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# Existence of nodal solutions of a nonlinear fourth-order two-point boundary value problem

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## Abstract

In this article, we give conditions on parameters  $k, l$  that the generalized eigenvalue problem  $x'''' + kx'' + lx = \lambda h(t)x$ ,  $0 < t < 1$ ,  $x(0) = x(1) = x'(0) = x'(1) = 0$  possesses an infinite number of simple positive eigenvalues  $\{\lambda_k\}_{k=1}^{\infty}$  and to each eigenvalue there corresponds an essential unique eigenfunction  $\psi_k$  which has exactly  $k - 1$  simple zeros in  $(0,1)$  and is positive near 0. It follows that we consider the fourth-order two-point boundary value problem  $x'''' + kx'' + lx = f(t,x)$ ,  $0 < t < 1$ ,  $x(0) = x(1) = x'(0) = x'(1) = 0$ , where  $f(t, x) \in C([0,1] \times \mathbb{R}, \mathbb{R})$  satisfies  $f(t, x) > 0$  for all  $x \neq 0$ ,  $t \in [0,1]$  and  $\lim_{|x| \rightarrow 0} f(t,x)/x = a(t)$ ,  $\lim_{|x| \rightarrow +\infty} f(t,x)/x = b(t)$  or  $\lim_{x \rightarrow -\infty} f(t,x)/x = 0$  and  $\lim_{x \rightarrow +\infty} f(t,x)/x = c(t)$  for some  $a(t), b(t), c(t) \in C([0,1], (0,+\infty))$  and  $t \in [0,1]$ . Furthermore, we obtain the existence and multiplicity results of nodal solutions for the above problem. The proofs of our main results are based upon disconjugate operator theory and the global bifurcation techniques.

**MSC (2000):** 34B15.

**Keywords:** disconjugacy theory, bifurcation, nodal solutions, eigenvalue

## 1 Introduction

The deformations of an elastic beam in equilibrium state with fixed both endpoints can be described by the fourth-order boundary value problem

$$\begin{aligned} x'''' + lx &= \lambda h(t)f(x), 0 < t < 1, \\ x(0) &= x(1) = x'(0) = x'(1) = 0, \end{aligned} \quad (1.1)$$

where  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $\lambda \in \mathbb{R}$  is a parameter and  $l$  is a given constant. Since the problem (1.1) cannot transform into a system of second-order equation, the treatment method of second-order system does not apply to the problem (1.1). Thus, existing literature on the problem (1.1) is limited. Recently, when  $l = 0$ , the existence and multiplicity of positive solutions of the problem (1.1) has been studied by several authors, see Agarwal and Chow [1], Ma and Wu [2], Yao [3,4] and Korman [5]. Especially, when  $l \neq 0$ ,  $l$  satisfying (H1) and  $h(t)$  satisfying (H2), Xu and Han [6] studied the existence of nodal solutions of the problem (1.1) by applying bifurcation techniques, where

(H1)  $l \in (-\pi^4, \pi^4/64)$  is given constant.

(H2)  $h \in C([0,1], [0, \infty))$  with  $h(t) \not\equiv 0$  on any subinterval of  $[0,1]$ .

Motivated by [6], we consider the existence of nodal solutions of general fourth-order boundary value problem

$$\begin{aligned} x'''' + kx'' + lx &= f(t, x), \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) &= 0, \end{aligned} \tag{1.2}$$

and under the assumptions:

(A1) One of following conditions holds

(i)  $k, l$  satisfying  $(k, l) \in \{(k, l) \mid k \in (-\infty, 0], l \in (0, \infty)\} \setminus \left\{ \left(0, \frac{\pi^4}{64}\right) \right\} \cup \{(k, l) \mid k \in (-\infty, \pi^2), l \in (-\infty, 0]\}$

are given constants with

$$\pi^2(k - \pi^2) < l \leq \frac{1}{4} \left( k - \frac{\pi^2}{4} \right)^2; \tag{1.3}$$

(ii)  $k, l$  satisfying  $(k, l) \in \left\{ (k, l) \mid k \in \left(0, \frac{\pi^2}{2}\right), l \in (0, \infty) \right\}$  are given constants with

$$\frac{1}{4} \left( \pi^2 k - \frac{\pi^4}{4} \right) < l \leq \frac{1}{4} k^2. \tag{1.4}$$

(A2)  $f(t, x) \in C([0,1] \times \mathbb{R}, \mathbb{R})$  satisfies  $f(t, x)x > 0$  for all  $x \neq 0$  and  $t \in [0,1]$ .

(A3) There exists  $a(t) \in C([0,1], (0, \infty))$  such that

$$\lim_{|x| \rightarrow 0} \frac{f(t, x)}{x} = a(t), \quad \forall t \in [0, 1]. \tag{1.5}$$

(A4) There exists  $b(t) \in C([0,1], (0, \infty))$  such that

$$\lim_{|x| \rightarrow \infty} \frac{f(t, x)}{x} = b(t), \quad \forall t \in [0, 1]. \tag{1.6}$$

(A5) There exists  $c(t) \in C([0,1], (0, \infty))$  such that

$$\lim_{x \rightarrow -\infty} \frac{f(t, x)}{x} = 0, \quad \lim_{x \rightarrow +\infty} \frac{f(t, x)}{x} = c(t), \quad \forall t \in [0, 1]. \tag{1.7}$$

However, in order to use bifurcation technique to study the nodal solutions of the problem (1.2), we first prove that the generalized eigenvalue problem

$$\begin{aligned} x'''' + kx'' + lx &= \lambda h(t)x, \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) &= 0, \end{aligned} \tag{1.8}$$

(where  $h$  satisfies (H2)) has an infinite number of positive eigenvalues

$$0 < \lambda_1(h) < \lambda_2(h) < \dots < \lambda_k(h) < \lambda_{k+1}(h) < \dots \tag{1.9}$$

and each eigenvalue corresponding an essential unique eigenfunction  $\psi_k$  which has exactly  $k - 1$  simple zeros in  $(0,1)$  and is positive near 0. Fortunately, Elias [7] developed a theory on the eigenvalue problem

$$\begin{aligned} \mathcal{L}\gamma + \lambda h(t)\gamma &= 0, \\ (\mathcal{L}_i\gamma)(a) &= 0, \quad i \in \{i_1, \dots, i_k\}, \\ (\mathcal{L}_j\gamma)(b) &= 0, \quad j \in \{j_1, \dots, j_{n-k}\}, \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} \mathcal{L}_0\gamma &= \rho_0\gamma, \\ \mathcal{L}_i\gamma &= \rho_i(\mathcal{L}_{i-1}\gamma)', \quad i = 1, \dots, n, \\ \mathcal{L}\gamma &= \mathcal{L}_n\gamma, \end{aligned} \tag{1.11}$$

and  $\rho_i \in C^{n-i}[a, b]$  with  $\rho_i > 0$  ( $i = 0, 1, \dots, n$ ) on  $[a, b]$ .  $\mathcal{L}_0\gamma, \dots, \mathcal{L}_{n-1}\gamma$  are called the quasi-derivatives of  $\gamma(t)$ . To apply Elias's theory, we have to prove that (1.8) can be rewritten to the form of (1.10), that is, the linear operator

$$L[x] := x'''' + kx'' + lx \tag{1.12}$$

has a factorization of the form

$$L[x] = l_4 \left( l_3 \left( l_2 \left( l_1 (l_0 x)' \right)' \right)' \right)' \tag{1.13}$$

on  $[0,1]$ , where  $l_i \in C^{4-i}[0,1]$  with  $l_i > 0$  ( $i = 0, 1, 2, 3, 4$ ) on  $[0, 1]$ , and  $x(0) = x(1) = x'(0) = x'(1) = 0$  if and only if

$$(l_0x)(0) = (l_0x)(1) = (l_1x)(0) = (l_1x)(1) = 0. \tag{1.14}$$

This can be achieved under (A1) by using the *disconjugacy theory* in [8].

