tau)}{\partial t} y(\tau)d\tau. \tag{2.5}$$

Lemma 2.2. [5] Assume that (H_1) and (H_2) hold. Then one has
(*i*) $Q_i(t, s) \ge 0$, $\forall t, s \in [0, 1]$; $Q_i(t, s) > 0$, $\forall t, s \in (0, 1)$;
(*ii*) $G_i(t, s) \ge b_i G_i(t, t) G_i(s, s)$, $\forall t, s \in [0, 1]$;
(*iii*) $G_i(t, s) \le c_i G_i(s, s)$, $\forall t, s \in [0, 1]$.

where $b_1 = 1$, $b_2 = \sqrt{A}\sin\sqrt{A}$; $c_1 = 1$, $c_2 = \dfrac{1}{\sin\sqrt{A}}$.

Let

$$d_i = \min_{\frac{1}{4}\le t\le\frac{3}{4}} b_i G_i(t,t), (i = 1,2); \xi = \frac{\min_{\frac{1}{4}\le t\le\frac{3}{4}} \omega_2(t)}{\max_{\frac{1}{4}\le t\le\frac{3}{4}} \omega_2(t)}.$$

Lemma 2.3. [5] Suppose that (H_1) and (H_2) hold and w_2, d_i, ξ_i are given as above. Then

one has

(*i*) $\max_{0\le t\le 1}\omega_2(t) = \omega_2\left(\frac{1}{2}\right)$;

(*ii*) $0 < d_i < 1, \quad 0 < \xi < 1$.

Lemma 2.4. If $y(t) \in C[0, 1]$ and $y(t) \ge 0$, then the unique solution $u(t)$ of problem (2.1)

satisfies

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) \ \ge d_1||u||_0, \ \min_{\frac{1}{4}\le t\le\frac{3}{4}} (-u''(t)) \ge \frac{d_2\xi}{c_2}||u''||_0.$$

Proof. By (2.2) and (*iii*) of Lemma 2.2, we get

$$\begin{aligned} u(t) &= \int_0^1\int_0^1 Q_1(t,\ s)Q_2(s,\ \tau)y(\tau)d\tau ds \\ &\le \int_0^1\int_0^1 \left[c_1G_1(s,\ s) + \omega_1\int_0^1 G_1(s,\ x)p(x)dx\right]Q_2(s,\ \tau)y(\tau)d\tau ds \\ &= \int_0^1\int_0^1 \left[G_1(s,\ s) + \omega_1\int_0^1 G_1(s,\ x)p(x)dx\right]Q_2(s,\ \tau)y(\tau)d\tau ds \\ &= \int_0^1\int_0^1 Q_1(s,\ s)Q_2(s,\ \tau)d\tau ds. \end{aligned}$$

So,

$$||u||_0 \le \int_0^1\int_0^1 Q_1(s,s)Q_2(s,\tau)d\tau ds.$$

Using (*ii*) of Lemma 2.2, we have

$$\begin{aligned} \min_{\frac{1}{4}\le t\le\frac{3}{4}} u(t) &= \min_{\frac{1}{4}\le t\le\frac{3}{4}} \int_0^1\int_0^1 Q_1(t,\ s)Q_2(s,\ \tau)y(\tau)d\tau ds \\ &\ge \min_{\frac{1}{4}\le t\le\frac{3}{4}} \int_0^1\int_0^1 [b_1G_1(t,\ t)G_1(s,\ s) \\ &+ \omega_1\int_0^1 G_1(s,\ x)p(x)dx]Q_2(s,\ \tau)y(\tau)d\tau ds \\ &= \int_0^1\int_0^1 \left[d_1G_1(s,\ s) + \omega_1\int_0^1 G_1(s,\ x)p(x)dx\right]Q_2(s,\ \tau)y(\tau)d\tau ds \\ &\ge d_1\int_0^1\int_0^1 \left[G_1(s,\ s) + \omega_1\int_0^1 G_1(s,\ x)p(x)dx\right]Q_2(s,\ \tau)y(\tau)d\tau ds \\ &= d_1\int_0^1\int_0^1 Q_1(s,\ s)Q_2(s,\ \tau)y(\tau)d\tau ds \\ &\ge d_1||u||_0. \end{aligned}$$

By (2:4) and (*iii*) of Lemma 2.2, we get

$$\begin{aligned}
\max_{\frac{1}{4}\le t\le\frac{3}{4}}(-u''(t)) &= \max_{\frac{1}{4}\le t\le\frac{3}{4}}\int_0^1 Q_2(t,\ \tau)y(\tau)d\tau\\
&\le \int_0^1\left[c_2G_2(\tau,\ \tau)+\max_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)\int_0^1 G_2(\tau,\ x)q(x)dx\right]y(\tau)d\tau\\
&\le c_2\max_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)\int_0^1\left[G_2(\tau,\ \tau)+\int_0^1 G_2(\tau,\ x)q(x)dx\right]y(\tau)d\tau.
\end{aligned}$$

So,

$$||u''||_0\le c_2\max_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)\int_0^1\left[G_2(\tau,\ \tau)+\int_0^1 G_2(\tau,\ x)q(x)dx\right]y(\tau)d\tau.$$

Using (*ii*) of Lemma 2.2, we have

$$\begin{aligned}
\min_{\frac{1}{4}\le t\le\frac{3}{4}}(-u''(t)) &= \min_{\frac{1}{4}\le t\le\frac{3}{4}}\int_0^1 Q_2(t,\ \tau)y(\tau)d\tau\\
&\ge \min_{\frac{1}{4}\le t\le\frac{3}{4}}\int_0^1\left[b_2G_2(t,\ t)G_2(\tau,\ \tau)+\omega_2(t)\int_0^1 G_2(\tau,\ x)q(x)dx\right]y(\tau)d\tau\\
&\ge \int_0^1\left[b_2G_2(t,\ t)G_2(\tau,\ \tau)+\min_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)\int_0^1 G_2\ (\tau,\ x)q(x)dx\right]y(\tau)d\tau\\
&= \int_0^1\left[d_2G_2(\tau,\ \tau)+\min_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)\int_0^1 G_2(\tau,\ x)q(x)dx\right]y(\tau)d\tau\\
&\ge d_2\min_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)\ \int_0^1\left[G_2(\tau,\ \tau)+\int_0^1 G_2(\tau,\ x)q(x)dx\right]y(\tau)d\tau\\
&\ge \frac{d_2}{c_2}\frac{\min_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)}{\max_{\frac{1}{4}\le t\le\frac{3}{4}}\omega_2(t)}||u''||_0\\
&\ge \frac{d_2\xi}{c_2}||u''||_0.
\end{aligned}$$

The proof is completed.

Let X be a Banach space and $K\subset X$ a cone. Suppose α, β: $\times\to R^+$ are two continuous convex functionals satisfying $\alpha(\lambda u) = |\lambda|\alpha(u)$, $\beta(\lambda u) = |\lambda|\beta(u)$, for $u\in X$, $\lambda\in R$, and $||u||\le M\max\{\alpha(u), \beta(u)\}$, for $u\in X$ and $\alpha(u)\le\alpha(v)$ for $u, v\in K$, $u\le v$, where $M>0$ is a constant.

Theorem 2.1. [12] Let $r_2 > r_1 > 0$, $L > 0$ be constants and

$$\Omega_i = \{u\in X\ :\ \alpha(u) < r_i,\ \beta(u) < L\},\quad i = 1, 2,$$

two bounded open sets in X. Set

$$D_i = \{u\in X\ :\ \alpha(u) = r_i\},\quad i = 1, 2.$$

Assume T: $K\to K$ is a completely continuous operator satisfying

(A_1) $\alpha(Tu) < r_1$, $u \in D_1 \cap K$; $\alpha(Tu) > r_2$, $u \in D_2 \cap K$;

(A_2) $\beta(Tu) < L$, $u \in K$;

(A_3) there is a $p \in (\Omega_2 \cap K) \setminus \{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(u + \lambda p) \geq \alpha(u)$, for all $u \in K$ and $\lambda \geq 0$.

Then T has at least one fixed point in $(\Omega_2 \backslash \overline{\Omega}_1) \cap K$.

3 The main results

Let $X = C^4[0, 1]$ be the Banach space equipped with the norm $||u|| = \max_{t\in[0,1]}|u(t)| + \max_{t\in[0,1]}|u'(t)| + \max_{t\in[0,1]}|u''(t)| + \max_{t\in[0,1]}|u'''(t)|$, and $K = \left\{ u \in X : u(t) \geq 0,\ u''(t) \leq 0,\ \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq d_1||u||_0,\ \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} (-u''(t)) \geq \frac{d_2\xi}{c_2}||u''||_0 \right\}$ is a cone in X.