The rest of the article is arranged as follows: In Section 2, we state some disconjugacy theory which can be used in this article, and then show that (A1) implies the equation

$$L[x] = 0 \tag{1.15}$$

is disconjugate on  $[0, 1]$ , and establish some preliminary properties on the eigenvalues and eigenfunctions of the generalized eigenvalue problem (1.8). Finally in Section 3, we state and prove our main results (Theorems 3.1 and 3.2 ).

*Remark 1.1.* If we let  $k = 0$ , then the condition (A1) reduces to (H1) in [6].

*Remark 1.2.* Since the function  $f(t, x)$  is more general than the function  $h(t)f(x)$  in [6], then the problem considered in this article is more general than the problem in [6].

*Remark 1.3.* If we let  $k = 0$  and  $f(t, x) = \lambda h(t)f(x)$ , then Theorem 3.2 reduces to [[6], Theorem 3.1].

*Remark 14.* For other results on the existence and multiplicity of positive solutions and nodal solutions for the boundary value problems of fourth-order ordinary differential equations based on bifurcation techniques, see [9-14]s and their references.

## 2 Preliminary results

Let

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y = 0 \tag{2.1}$$

be  $n$ th-order linear differential equation whose coefficients  $p_k(\cdot)$  ( $k = 1, \dots, n$ ) are continuous on an interval  $I$ .

*Definition 2.1* [[8], Definition 0.2, p. 2]. Equation (2.1) is said to be *disconjugate* on an interval  $I$  if no nontrivial solution has  $n$  zeros on  $I$ , multiple zeros being counted according to their multiplicity.

**Lemma 2.2** [[8], Theorem 0.7, p. 3]. *Equation (2.1) is disconjugate on a compact interval  $I$  if and only if there exists a basis of solutions  $y_0, \dots, y_{n-1}$  such that*

$$w_k := w_k(y_0, \dots, y_{k-1}) = \begin{vmatrix} y_0 & \dots & y_{k-1} \\ \vdots & & \vdots \\ y_0^{(k-1)} & \dots & y_{k-1}^{(k-1)} \end{vmatrix} > 0 \quad (k = 1, \dots, n) \tag{2.2}$$

on  $I$ . A disconjugate operator  $L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_n(t)y$  can be written as

$$L[y] = \rho_n D(\rho_{n-1}(\dots D(\rho_1 D(\rho_0 y)) \dots)), \quad D = \frac{d}{dt} \tag{2.3}$$

where  $\rho_k \in C^{n-k}(I)$  ( $k = 0, 1, \dots, n$ ) and

$$\rho_0 = \frac{1}{w_1}, \rho_1 = \frac{w_1^2}{w_2}, \rho_k = \frac{w_k^2}{w_{k-1}w_{k+1}}, \quad k = 2, \dots, n-1, \tag{2.4}$$

and  $\rho_0\rho_1 \dots \rho_n \equiv 1$ .

**Lemma 2.3** [[8], Theorem 0.13, p. 9]. *Green's function  $G(t,s)$  of the disconjugate equation (2.3) and the two-point boundary value conditions*

$$\begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \dots, k-1, \\ y^{(j)}(b) &= 0, \quad j = 0, \dots, n-k-1 \end{aligned} \tag{2.5}$$

satisfies

$$(-1)^{n-k}G(t,s) > 0, \quad \forall (t,s) \in (a,b) \times (a,b). \tag{2.6}$$

Now using Lemmas 2.2 and 2.3, we will prove some preliminary results.

**Theorem 2.4.** *Let (A1) hold. Then*

(i)  $L[x] = 0$  is disconjugate on  $[0,1]$ , and  $L[x]$  has a factorization

$$L[x] = \rho_4 \left( \rho_3 \left( \rho_2 \left( \rho_1 (\rho_0 x)' \right)' \right)' \right)' \tag{2.7}$$

where  $\rho_k \in C^{4-k}[0,1]$  with  $\rho_k > 0$  ( $k = 0, 1, 2, 3, 4$ ).

(ii)  $x(0) = x(1) = x'(0) = x'(1) = 0$  if and only if

$$(L_0 x)(0) = (L_1 x)(0) = (L_0 x)(1) = (L_1 x)(1) = 0, \tag{2.8}$$

where

$$\begin{aligned} L_0x &= \rho_0x, \\ L_i x &= \rho_i(L_{i-1}x)', \quad i = 1, 2, 3, 4. \end{aligned} \tag{2.9}$$

*Proof of Theorem 2.4.* We divide the proof into nine cases.

$$\text{Case 1. } (k, l) \in \left\{ (k, l) \mid k \in (-\infty, 0], l \in \left( \frac{1}{4}k^2, \frac{1}{4}\left(k - \frac{\pi^2}{4}\right)^2 \right) \right\} \setminus \left\{ \left(0, \frac{\pi^4}{64}\right) \right\}.$$

In the case, we have corresponding  $L[x] = x'''' + kx'' + lx = 0$  that the equation  $\lambda^4 + k\lambda^2 + lx = 0$  has 4 roots  $\lambda_1 = m_1 + m_2i, \lambda_2 = m_1 - m_2i, \lambda_3 = -m_1 + m_2i,$  and  $\lambda_4 = -m_1 - m_2i,$  where

$$m_1 = \frac{\sqrt{-k + \sqrt{4l}}}{2}, \quad m_2 = \frac{\sqrt{k + \sqrt{4l}}}{2}, \quad m_2 \leq m_1. \tag{2.10}$$

Combining  $\frac{1}{4}k^2 < l \leq \frac{1}{4}\left(k - \frac{\pi^2}{4}\right)^2$  with (2.10), we have  $0 < m_2 \leq \frac{\pi}{4}$ . Thus, we get that either the following (1) or (2) holds:

- (1)  $0 \leq m_2 t < \frac{\pi}{4} \Rightarrow \tan m_2 t < 1 \leq \frac{m_1}{m_2},$  for  $m_2 \in \left(0, \frac{\pi}{4}\right), t \in [0, 1];$
- (2)  $0 \leq m_2 t \leq \frac{\pi}{4} \Rightarrow \tan m_2 t \leq 1 < \frac{m_1}{m_2},$  for  $m_2 = \frac{\pi}{4}, t \in [0, 1].$

Furthermore, it is easy to check that

$$\cos m_2 t > 0, \quad m_1 \cos m_2 t - m_2 \sin m_2 t > 0, \quad \forall t \in [0, 1]. \tag{2.11}$$

Take

$$\begin{aligned} x_0(t) &= e^{-m_1 t} \cos m_2 t, \quad x_1(t) = e^{-m_1 t} \sin m_2 t, \\ x_2(t) &= e^{m_1 t} \cos m_2 t, \quad x_3(t) = e^{m_1 t} \sin m_2 t. \end{aligned} \tag{2.12}$$