Define two continuous convex functionals $\alpha(u) = \max_{t\in[0,1]}|u(t)| + \max_{t\in[0,1]}|u''(t)|$ and $\beta(u) = \max_{t\in[0,1]}|u'(t)| + \max_{t\in[0,1]}|u'''(t)|$, for each $u \in X$, then $||u|| \leq 2\max\{\alpha(u), \beta(u)\}$ and $\alpha(\lambda u) = |\lambda|\alpha(u)$, $\beta(\lambda u) = |\lambda|\beta(u)$, for $u \in X$, $\lambda \in R$; $\alpha(u) \leq \alpha(v)$ for $u, v \in K$, $u \leq v$.

In the following, we denote

$$
\begin{aligned}
B &= \int_0^1 \int_0^1 Q_1(s,\ s)Q_2(s,\ \tau)d\tau ds,\\
D &= \int_0^1 \left[G_2(\tau,\ \tau) + \omega_2\left(\tfrac{1}{2}\right)\int_0^1 G_2(\tau,\ x)q(x)dx \right] d\tau,\\
F &= \frac{1}{\sin\sqrt{A}}\int_0^1 \sin\sqrt{A}\tau d\tau\\
&\quad + \frac{\sqrt{A}\int_0^1\int_0^1 G_2(\tau, x)q(x)dxd\tau}{\sin\sqrt{A} - \int_0^1 q(x)\sin\sqrt{A}xdx - \int_0^1 q(x)\sin\sqrt{A}(1-x)dx},\\
\eta_0 &= \frac{1}{B + c_2D},\ \eta_1 = \frac{1}{\int_{\frac{1}{4}}^{\frac{3}{4}} Q_2\left(\frac{1}{2}, \tau\right)d\tau},\ \eta_2 = \frac{2}{3c_2D + 4F},\ \theta = \min\left\{\frac{d_1}{2},\ \frac{d_2\xi}{2c_2}\right\}.
\end{aligned}
$$

We will suppose that there are $L > b > \theta b > c > 0$ such that $f(t, u, v, u_0, v_0)$ satisfies the following growth conditions:

$$
\begin{aligned}
&(H_3) f(t,\ u,\ v,\ u_0,\ v_0) < \frac{c\eta_0}{\lambda}, \text{ for } (t,\ u,\ v,\ u_0,\ v_0) \in [0, 1] \times [0,\ c] \times [-L,\ L] \times [-c,\ 0] \times [-L,\ L],\\
&(H_4) f(t,\ u,\ v,\ u_0,\ v_0) \geq \frac{b\eta_1}{\lambda}, \text{ for } (t,\ u,\ v,\ u_0,\ v_0) \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right] \times [\theta b,\ b] \times [-L,\ L] \times [-b,\ 0] \times [-L,\ L]\\
&\qquad \cup \left[\tfrac{1}{4}, \tfrac{3}{4}\right] \times [0,\ b] \times [-L,\ L] \times [-b,\ -\theta b] \times [-L,\ L],\\
&(H_5) f(t,\ u,\ v,\ u_0,\ v_0) < \frac{L\eta_2}{\lambda}, \text{ for } (t,\ u,\ v,\ u_0,\ v_0) \in [0, 1] \times [0,\ b] \times [-L,\ L] \times [-b,\ 0] \times [-L,\ L].
\end{aligned}
$$

Let $f_1(t,\ u,\ v,\ u_0,\ v_0) = f_1\left(t,\ u^*,\ v^*,\ u_0^*,\ v_0^*\right)$, where

$$
\begin{aligned}
&u^* = \min\{\max(u, 0), b\},\ v^* = \min\{\max(v, -\mathrm{L}), \mathrm{L}\},\\
&u_0^* = \min\{\max(u_0,\ -b),\ 0\},\ v_0^* = \min\{\max(v,\ -L),\ L\}.
\end{aligned}
$$

We denote

$$
(Tu)(t) = \lambda \int_0^1 \int_0^1 Q_1(t,\ s)Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds, \tag{3.1}
$$

$$
\begin{aligned}
(Tu)'(t) = \lambda \Bigg[& \int_t^1 \int_0^1 Q_2(s, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau ds \\
& - \int_0^1 \int_0^1 s Q_2(s, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau ds \Bigg],
\end{aligned}
\tag{3.2}
$$

$$
(Tu)''(t) = -\lambda \int_0^1 Q_2(t, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau, \tag{3.3}
$$

$$
(Tu)'''(t) = -\lambda \int_0^1 \frac{\partial Q_2(t,\tau)}{\partial t} f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau. \tag{3.4}
$$

Lemma 3.1. Suppose that (H_1) and (H_2) hold. Then T: $K \to K$ is completely continuous.

Proof. For $u \in K$, by (3.1), (3.3) and Lemma 2.2, it is obviously that $Tu \geq 0$, $(Tu)'' \leq 0$. In view of $c_1 = 1$, $c_2 > 1$, so

$$
\begin{aligned}
||Tu||_0 &= \max_{t \in [0,1]} \left| \lambda \int_0^1 \int_0^1 Q_1(t, s) Q_2(s, \tau) f_1(t, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau ds \right| \\
&\leq \lambda \int_0^1 \int_0^1 [c_1 G_1(s, s) \\
&\quad + \omega_1 \int_0^1 G_1(s, x) p(x) dx] Q_2(s, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau ds \\
&= \lambda \int_0^1 \int_0^1 Q_1(s, s) Q_2(s, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau ds,
\end{aligned}
$$

$$
\begin{aligned}
||(Tu)''||_0 &= \max_{t \in [0,1]} \left| -\lambda \int_0^1 Q_2(t, \tau) f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau \right| \\
&\leq \lambda \int_0^1 [c_2 G_2(\tau, \tau) \\
&\quad + \max_{\frac{1}{4} \leq t \leq \frac{3}{4}} \omega_2(t) \int_0^1 G_2(\tau, x) q(x) dx] f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau \\
&\leq \max_{\frac{1}{4} \leq t \leq \frac{3}{4}} \omega_2(t) \int_0^1 \left[G_2(\tau, \tau) + \int_0^1 G_2(\tau, x) q(x) dx \right] \\
&\quad \times f_1(\tau, u(\tau), u'(\tau), u''(\tau), u'''(\tau)) d\tau.
\end{aligned}
$$

By Lemma 2.3, (3.1) and (3.3), we have

$$\begin{aligned}
\min_{\frac14\le t\le\frac34}(Tu)(t) &= \min_{\frac14\le t\le\frac34}\lambda\int_0^1\int_0^1 Q_1(t,\ s)Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\\
&\ge \lambda\int_0^1\int_0^1 [b_1G_1(t,\ t)G_1(s,\ s)\\
&\quad+\omega_1\int_0^1 G_1(s,\ x)p(x)dx]Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\\
&= \lambda\int_0^1\int_0^1\left[d_1G_1(s,\ s)+\omega_1\int_0^1 G_1(s,\ x)p(x)dx\right]\\
&\quad\times\ Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\\
&\ge d_1\lambda\int_0^1\int_0^1 Q_1(s,\ s)Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\\
&\ge d_1||Tu||_0,
\end{aligned}$$

$$\begin{aligned}
\min_{\frac14\le t\le\frac34}(-(Tu)''(t)) &= \min_{\frac14\le t\le\frac34}\int_0^1 Q_2(t,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\\
&\ge \min_{\frac14\le t\le\frac34}\int_0^1 [b_2G_2(t,\ t)G_2(\tau,\ \tau)\\
&\quad+\omega_2(t)\int_0^1 G_2(\tau,\ x)q(x)dx]f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\\
&\ge \int_0^1 [b_2G_2(t,\ t)G_2(\tau,\ \tau)\\
&\quad+\min_{\frac14\le t\le\frac34}\omega_2(t)\int_0^1 G_2(\tau,\ x)q(x)dx]f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\\
&= \int_0^1 [d_2G_2(\tau,\ \tau)\\
&\quad+\min_{\frac14\le t\le\frac34}\omega_2(t)\int_0^1 G_2(\tau,\ x)q(x)dx]f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\\
&\ge d_2\min_{\frac14\le t\le\frac34}\omega_2(t)\int_0^1 [G_2(\tau,\ \tau)\\
&\quad+\int_0^1 G_2(\tau,\ x)q(x)dx]f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\\
&\ge \frac{d_2}{c_2}\frac{\min_{\frac14\le t\le\frac34}\omega_2(t)}{\max_{\frac14\le t\le\frac34}\omega_2(t)}||(Tu)''||_0\\
&\ge \frac{d_2\xi}{c_2}||(Tu)''||_0.
\end{aligned}$$

So we can get $T(K) \subset K$: Let $B \subset K$ is bounded, it is clear that $T(B)$ is bounded. Using f_1, $Q_1(t, s)$, $Q_2(t, s)$ is continuous, we show that $T(B)$ is equicontinuous. By the Arzela-Ascoli theorem, a standard proof yields $T: K \to K$ is completely continuous.