It is easy to check that  $x_0(t), x_1(t), x_2(t),$  and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ . By simple computation, we have

$$\begin{aligned} w_1 &= \frac{\cos m_2 t}{e^{m_1 t}}, \quad w_2 = \frac{m_2}{e^{2m_1 t}}, \\ w_3 &= \frac{4m_1 m_2 (m_1 \cos m_2 t - m_2 \sin m_2 t)}{e^{m_1 t}}, \quad w_4 = 16m_1^2 m_2^2 (m_1^2 + m_2^2). \end{aligned} \tag{2.13}$$

This together with (2.11) implies that  $w_i > 0 (i=1, 2, 3, 4)$  on  $[0,1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0,1]$ , and  $L[x]$  has a factorization

$$\begin{aligned} x'''' + kx'' - lx &= \frac{4m_1 m_2 (m_1^2 + m_2^2) e^{m_1 t}}{m_1 \cos m_2 t - m_2 \sin m_2 t} \left( \frac{(m_1 \cos m_2 t - m_2 \sin m_2 t)^2}{m_2 (m_1^2 + m_2^2)} \right. \\ &\times \left. \left( \frac{m_2}{4m_1 \cos m_2 t (m_1 \cos m_2 t - m_2 \sin m_2 t) e^{2m_1 t}} \left( \frac{\cos^2 m_1 t}{m_2} \left( \frac{e^{m_1 t}}{\cos m_2 t} x \right) \right) \right) \right)', \end{aligned} \tag{2.14}$$

and accordingly

$$L_0x = \rho_0x = \frac{e^{m_1t}}{\cos m_2t}x, \\ L_1x = \frac{\cos^2 m_2t}{m_2} \left( \frac{e^{m_1t}}{\cos m_2t}x \right)' = \frac{e^{m_1t}}{m_2} (m_1 \cos m_2t + m_2 \sin m_2t)x + \frac{e^{m_1t} \cos m_2t}{m_2} x'. \quad (2.15)$$

Using (2.15), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

*Case 2.*  $k \in (-\infty, 0)$  and  $l = \frac{1}{4}k^2$ .

In the case, applying the similar method used in *Case 1*, we take

$$x_0(t) = e^{-mt}, \quad x_1(t) = te^{-mt}, \quad x_2(t) = e^{mt}, \quad x_3(t) = te^{mt}, \quad (2.16)$$

where  $m = \sqrt{-\frac{k}{2}}$ .

It is easy to check that  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ .

By simple computation, we have

$$w_1 = e^{-mt}, \quad w_2 = e^{-2mt}, \quad w_3 = 4m^2e^{-mt}, \quad w_4 = 16m^4. \quad (2.17)$$

Clearly,  $w_i > 0$  ( $i = 1, 2, 3, 4$ ) on  $[0, 1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0, 1]$ , and  $L[x]$  has a factorization

$$x'''' + kx'' - lx = \frac{4m^2e^{mt}}{1} \left( \frac{1}{1} \left( \frac{1}{4m^2e^{2mt}} \left( \frac{1}{1} \left( \frac{e^{mt}}{1}x \right)' \right)' \right)' \right)' \quad (2.18)$$

and accordingly

$$L_0x = \rho_0x = e^{mt}x, \quad L_1x = \rho_1(L_0x)' = e^{mt}(mx + x'). \quad (2.19)$$

Using (2.19), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

*Case 3.*  $k \in (-\infty, 0)$  and  $0 < l < \frac{1}{4}k^2$ .

In the case, we take

$$x_0(t) = e^{-m_2t}, \quad x_1(t) = e^{m_2t}, \quad x_2(t) = e^{-m_1t}, \quad x_3(t) = e^{m_1t}, \quad (2.20)$$

where  $m_1 = \sqrt{\frac{-k + \sqrt{k^2 - 4l}}{2}} > 0$ ,  $m_2 = \sqrt{\frac{-k - \sqrt{k^2 - 4l}}{2}} > 0$ ,  $m_1 > m_2$ ,

It is easy to check that  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ .

By simple computation, we have

$$w_1 = e^{-m_2t}, \quad w_2 = 2m_2, \\ w_3 = 2m_2(m_1^2 - m_2^2)e^{-m_1t}, \quad w_4 = 4m_1m_2(m_1^2 - m_2^2)^2. \quad (2.21)$$

Clearly,  $w_i > 0$  ( $i = 1, 2, 3, 4$ ) on  $[0, 1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0, 1]$ , and  $L[x]$  has a factorization

$$\begin{aligned}
 & x'''' + kx'' + lx \\
 &= 2m_1(m_1^2 - m_2^2)e^{m_1 t} \left( \frac{1}{2m_1 e^{2m_1 t}} \left( \frac{2m_2 e^{(m_1+m_2)t}}{m_1^2 - m_2^2} \left( \frac{1}{2m_2 e^{2m_2 t}} (e^{m_2 t} x)' \right)' \right)' \right)' \quad (2.22)
 \end{aligned}$$

and accordingly

$$L_0 x = \rho_0 x = e^{m_2 t} x, \quad L_1 x = \rho_1 (L_0 x) = \frac{1}{2m_2 e^{m_2 t}} (m_2 x + x'). \quad (2.23)$$

Using (2.23), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

*Case 4.*  $k \in (-\infty, 0)$ ,  $l = 0$ .

In the case, we take

$$x_0(t) = 1, \quad x_1(t) = 1 + t, \quad x_2(t) = e^{-mt}, \quad x_3(t) = e^{mt}, \quad (2.24)$$

where  $m = \sqrt{-k} > 0$ .

It is easy to check that  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ .

By simple computation, we have

$$w_1 = 1, \quad w_2 = 1, \quad w_3 = m^2 e^{-mt}, \quad w_4 = 2m^5. \quad (2.25)$$

Clearly,  $w_i > 0$  ( $i = 1, 2, 3, 4$ ) on  $[0, 1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0, 1]$ , and  $L[x]$  has a factorization

$$x'''' + kx'' + lx = 2m^3 e^{mt} \left( \frac{1}{2m e^{2mt}} \left( \frac{e^{mt}}{m^2} \left( \frac{1}{1} \left( \frac{1}{1} x \right)' \right)' \right)' \right)' \quad (2.26)$$

and accordingly

$$L_0 x = \rho_0 x = x, \quad L_1 x = x'. \quad (2.27)$$

Using (2.27), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

*Case 5.*  $k = 0$ ,  $l = 0$ . The case is obvious.

*Case 6.*  $k \in (0, \pi^2)$ ,  $l = 0$ .

In the case, we take

$$x_0(t) = 1, \quad x_1(t) = 1 + t, \quad x_2(t) = -\sin m(t + \sigma), \quad x_3(t) = \cos m(t + \sigma), \quad (2.28)$$

where  $m = \sqrt{k} > 0$ ,  $\sigma$  is a positive constant. Clearly,  $m \in (0, \pi)$  and then

$$\sin m(t + \sigma) > 0, \quad \forall t \in [0, 1]. \quad (2.29)$$

It is easy to check that  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ .

By simple computation, we have

$$w_1 = 1, \quad w_2 = 1, \quad w_3 = m_2 \sin m(t + \sigma), \quad w_4 = m^5. \quad (2.30)$$

Clearly,  $w_i > 0$  ( $i = 1, 2, 3, 4$ ) on  $[0, 1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0,1]$ , and  $L[x]$  has a factorization

$$x'''' + kx'' + lx = \frac{m^3}{\sin m(t + \sigma)} \left( \frac{\sin^2 m(t + \sigma)}{m} \left( \frac{1}{m^2 \sin m(t + \sigma)} \left( \frac{1}{1} \left( \frac{1}{1} x \right)' \right)' \right)' \right)' \quad (2.31)$$

and accordingly

$$L_0x = \rho_0x = x, \quad L_1x = x'. \quad (2.32)$$

Using (2.32), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

*Case 7.*  $\pi^2 (k - \pi^2) < l < 0$ .

In the case, we take

$$x_0(t) = e^{-m_1t}, \quad x_1(t) = e^{m_1t}, \quad x_2(t) = -\sin m_2(t + \sigma), \quad x_3(t) = \cos m_2(t + \sigma), \quad (2.33)$$

where  $m_1 = \sqrt{\frac{-k + \sqrt{k^2 - 4l}}{2}} > 0$ ,  $m_2 = \sqrt{\frac{k + \sqrt{k^2 - 4l}}{2}} > 0$ ,  $\sigma$  is a positive constant.

Clearly,  $m_2 \in (0, \pi)$  and then

$$\sin m_2(t + \sigma) > 0, \quad \forall t \in [0, 1]. \quad (2.34)$$

It is easy to check that  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ . By simple computation, we have

$$\begin{aligned} w_1 &= e^{-m_1t}, \quad w_2 = 2m_1, \\ w_3 &= 2m_1(m_1^2 + m_2^2) \sin m_2(t + \sigma), \quad w_4 = 2m_1m_2(m_1^2 + m_2^2)^2. \end{aligned} \quad (2.35)$$

Clearly,  $w_i > 0$  ( $i = 1, 2, 3, 4$ ) on  $[0, 1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0,1]$ , and  $L[x]$  has a factorization

$$\begin{aligned} x'''' + kx'' + lx &= \frac{m_2(m_1^2 + m_2^2)}{\sin m_2(t + \sigma)} \left( \frac{\sin^2 m_2(t + \sigma)}{m_2} \left( \frac{2m_1e^{m_1t}}{(m_1^2 + m_2^2) \sin m_2(t + \sigma)} \left( \frac{1}{2m_1e^{2m_1t}} (e^{m_1t}x)' \right)' \right)' \right)' \end{aligned} \quad (2.36)$$

and accordingly

$$L_0x = \rho_0x = e^{m_1t}x, \quad L_1x = \rho_1(L_0x) = \frac{1}{2m_1e^{m_1t}}(m_1x + x'). \quad (2.37)$$

Using (2.37), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

*Case 8.*  $k \in \left(0, \frac{\pi^2}{2}\right)$  and  $l = \frac{1}{4}k^2$

In the case, we take

$$\begin{aligned} x_0(t) &= \cos m(t + \sigma), \quad x_1(t) = \sin m(t + \sigma), \\ x_2(t) &= -t \cos m(t + \sigma), \quad x_3(t) = -t \sin m(t + \sigma), \end{aligned} \quad (2.38)$$

where  $m = \sqrt{\frac{k}{2}}$ ,  $\sigma$  is a positive constant. Clearly,  $m \in \left(0, \frac{\pi}{2}\right)$  and then

$$\cos m(t + \sigma) > 0, \quad \sin m(t + \sigma) > 0, \quad \forall t \in [0, 1]. \quad (2.39)$$

It is easy to check that  $x_0(t)$ ,  $x_1(t)$ ,  $x_2(t)$ , and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ . By simple computation, we have

$$w_1 = \cos m(t + \sigma), \quad w_2 = m, \quad w_3 = 2m^2 \sin m(t + \sigma), \quad w_4 = 4m^4. \quad (2.40)$$

Clearly,  $w_i > 0$  ( $i = 1, 2, 3, 4$ ) on  $[0, 1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0, 1]$ , and  $L[x]$  has a factorization

$$\begin{aligned} & x'''' + kx'' + lx \\ &= \frac{2m^2}{\sin m(t + \sigma)} \left( \frac{\sin^2 m(t + \sigma)}{m} \left( \frac{1}{\sin 2m(t + \sigma)} \left( \frac{\cos^2 m(t + \sigma)}{m} \left( \frac{1}{\cos m(t + \sigma)} x \right) \right) \right) \right)' \end{aligned} \quad (2.41)$$

and accordingly

$$\begin{aligned} L_0 x &= \rho_0 x = \frac{1}{\cos m(t + \sigma)} x, \\ L_1 x &= \rho_1 (L_0 x)' = \frac{1}{m} (m \sin m(t + \sigma) \cdot x + \cos m(t + \sigma) \cdot x'). \end{aligned} \quad (2.42)$$

Using (2.42), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

Case 9.  $k \in \left(0, \frac{\pi^2}{2}\right), l \in (0, \infty), \frac{1}{4} \left(\pi^2 k - \frac{\pi^4}{4}\right) < l < \frac{1}{4} k^2$

In the case, we take

$$x_0(t) = \cos m_1 t, \quad x_1(t) = \sin m_1 t, \quad x_2(t) = -\cos m_2 t, \quad x_3(t) = -\sin m_2 t, \quad (2.43)$$

where  $m_1 = \sqrt{\frac{k - \sqrt{k^2 - 4l}}{2}}, m_2 = \sqrt{\frac{k + \sqrt{k^2 - 4l}}{2}}$  Clearly,  $m_1 \in \left(0, \frac{\pi}{2}\right), m_2 \in \left(0, \frac{\pi}{2}\right),$

$m_1 < m_2$  and then

$$\cos m_1 t > 0, \quad \cos m_2 t > 0, \quad \forall t \in [0, 1]. \quad (2.44)$$

It is easy to check that  $x_0(t), x_1(t), x_2(t),$  and  $x_3(t)$  form a basis of solutions of  $L[x] = 0$ .

By simple computation, we have

$$w_1 = \cos m_1 t, \quad w_2 = m_1, \quad w_3 = m_1 (m_2^2 - m_1^2) \cos m_2 t, \quad w_4 = m_1 m_2 (m_2^2 - m_1^2)^2 \quad (2.45)$$

Clearly,  $w_i > 0$  ( $i = 1, 2, 3, 4$ ) on  $[0, 1]$ .

By Lemma 2.2,  $L[x] = 0$  is disconjugate on  $[0, 1]$ , and  $L[x]$  has a factorization

$$\begin{aligned} & x'''' + kx'' + lx \\ &= \frac{m_2 (m_2^2 - m_1^2)}{\cos m_2 t} \left( \frac{\cos^2 m_2 t}{m_2} \left( \frac{m_1}{(m_2^2 - m_1^2) \cos m_1 t \cos m_2 t} \left( \frac{\cos^2 m_1 t}{m_1} \left( \frac{1}{\cos m_1 t} x \right) \right) \right) \right)' \end{aligned} \quad (2.46)$$

and accordingly

$$L_0 x = \rho_0 x = \frac{1}{\cos m_1 t} x, \quad L_1 x = \rho_1 (L_0 x)' = \frac{1}{m_1} (\cos m_1 t \cdot x' + m_1 \sin m_1 t \cdot x). \quad (2.47)$$

Using (2.47), we conclude that  $x(0) = x(1) = x'(0) = x'(1) = 0$  is equivalent to (2.8).