Theorem 3.1. Suppose that (H_1)-(H_5) hold. Then BVP (1.1) has at least one positive solution $u(t)$ satisfying

$$c < \alpha(u) < b, \quad \beta(u) < L.$$

Proof. Take

$$\Omega_1 = \{u \in X\ :\ \alpha(u) < c,\quad \beta(u) < L\},\quad \Omega_2 = \{u \in X\ :\ \alpha(u) < b,\quad \beta(u) < L\},$$

two bounded open sets in X, and

$$D_1 = \{u \in X \ : \ \alpha(u) = c\}, \quad D_2 = \{u \in X \ : \ \alpha(u) = b\}.$$

By Lemma 3.1, $T\colon K \to K$ is completely continuous. Let $p = \frac{b}{2} \in (\Omega_2 \cap K)\backslash\{0\}$, $\alpha(p) \neq 0$. It is easy to see that $\alpha(u + \lambda p) \geq \alpha(u)$, for all $u \in K$ and $\lambda \geq 0$.

Let $u \in D_1$, we have

$$\begin{aligned}
||Tu||_0 &= \max_{t\in[0,1]} \left|\lambda \int_0^1 \int_0^1 Q_1(t,\ s)Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\right| \\
&\leq \lambda \int_0^1 \int_0^1 \left[c_1 G_1(s,\ s) + \omega_1 \int_0^1 G_1(s,\ x)p(x)dx\right] Q_2(s,\ \tau)d\tau ds \times \frac{c\eta_0}{\lambda} \\
&= c\eta_0 \int_0^1 \int_0^1 Q_1(s,\ s)Q_2(s,\ \tau)d\tau ds \\
&= Bc\eta_0,
\end{aligned}$$

$$\begin{aligned}
||(Tu)''||_0 &= \max_{t\in[0,1]} \left|-\lambda \int_0^1 Q_2(t,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\right| \\
&< \lambda \int_0^1 \left[c_2 G_2(\tau,\ \tau) + \omega_2 \left(\tfrac{1}{2}\right) \int_0^1 G_2(\tau,\ x)q(x)dx\right] d\tau \times \frac{c\eta_0}{\lambda} \\
&\leq c_2 c\eta_0 \int_0^1 \left[G_2(\tau,\ \tau) + \omega_2 \left(\tfrac{1}{2}\right) \int_0^1 G_2(\tau,\ x)q(x)dx\right] d\tau \\
&= c_2 Dc\eta_0,
\end{aligned}$$

Hence, for $u \in D_1 \cap K$, $\alpha(u) = c$, we get

$$\alpha(Tu) = ||Tu||_0 + ||(Tu)''||_0 < Bc\eta_0 + c_2 Dc\eta_0 = (B + c_2 D)c\eta_0 = c.$$

Whereas for $u \in D_2 \cap K$, $\alpha(u) = b$, there is $||u||_0 \geq \frac{b}{2}$ or $||u''||_0 \geq \frac{b}{2}$, By Lemma 2.4, we get

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq d_1 ||u||_0 \geq \frac{d_1 b}{2} \text{ or } \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} (-u''(t)) \geq \frac{d_2\xi}{c_2}||u''||_0 \geq \frac{d_2\xi b}{2c_2}.$$

Therefore, using (H_4) and (3.3), we have

$$\begin{aligned}
\left|(Tu)''\left(\tfrac{1}{2}\right)\right| &= \left|\lambda \int_0^1 Q_2\left(\tfrac{1}{2},\ \tau\right) f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\right| \\
&> \left|\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} Q_2\left(\tfrac{1}{2},\ \tau\right) f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\right| \\
&\geq \lambda \times \frac{b\eta_1}{\lambda} \int_{\frac{1}{4}}^{\frac{3}{4}} Q_2\ \left(\tfrac{1}{2},\ \tau\right) d\tau \\
&= b.
\end{aligned}$$

Hence,

$$\alpha(Tu) \geq \left|(Tu)''\left(\tfrac{1}{2}\right)\right| > b.$$

By (3.2), (3.4), and (H_5), for $u \in K$, we have

$$\begin{aligned}
\|(Tu)'\|_0 &= \max_{t\in[0,1]}\left|\lambda\int_t^1\int_0^1 Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\right.\\
&\qquad\left.-\lambda\int_0^1\int_0^1 sQ_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\right|\\
&\le \max_{t\in[0,1]}\left|\lambda\int_t^1\int_0^1 Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\right|\\
&\quad+\max_{t\in[0,1]}\left|\lambda\int_0^1\int_0^1 sQ_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\right|\\
&\le \lambda\left|\int_0^1\int_0^1 (1+s)Q_2(s,\ \tau)f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau ds\right|\\
&\le \lambda\times\frac{\eta_2 L}{\lambda}\left|\int_0^1\int_0^1(1+s)\left[c_2G_2(\tau,\ \tau)+\omega_2\left(\tfrac{1}{2}\right)\int_0^1 G_2(\tau,\ x)q(x)dx\right]d\tau ds\right|\\
&\le \eta_2 L\times\frac{3}{2}c_2\int_0^1\left[G_2(\tau,\ \tau)+\omega_2\left(\tfrac{1}{2}\right)\int_0^1 G_2(\tau,\ x)q(x)dx\right]d\tau\\
&= \frac{3}{2}c_2 D\eta_2 L,
\end{aligned}$$

$$\begin{aligned}
\|(Tu)'''\|_0 &= \max_{t\in[0,1]}\left|\lambda\int_0^1\frac{\partial Q_2(t,\tau)}{\partial t}f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))d\tau\right|\\
&\le 2\lambda\int_0^1\left[\frac{\sin\sqrt{A}\tau}{\sin\sqrt{A}}+\frac{\sqrt{A}\int_0^1 G_2(\tau,x)q(x)dx}{\sin\sqrt{A}-\int_0^1 q(x)\sin\sqrt{A}xdx-\int_0^1 q(x)\sin\sqrt{A}(1-x)dx}\right]\\
&\qquad\times|f_1(\tau,\ u(\tau),\ u'(\tau),\ u''(\tau),\ u'''(\tau))|d\tau\\
&\qquad< \lambda 2F\times\frac{\eta_2 L}{\lambda}\\
&\qquad= 2F\eta_2 L.
\end{aligned}$$

So,

$$\beta(Tu)=\|(Tu)'\|_0+\|(Tu)'''\|_0<\frac{3}{2}c_2D\eta_2L+2F\eta_2L=\left(\frac{3}{2}c_2D+2F\right)\eta_2L=L.$$

Theorem 2.1 implies there is $(\Omega_2\backslash\overline{\Omega}_1)\cap K$ such that $u = Tu$. So, $u(t)$ is a positive solution for BVP (1.1) satisfying

$$c<\alpha(u)<b,\quad \beta(u)<L.$$

Thus, Theorem 3.1 is completed.

4 Example

Example 4.1. Consider the following boundary value problem

$$\begin{cases} u^{(4)}(t)+\frac{\pi^2}{9}u''(t)=\pi^2 f(t,\ u(t),\ u'(t),\ u''(t),\ u'''(t)), & 0<t<1,\\ u(0)=u(1)=\displaystyle\int_0^1 su(s)ds, & \\ u''(0)=u''(1)=0, & \end{cases} \tag{4.1}$$