This completes the proof of Theorem 2.4.

*Remark 2.5.* If condition (A1) does not hold, the results of Theorem 2.4 cannot be obtained. For example, in the case of  $L[x] = 0$  with  $k = -\pi^2, l = \pi^4, t = \frac{5}{6} \in [0, 1],$  we have  $l > \frac{1}{4} \left(k - \frac{\pi^2}{4}\right)^2, k \in (-\infty, 0).$  Applying the similar method to prove case 1 in

Theorem 2.4, we conclude that

$$x_0(t) = e^{-\frac{\sqrt{3}\pi}{2}t} \cos \frac{\pi}{2}t, \quad x_1(t) = e^{-\frac{\sqrt{3}\pi}{2}t} \sin \frac{\pi}{2}t, \quad x_2(t) = e^{-\frac{\sqrt{3}\pi}{2}t} \cos \frac{\pi}{2}t, \quad x_3(t) = e^{-\frac{\sqrt{3}\pi}{2}t} \sin \frac{\pi}{2}t$$

form a basis of solutions of  $L[x] = 0$ . By simple computation, we have

$$w_1 = \frac{\cos \frac{\pi}{2}t}{e^{\frac{\sqrt{3}\pi}{2}t}}, \quad w_2 = \frac{\pi}{2}e^{-\sqrt{3}\pi t}, \quad w_3 = \frac{\sqrt{3}}{2}\pi^3 \left( \sqrt{3} \cos \frac{\pi}{2}t - \sin \frac{\pi}{2}t \right) e^{-\frac{\sqrt{3}\pi}{2}t}, \quad w_4 = 3\pi^6.$$

From  $\frac{\pi}{2}t = \frac{5}{12}\pi$ , we easily get that  $\tan \frac{\pi}{2}t = 2 + \sqrt{3} > \sqrt{3}$ . Furthermore,  $w_3 < 0$ . Thus, Theorem 2.4 does not hold in this case.

*Remark 2.6.* In the following, consider  $L[x] = 0$ , for  $k, l$  are given constants, by the similar method in Remark 2.5, we may gain the location of  $(k, l)$  in the  $(k, l)$ -plane and the results of  $w_3$  or  $w_1$  corresponding  $k, l$ :  $k = \frac{\pi^2}{16}, l = \frac{4\pi^2}{16^2}, t = \frac{2}{\sqrt{5}}, l > \frac{1}{4} \left( k - \frac{\pi^2}{4} \right)^2$  and  $w_3 < 0$ ;  $k = \frac{\pi^2}{16}, l = \frac{9\pi^4}{4 \times 16^2}, t = 1, l = \frac{1}{4} \left( k - \frac{\pi^2}{4} \right)^2$  and  $w_3 < 0$ ;  $k = \frac{\pi^2}{16}, l = \frac{\pi^4}{16^2}, t = \frac{4}{3\sqrt{3}}, \frac{1}{4}k^2 < l < \frac{1}{4} \left( k - \frac{\pi^2}{4} \right)^2$  and  $w_3 = 0$ ;  $k = \pi^2, l = \frac{\pi^4}{4}, t = \frac{2\sqrt{2}}{3}, l = \frac{1}{4}k^2$  and  $w_1 < 0$ ;  $k = \pi^2, l = \frac{\pi^4}{8}, t = \sqrt{2 - \sqrt{2}}, 0 < l < \frac{1}{4}k^2$  and  $w_3 < 0$ ;  $k = 4\pi^2, l = 0, t = \frac{3}{4}$  and  $w_3 < 0$ ;  $k = \frac{\pi^2}{2}, l = -\frac{\pi^4}{2}, t = 1, l = \pi^2(k - \pi^2)$  and  $w_3 = 0$ ;  $k = -\pi^2, l = -8\pi^4, t = \frac{\sqrt{2}}{2}, l < \pi^2(k - \pi^2)$  and  $w_3 < 0$ . Furthermore, it follows that the conclusion of Theorem 2.4 cannot be yielded in the cases.

**Theorem 2.7.** *Let (A1) hold and  $h$  satisfy (H2). Then*

(i) *The problem (1.8) has an infinite number of positive eigenvalue*

$$\lambda_1(h) < \lambda_2(h) < \dots < \lambda_k(h) < \lambda_{k+1}(h) < \dots$$

(ii)  $\lambda_k(h) \rightarrow \infty$  as  $k \rightarrow \infty$ .

(iii) *To each eigenvalue  $\lambda_k(h)$  there corresponds an essential unique eigenfunction  $\psi_k$  which has exactly  $k - 1$  simple zeros in  $(0, 1)$  and is positive near 0.*

(iv) *Given an arbitrary subinterval of  $[0, 1]$ , then an eigenfunction which belongs to a sufficiently large eigenvalue change its sign in that subinterval.*

(v) *For each  $k \in \mathbb{N}$ , the geometric multiplicity of  $\lambda_k(h)$  is 1.*

*Proof of Theorem 2.7.* (i)-(iv) are immediate consequences of Elias [[7], Theorem 1-5] and Theorem 2.4, we only prove (v).

Let

$$\hat{L}x := x'''' + kx'' + lx, \quad x \in D(\hat{L}), \tag{2.48}$$

with

$$D(\hat{L}) = \{x \in C^4[0, 1] | x(0) = x(1) = x'(0) = x'(1) = 0\}.$$

To show (v), it is enough to prove

$$\ker(\hat{L} - \lambda_k(h)h(\cdot))^2 = \ker(\hat{L} - \lambda_k(h)h(\cdot)).$$

Clearly

$$\ker(\hat{L} - \lambda_k(h)h(\cdot))^2 \supseteq \ker(\hat{L} - \lambda_k(h)h(\cdot)). \tag{2.49}$$

Suppose on the contrary that the geometric multiplicity of  $\lambda_k(h)$  is greater than 1. Then there exists  $x \in \ker(\hat{L} - \lambda_k(h)h(\cdot))^2 \setminus \ker(\hat{L} - \lambda_k(h)h(\cdot))$  and subsequently

$$\hat{L}x - \lambda_k(h)h(t)x = \gamma \psi_k \tag{2.50}$$

for some  $\gamma \neq 0$ . Multiplying both sides of (2.50) by  $\psi_k(t)$  and integrating from 0 to 1, we deduce that

$$0 = \gamma \int_0^1 [\psi_k(t)]^2 dt, \tag{2.51}$$

which is a contradiction !

**Theorem 2.8** (Maximum principle). *Let (A1) hold. Let  $e \in C[0,1]$  with  $e \geq 0$  on  $[0,1]$  and  $e \not\equiv 0$  on any compact subinterval in  $[0,1]$ . If  $x \in C^4[0,1]$  satisfies*

$$\begin{aligned} x'''' + kx'' + lx &= e(t), \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) &= 0, \end{aligned} \tag{2.52}$$

Then  $x > 0$  on  $(0,1)$ .