where

$$f(t,u,v,u_0,v_0)=\begin{cases}\frac{1}{20}(u-u_0)+\frac{1}{2}|\cos(v+v_0)|,\\ \quad (t,u,v,u_0,v_0)\in[0,1]\times[0,2]\times[-16000,16000]\times[-2,0]\times[-16000,16000],\\ \frac{1}{20}(2-u_0)(3-u)+\frac{27}{2}(3-u_0)(u-2)+\frac{1}{2}|\cos(v+v_0)|,\\ \quad (t,u,v,u_0,v_0)\in[0,1]\times[2,3]\times[-16000,16000]\times[-2,0]\times[-16000,16000],\\ \frac{1}{20}(u+2)(u_0+3)-\frac{27}{2}(u+3)(u_0+2)+\frac{1}{2}|\cos(v+v_0)|,\\ \quad (t,u,v,u_0,v_0)\in[0,1]\times[0,2]\times[-16000,16000]\times[-3,-2]\times[-16000,16000],\\ \frac{1}{5}(3-u)(u_0+3)+\frac{135}{2}(u-2)(u_0+3)-\frac{27}{2}(u+3)(u_0+2)+\frac{1}{2}|\cos(v+v_0)|,\\ \quad (t,u,v,u_0,v_0)\in[0,1]\times[2,3]\times[-16000,16000]\times[-3,-2]\times[-16000,16000],\\ \frac{27}{2}(u-u_0)+\frac{1}{2}|\cos(v+v_0)|,\\ \quad (t,u,v,u_0,v_0)\in[0,1]\times[3,40]\times[-16000,16000]\times[-40,0]\times[-16000,16000],\\ \qquad\qquad \cup[0,1]\times[0,40]\times[-16000,16000]\times[-40,-3]\times[-16000,16000].\end{cases}$$

In this problem, we know that $A=\frac{\pi^2}{9},\ \lambda=\pi^2,\ p(t)=t,\ q(t)=0$, then we can get $b_1=1, b_2=\frac{\sqrt{3}\pi}{6},\ c_1=1,\ c_2=\frac{2\sqrt{3}}{3},\ \omega_1=2,\ \omega_2=\frac{2\sqrt{3}\sin\frac{\pi}{3}(1+t)}{3},\ d_1=\frac{3}{16},\ d_2=\frac{\sqrt{3}-1}{4},\ \xi=\frac{\sqrt{2+\sqrt{3}}}{2}$. Further more, we obtain

$$B=\frac{1944\sqrt{3}-972\pi-9\pi^3}{4\pi^5},\quad D=\frac{9-\sqrt{3}\pi}{2\pi^2},\quad F=\frac{\sqrt{3}}{\pi}.$$

then $\eta_0=\dfrac{12\pi^5}{5832\sqrt{3}-2916\pi-27\pi^3+36\sqrt{3}\pi^3-12\pi^4},\quad \eta_1=\dfrac{\pi^2}{3\sqrt{6+3\sqrt{3}}-9},$
$\theta=\min\left\{\frac{d_1}{2},\ \frac{d_2\xi}{2c_2}\right\}=\frac{\sqrt{2+\sqrt{3}}(3-\sqrt{3})}{32},\quad \theta=\min\left\{\frac{d_1}{2},\ \frac{d_2\xi}{2c_2}\right\}=\frac{\sqrt{2+\sqrt{3}}(3-\sqrt{3})}{32},$
$\theta b\approx 3.06>3$.

If we take $c=2,\ b=40,\ L=16000$, then we get

$$\begin{aligned}&f(t,u,v,u_0,v_0)=\frac{1}{20}(u-u_0)+\frac{1}{2}|\cos(v+v_0)|\le 0.7<\frac{c\eta_0}{\lambda}\approx 0.8,\\ &\text{for}\,(t,u,v,u_0,v_0)\in[0,1]\times[0,2]\times[-16000,16000]\times[-2,0]\times[-16000,16000],\\ &f(t,u,v,u_0,v_0)=\frac{27}{2}(u-u_0)+\frac{1}{2}|\cos(v+v_0)|\ge 40>\frac{b\eta_1}{\lambda}\approx 38,\\ &\text{for}\,(t,u,v,u_0,v_0)\in\left[\tfrac{1}{4},\tfrac{3}{4}\right]\times[\theta b,\ 40]\times[-16000,16000]\times[-40,0]\times[-16000,16000]\\ &\qquad\qquad\cup\left[\frac{1}{4},\ \frac{3}{4}\right]\times[0,40]\times[-16000,16000]\times[-40,\ -\theta b]\times[-16000,16000],\\ &f(t,u,v,u_0,v_0)\le 1080.5<\frac{L\eta_2}{\lambda}\approx 1146,\\ &\text{for}\,(t,u,v,u_0,v_0)\in[0,1]\times[0,40]\times[-16000,16000]\times[-40,0]\times[-16000,16000].\end{aligned}$$

Then all the conditions of Theorem 3.1 are satisfied. Therefore, by Theorem 3.1 we know that boundary value problem (4.1) has at least one positive solution $u(t)$ satisfying

$$2<\alpha(u)<40,\ \beta(u)<16000.$$

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Authors' contributions
The authors declare that the work was realized in collaboration with same responsibility.
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