**Proof.** When (A1) holds, the homogeneous problem

$$\begin{aligned} x'''' + kx'' + lx &= 0, \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) &= 0 \end{aligned} \tag{2.53}$$

has only trivial solution. So the boundary value problem (2.52) has a unique solution which may be represented in the form

$$x(t) = \int_0^1 G(t,s)e(s) ds, \tag{2.54}$$

where  $G(t, s)$  is Green's function. By Theorem 2.4 and Lemma 2.3 (take  $n = 4, k = 2$ ), we have

$$(-1)^{4-2}G(t,s) > 0, \quad \forall (t,s) \in (0,1) \times (0,1), \tag{2.55}$$

that is,  $G(t, s) > 0$ , for all  $(t, s) \in (0,1) \times (0,1)$ .

Using (2.54), when  $e \geq 0$  on  $[0,1]$  and  $e \not\equiv 0$  on any compact subinterval in  $[0,1]$ , then  $x > 0$  on  $(0,1)$ .

### 3 Main results

**Theorem 3.1.** *Let (A1), (A2), (A3) and (A4) hold. Assume that either (i) or (ii) holds for some  $k \in \mathbb{N}$  and  $j \in \{0\} \cup \mathbb{N}$ :*

$$\begin{aligned} (i) \lambda_k(b) &< \dots < \lambda_{k+j}(b) < 1 < \lambda_k(a) < \dots < \lambda_{k+j}(a); \\ (ii) \lambda_k(a) &< \dots < \lambda_{k+j}(a) < 1 < \lambda_k(b) < \dots < \lambda_{k+j}(b). \end{aligned} \tag{3.1}$$

Then the problem (1.2) has  $2(j + 1)$  solutions  $x_{k+i}^+, x_{k+i}^-, i = 0, \dots, j, x_{k+i}^+$  has exactly  $k+i-1$  zeros in  $(0,1)$  and is positive near  $t = 0$ , and  $x_{k+i}^-$  has exactly  $k + i - 1$  zeros in  $(0,1)$  and is negative near  $t = 0$ .

**Theorem 3.2** Let (A1), (A2), (A3) and (A5) hold. Assume that for some  $k \in \mathbb{N}$ ,

$$\lambda_k(a) < 1. \tag{3.2}$$

Then there are at least  $2k - 1$  nontrivial solutions of the problem (1.2). In fact, there exist solutions  $\omega_1, \dots, \omega_k$  such that for  $1 \leq j \leq k$ ,  $\omega_j$  has exactly  $j - 1$  simple zeros on the open interval  $(0,1)$  and  $\omega_j''(0) < 0$  and there exist solutions  $z_2, \dots, z_k$  such that for  $2 \leq j \leq k$ ,  $z_j$  has exactly  $j - 1$  simple zeros on the open interval  $(0,1)$  and  $z_j''(0) > 0$ .

Let  $Y = C[0,1]$  with the norm

$$\|x\|_\infty = \max_{t \in [0,1]} |x|. \tag{3.3}$$

Let  $E = \{x \in C^2[0, 1] | x(0) = x(1) = x'(0) = x'(1) = 0\}$  with the norm

$$\|x\|_E = \max \{ \|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty \}. \tag{3.4}$$

Then  $\hat{L}^{-1} : Y \rightarrow E$  is completely continuous. Here  $\hat{L}$  is given as in (2.48).

Let  $\zeta(\cdot, \cdot), \xi_1(\cdot, \cdot), \xi_2(\cdot, \cdot) \in C([0,1] \times \mathbb{R}, \mathbb{R})$  be such that

$$\begin{aligned} f(t, x) &= a(t)x + \zeta(t, x), \quad f(t, x) = b(t)x + \xi_1(t, x), \\ f(t, x) &= c(t)x^+ + \xi_2(t, x), \quad \forall (t, x) \in [0, 1] \times \mathbb{R}. \end{aligned} \tag{3.5}$$

Here  $x^+ = \max\{x, 0\}$ .

Clearly,

$$\lim_{|x| \rightarrow 0} \frac{\zeta(t, x)}{x} = 0, \quad \lim_{|x| \rightarrow \infty} \frac{\xi_1(t, x)}{x} = 0, \quad \lim_{|x| \rightarrow \infty} \frac{\xi_2(t, x)}{x} = 0 \tag{3.6}$$

uniformly for  $t \in [0,1]$ .

Let

$$\begin{aligned} \bar{\xi}_1(x) &= \max \{ |\xi_1(t, s)| : 0 \leq |s| \leq x, t \in [0, 1] \}, \\ \bar{\xi}_2(x) &= \max \{ |\xi_2(t, s)| : 0 \leq |s| \leq x, t \in [0, 1] \}, \end{aligned} \tag{3.7}$$

then  $\bar{\xi}_1$  and  $\bar{\xi}_2$  are nondecreasing and

$$\lim_{x \rightarrow \infty} \frac{\bar{\xi}_1(x)}{x} = 0, \quad \lim_{x \rightarrow \infty} \frac{\bar{\xi}_2(x)}{x} = 0. \tag{3.8}$$

Let us consider

$$\hat{L}x = \lambda a(t)x + \lambda \zeta(t, x) \tag{3.9}$$

as a bifurcation problem from the trivial solution  $x \equiv 0$ .

Equation (3.9) can be converted to the equivalent equation

$$x(t) = \lambda \hat{L}^{-1} [a(\cdot)x(\cdot)](t) + \lambda \hat{L}^{-1} [\zeta(\cdot, x(\cdot))](t). \tag{3.10}$$

Clearly, the compactness of  $\hat{L}^{-1}$  together with (3.6) imply that

$$\left\| \hat{L}^{-1} [\zeta(\cdot, x(\cdot))] \right\|_E = o(\|x\|_E), \quad \text{as } \|x\|_E \rightarrow 0.$$

Let  $S_k^+$  denotes the set of functions in  $E$  which have exactly  $k - 1$  interior nodal (i.e., non-degenerate) zeros in  $(0,1)$  and are positive near  $t = 0$ , set  $S_k^- = -S_k^+$ , and

$\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ . They are disjoint and open sets in  $E$ . Finally, let  $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$  and  $\Phi_k = \mathbb{R} \times S_k$ .

The results of Rabinowitz [15] for (3.9) can be stated as follows: For each integer  $k \geq 1$  and each  $v = \{+, -\}$ , there exists a continuum  $C_k^v \subseteq \Phi_k^v$  of solution of (3.9), joining  $(\lambda_k(a), 0)$  to infinity in  $\Phi_k^v$ . Moreover,  $C_k^v \setminus (\lambda_k(a), 0) \subset \Phi_k^v$ .

Notice that we have used the fact that if  $x$  is a nontrivial solution of (3.9), then all zeros of  $x$  on  $(0, 1)$  are simple under (A1), (A2), (A3), and (A4).

In fact, (3.9) can be rewritten to

$$\hat{L}x = \lambda \hat{a}(t)x, \tag{3.11}$$

where

$$\hat{a}(t) = \begin{cases} \frac{f(t, x(t))}{x(t)}, & x(t) \neq 0, \\ a(t), & x(t) = 0. \end{cases}$$

Clearly  $\hat{a}(t)$  satisfies (H2). So Theorem 2.7 (iii) yields that all zeros of  $x$  on  $(0,1)$  are simple.

*Proof of Theorem 3.1.* We first prove the theorem when  $j = 0$ .

It is clear that any solution of (3.9) of the form  $(1, x)$  yields solutions  $x$  of (1.2). We will show that  $C_k^v$  crosses the hyperplane  $\{1\} \times E$  in  $\mathbb{R} \times E$ . To do this, it is enough to show that  $C_k^v$  joins  $(\lambda_k(a), 0)$  to  $(\lambda_k(b), \infty)$ . Let  $(\mu_n, x_n) \in C_k^v$  satisfy

$$\mu_n + \|x_n\|_E \rightarrow \infty. \tag{3.12}$$

We note that  $\mu_n > 0$  for all  $n \in \mathbb{N}$  since  $(0, 0)$  is the only solution of (3.9) for  $\lambda = 0$  and  $C_k^v \cap (\{0\} \times E) = \emptyset$ .

*Case 1.*  $\lambda_k(b) < 1 < \lambda_k(a)$ .

In this case, we show that

$$(\lambda_k(b), \lambda_k(a)) \subseteq \{\lambda \in \mathbb{R} : \exists(\lambda, x) \in C_k^v\}.$$

We divide the proof into two steps.

*Step 1.* We show that if there exists a constant number  $M > 0$  such that

$$\mu_n \in (0, M], \tag{3.13}$$

then  $C_k^v$  joins  $(\lambda_k(a), 0)$  to  $(\lambda_k(b), \infty)$ .

In this case  $\|x_n\|_E \rightarrow \infty$ . We divide the equation

$$\hat{L}x_n = \mu_n b(t)x_n + \mu_n \xi_1(t, x_n), \quad t \in (0, 1) \tag{3.14}$$

by  $\|x_n\|_E$  and set  $y_n = \frac{x_n}{\|x_n\|_E}$ . Since  $y_n$  is bounded in  $C^2[0,1]$ , choosing a subsequence and relabeling if necessary, we have that  $y_n \rightarrow y$  for some  $y \in E$  with  $\|y\|_E = 1$ . Moreover, from (3.8) and the fact that  $\xi_1$  is nondecreasing, we have that

$$\lim_{n \rightarrow \infty} \frac{|\xi_1(t, x_n(t))|}{\|x_n\|_E} = 0, \quad \forall t \in [0, 1], \tag{3.15}$$

since

$$\frac{|\xi_1(t, x_n(t))|}{\|x_n\|_E} \leq \frac{\bar{\xi}_1(|x_n(t)|)}{\|x_n\|_E} \leq \frac{\bar{\xi}_1(\|x_n(t)\|_\infty)}{\|x_n\|_E} \leq \frac{\bar{\xi}_1(\|x_n(t)\|_E)}{\|x_n\|_E}, \quad \forall t \in [0, 1]. \quad (3.16)$$

Thus

$$\gamma(t) = \hat{L}^{-1} [\mu b(\cdot)\gamma(\cdot)](t), \quad (3.17)$$

where  $\mu := \lim_{n \rightarrow \infty} \mu_n$ , again choosing a subsequence and relabeling if necessary. Thus

$$\hat{L}\gamma = \mu b(t)\gamma. \quad (3.18)$$

We claim that

$$\gamma \in S_k^v. \quad (3.19)$$

Suppose, to the contrary, that  $\gamma \notin S_k^v$ . Since  $\gamma \neq 0$  is a solution of (3.18), all zeros of  $\gamma$  in  $[0, 1]$  are simple. It follows that  $\gamma \in S_h^l \neq S_k^v$  for some  $h \in \mathbb{R}$  and  $l \in \{+, -\}$ . By the openness of  $S_h^l$  we know that there exists a neighborhood  $U(\gamma, \rho_0)$  such that

$$U(\gamma, \rho_0) \subset S_h^l,$$

which, together with the fact  $y_n \rightarrow \gamma$ , implies that exists  $n_0 \in \mathbb{N}$  such that

$$\gamma_n \in S_h^l, \quad n \geq n_0.$$

However, this contradicts the fact that  $\gamma_n \in S_k^v$ . Therefore,  $\gamma \in S_k^v$ .

Now, by Theorem 2.7, we obtain  $\mu = \lambda_k(b)$ .

Thus  $C_k^v$  joins  $(\lambda_k(a), 0)$  to  $(\lambda_k(b), \infty)$ .

*Step 2.* We show that there exists a constant number  $M > 0$  such that  $\mu_n \in (0, M]$ , for all  $n$ .

Suppose there is no such  $M$ . Choosing a subsequence and relabeling if necessary, it follows that

$$\lim_{n \rightarrow \infty} \mu_n = \infty. \quad (3.20)$$

Let

$$0 = \tau(0, n) < \tau(1, n) < \dots < \tau(k, n) = 1$$

denotes the zeros of  $x_n$ . Then there exists a subsequence  $\{\tau(1, n_m)\} \subseteq \{\tau(1, n)\}$  such that

$$\lim_{m \rightarrow \infty} \tau(1, n_m) := \tau(1, \infty).$$

Clearly

$$\lim_{m \rightarrow \infty} \tau(0, n_m) := \tau(0, \infty) = 0.$$

We claim that

$$\tau(1, \infty) - \tau(0, \infty) = 0. \quad (3.21)$$

Suppose, to the contrary, that

$$\tau(0, \infty) < \tau(1, \infty). \tag{3.22}$$

Define a function  $p: [0,1] \times [0, \infty) \rightarrow \mathbb{R}$  by

$$p(t, x) = \begin{cases} \frac{f(t, x)}{x}, & x \neq 0, t \in [0, 1], \\ a(t), & x = 0, t \in [0, 1]. \end{cases} \tag{3.23}$$

Then, by (A2), (A3), and (A4), there exist two positive numbers  $\rho_1$  and  $\rho_2$ , such that

$$\rho_1 \leq \frac{f(t, x)}{x} \leq \rho_2, \text{ for all } x \geq 0, t \in [0, 1]. \tag{3.24}$$

Using (3.22), (3.24), and the fact that  $\lim_{m \rightarrow \infty} \mu_{n_m} = \infty$ , we conclude that there exists a closed interval  $I_1 \subset (\tau(0, \infty), \tau(1, \infty))$  such that

$$\lim_{m \rightarrow \infty} \mu_{n_m} p(t, x_{n_m}(t)) = \infty$$

uniformly for  $t \in I_1$ .

However, since  $x_{n_m}$  satisfies

$$\hat{L}x_{n_m}(t) = \mu_{n_m} p(t, x_{n_m}(t)) x_{n_m}(t),$$

the proof of Lemma 4 in [7] (see also the remarks in the final paragraph in [[7], p. 43]), shows that for all  $n$  sufficiently large,  $x_{n_m}$  must change sign on  $I_1$ . However, this contradicts the fact that for all  $m$  sufficiently large we have  $I_1 \subset (\tau(0, n_m), \tau(1, n_m))$  and

$$v x_{n_m}(t) > 0, t \in (\tau(0, n_m), \tau(1, n_m)).$$

Thus, (3.21) holds.

Next, we work with  $(\tau(1, n_m), \tau(2, n_m))$ . It is easy to see that there is a subsequence  $\tau(2, n_{m_j}) \subseteq \tau(2, n_m)$  such that

$$\lim_{j \rightarrow \infty} \tau(2, n_{m_j}) := \tau(2, \infty).$$

Clearly

$$\lim_{j \rightarrow \infty} \tau(1, n_{m_j}) = \tau(1, \infty). \tag{3.25}$$

We claim that

$$\tau(2, \infty) - \tau(1, \infty) = 0. \tag{3.26}$$

Suppose, to the contrary, that  $\tau(1, \infty) < \tau(2, \infty)$ . Then, from (3.23), (3.24), and the fact that  $\lim_{j \rightarrow \infty} \mu_{n_{m_j}} = \infty$ , there exists a closed interval  $I_2 \subset (\tau(1, \infty), \tau(2, \infty))$  such that

$$\lim_{j \rightarrow \infty} \mu_{n_{m_j}} p(t, x_{n_{m_j}}(t)) = \infty$$

uniformly for  $t \in I_2$ .

This implies the solution  $x_{n_{m_j}}$  of the equation

$$\hat{L}x_{n_{m_j}}(t) = \mu_{n_{m_j}} p(t, x_{n_{m_j}}(t)) x_{n_{m_j}}(t)$$

must change sign on  $I_2$ . However, this contradicts the fact that for all  $j$  sufficiently large we have  $I_2 \subset (\tau(1, n_{m_j}), \tau(2, n_{m_j}))$  and

$$v x_{n_{m_j}}(t) > 0, \quad t \in (\tau(1, n_{m_j}), \tau(2, n_{m_j})).$$

Therefore, (3.26) holds.

By a similar argument to obtain (3.21) and (3.26), we can show that for each  $l \in \{2, \dots, k-1\}$ ,

$$\tau(l+1, \infty) - \tau(l, \infty) = 0. \tag{3.27}$$

Taking a subsequence and relabeling it as  $\{(\mu_n, x_n)\}$ , if necessary, it follows that for each  $l \in \{0, \dots, k-1\}$ ,

$$\lim_{n \rightarrow \infty} (\tau(l+1, n) - \tau(l, n)) = 0. \tag{3.28}$$

But this is impossible since

$$1 = \tau(k, n) - \tau(0, n) = \sum_{l=0}^{k-1} (\tau(l+1, n) - \tau(l, n)) \tag{3.29}$$

for all  $n$ . Therefore,

$$|\mu_n| \leq M$$

for some constant number  $M > 0$ , independent of  $n \in \mathbb{N}$ .

Case 2.  $\lambda_k(a) < 1 < \lambda_k(b)$ .

In this case, if  $(\mu_n, x_n) \in C_k^v$  is such that

$$\lim_{n \rightarrow \infty} (\mu_n + \|x_n\|_E) = \infty \tag{3.30}$$

and

$$\lim_{n \rightarrow \infty} \mu_n = \infty, \tag{3.31}$$

then

$$(\lambda_k(a)m, \lambda_k(b)) \subseteq \{\lambda \in (0, \infty) \mid (\lambda, x) \in C_k^v\} \tag{3.32}$$

and, moreover,  $(\{1\} \times E) \cap C_k^v \neq \emptyset$ .

Assume that there exists  $M > 0$  such that for all  $n \in \mathbb{N}$ ,

$$\mu_n \in (0, M]. \tag{3.33}$$

Applying a similar argument to that used in step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$(\mu_n, x_n) \rightarrow (\lambda_k(b), \infty), \quad n \rightarrow \infty. \tag{3.34}$$

Again  $C_k^v$  joins  $(\lambda_k(a), 0)$  to  $(\lambda_k(b), \infty)$  and the result follows.

Finally, let  $j \in \mathbb{N}$ . By repeating the arguments used in the proof of the case  $j = 0$ , we see that for each  $v \in \{+, -\}$  and each  $i \in \{k, k+1, \dots, k+j\}$ ,

$$C_i^v \cap (\{1\} \times E) \neq \emptyset. \tag{3.35}$$

The result follows.

*Proof of Theorem 3.2.*

We only need to show that

$$\begin{aligned} C_j^- \cap (\{1\} \times E) &\neq \emptyset, \quad j = 1, 2, \dots, k, \\ C_j^+ \cap (\{1\} \times E) &\neq \emptyset, \quad j = 2, \dots, k. \end{aligned}$$

Suppose on the contrary that

$$C_i^l \cap (\{1\} \times E) = \emptyset, \quad \text{for some } (i, l) \in \Gamma, \tag{3.36}$$

where

$$\Gamma := \{(j, \nu) | j \in \{2, \dots, k\} \text{ as } \nu = +, j \in \{1, 2, \dots, k\} \text{ as } \nu = -\}. \tag{3.37}$$

Since  $C_i^l$  joins  $(\lambda_{\nu_i}(a), 0)$  to infinity in  $\Phi_i^l$  and  $(\lambda, x) = (0, 0)$  is the unique solution of  $(3.9)_{\lambda=0}$  in  $E$ , there exists a sequence  $\{(\mu_n, x_n)\} \subset C_i^l$  such that  $\mu_n \in (0, 1)$  and  $\|x_n\|_E \rightarrow \infty$  as  $n \rightarrow \infty$ . We may assume that  $\mu_n \rightarrow \mu \in [0, 1]$  as  $n \rightarrow \infty$ . Let  $y_n = x_n / \|x_n\|_E$ ,  $n \geq 1$ . From the fact

$$\hat{L}x_n(t) = \mu_n c(t)(x_n)^+(t) + \mu_n \xi_2(t, x_n(t)). \tag{3.38}$$

We have that

$$y_n(t) = \mu_n \hat{L}^{-1}[c(\cdot)(y_n)^+](t) + \mu_n \hat{L}^{-1} \left[ \frac{\xi_2(\cdot, x_n)}{\|x_n\|_E} \right] (t). \tag{3.39}$$

Furthermore, since  $\hat{L}^{-1}|_E : E \rightarrow E$  is completely continuous, we may assume that there exists  $y \in E$  with  $\|y\|_E = 1$  such that  $\|y_n - y\|_E \rightarrow 0$  as  $n \rightarrow \infty$ . Since

$$\frac{|\xi_2(t, x_n)|}{\|x_n\|_E} \leq \frac{\bar{\xi}_2(\|x_n\|_\infty)}{\|x_n\|_E} \leq \frac{\bar{\xi}_2(\|x_n\|_E)}{\|x_n\|_E} \tag{3.40}$$

uniformly for  $t \in [0, 1]$ , we have from (3.39) and (3.8) that

$$y = \mu \hat{L}^{-1}[c(t)y^+], \tag{3.41}$$

that is,

$$\begin{aligned} y'''' + ky'' + ly &= \mu c(t)y^+, \quad 0 < t < 1, \\ y(0) = y(1) &= y'(0) = y'(1) = 0. \end{aligned} \tag{3.42}$$

By (A1), (A5), and (3.42) and the fact that  $\|y\|_E = 1$ , we conclude that  $\mu c(t)y^+ \not\equiv 0$  on any compact subinterval in  $[0, 1]$ , and consequently

$$\mu > 0, y^+ \not\equiv 0, \quad \text{on any compact subinterval in } [0, 1]. \tag{3.43}$$

By Theorem 2.8, we know that  $y(t) > 0$  in  $(0, 1)$ . This means  $\mu$  is the first eigenvalue of  $\hat{L}x = \lambda c(t)x$  and  $y$  is the corresponding eigenfunction. Hence  $y \in S_1^+$  and therefore, since  $S_1^+$  is open and  $\|y_n - y\|_E \rightarrow 0$ , we have that  $y_n \in S_1^+$  for  $n$  large. But this contradicts the assumption that  $(\mu_n, y_n) \in C_i^l$  and  $(i, l) \in \Gamma$ , so (3.36) is wrong, which completes the proof.

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#### Competing interests

The author declares that they have no competing interests.

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