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Global existence and asymptotic behavior of classical solutions to Goursat problem for diagonalizable quasilinear hyperbolic system

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Abstract

In this article, we investigate the global existence and asymptotic behavior of classical solutions to Goursat problem for diagonalizable quasilinear hyperbolic system. Under the assumptions that system is strictly hyperbolic and linearly degenerate, we obtain the global existence and uniqueness of C^1 solutions with the bounded $L^1 \cap L^\infty$ norm of the boundary data as well as their derivatives. Based on the existence result, we can prove that when t tends to in nity, the solutions approach a combination of piece-wised C^1 traveling wave solutions. As the important example, we apply the results to the chaplygin gas system.
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1 Introduction and main results

For the general first order quasilinear hyperbolic systems,

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0$$

the global existence of classical solutions of Cauchy problem has been established for lin-early degenerate characteristics or weakly linearly degenerate characteristics with various smallness assumptions on the initial data by Bressan [1], Li [2], Li and Zhou [3,4], Li and Peng [5,6], and Zhou [7]. The asymptotic behavior has been obtained by Kong and Yang [8], Dai and Kong [9,10]. For linearly degenerate diagonalizable quasi-linear hyperbolic systems with "large" initial data, asymptotic behavior of the global classical solutions has been ob-tained by Liu and Zhou [11]. For the initial-boundary value problem in the first quadrant Li and Wang [12] proved the global existence of classical solutions for weakly linearly degenerate positive eigenvalues with small and decay initial and boundary data. The asymptotic behavior of the global classical solutions is studied by Zhang [13]. The global existence and asymptotic behavior of classical solutions of the initial-boundary value problem of diagonal-izable quasilinear hyperbolic systems in the first quadrat was obtained in [14].

However, relatively little is known for the Goursat problem with characteristic boundaries. Global existence of the global classical solutions for the Goursat problem

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of reducible quasilinear hyperbolic system was obtained in [15]. Under the assumptions of boundary data is small and decaying, the global existence and asymptotic behavior to classical solutions can be obtained by Liu [16,17]. The asymptotic behavior of classical solutions of Goursat problem for reducible quasilinear hyperbolic system was shown in [18].

In this article, we consider the following diagonalizable quasilinear hyperbolic system:

$$\frac{\partial u_i}{\partial t} + \lambda_i(u)\frac{\partial u_i}{\partial x} = 0 \tag{1}$$

where $u = (u_1, \ldots, u_n)^T$ is unknown vector-valued function of (t, x). $\lambda_i(u)$ is given by C^2 vector-valued function of u and is linearly degenerate, i.e.,

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0 \tag{2}$$

The system (1) is strictly hyperbolic, i.e.,

$$\lambda_1(u) < \cdots < \lambda_m(u) < 0 < \lambda_{m+1}(u) < \cdots < \lambda_n(u) \tag{3}$$

Suppose that there exists a positive constant δ such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq \delta, \qquad i = 1, \ldots, n-1 \tag{4}$$

Consider the Goursat problem for the strictly quasilinear hyperbolic system (1), in which the solutions to system (1) is asked to satisfy the following characteristic boundary conditions:

$$x = x_1(t): \quad u = \phi(t) \tag{5}$$

and

$$x = x_n(t): \quad u = \psi(t); \tag{6}$$

where $x = x_1(t)$ and $x = x_n(t)$ are the leftmost and the rightmost characteristics passing through the origin $(t, x) = (0, 0)$, respectively, such that

$$\begin{cases} \frac{dx_1(t)}{dt} = \lambda_1\left(\phi(t)\right) \\ x_1(0) = 0 \end{cases} \tag{7}$$

and

$$\begin{cases} \frac{dx_n(t)}{dt} = \lambda_n\left(\psi(t)\right) \\ x_n(0) = 0 \end{cases} \tag{8}$$

moreover,

$$l_1(\phi(t))\phi'(t) \equiv 0 \tag{9}$$

and

$$l_n(\psi(t))\psi'(t) \equiv 0 \tag{10}$$

where $\varphi\,(t) = (\varphi_1(t), ..., \varphi_n(t))^T$ and $\psi(t) = (\psi_1(t), ..., \psi_n(t))^T$ are any given C^1 vector functions satisfying the conditions of C^1 compatibility at the origin (0, 0):

$$\phi(0) = \psi(0) \tag{11}$$

and

$$\lambda_n(\phi(0))\phi'(0) - \lambda_1(\psi(0))\psi'(0) + A(\phi(0))(\psi'(0) - \phi'(0)) = 0. \tag{12}$$

$l_i(u)$ be a left eigenvector corresponding to $\lambda_i(u)$ and $A(u) = \mathrm{diag}\{\lambda_1(\mathrm{u}), ..., \lambda_n(u)\}$.

Our goal in this article is to get the global existence and asymptotic behavior of the global classical solutions of the Goursat problem (1), (5), and (6) with "large" boundary data. With the assumptions that

$$\sup_{t\in R^+}\{|\phi'(t)| + |\psi'(t)|\} \doteq M < \infty, \qquad \sup_{t\in R^+}\{|\phi(t)| + |\psi(t)|\} \doteq M_0 < \infty \tag{13}$$

$$\int_0^{+\infty} \left\{|\phi(t)| + |\psi(t)|\right\} dt \doteq N_1 < \infty, \qquad \int_0^{+\infty} \left\{|\phi\prime(t)| + |\psi\prime(t)|\right\} dt \doteq N_2 < \infty \tag{14}$$

we can prove the following results:

Theorem 1.1. The above assumptions and the conditions of C^1 compatibility at the point (0, 0) are satisfied, the Goursat problem (1), (5), and (6) admits a unique global C^1 solutions $u = u(t, x)$ on the domain $D = \{(t, x)|\ t \geq 0, x_1(t) \leq \times \leq x_n(t)\}$.

If the leftmost and rightmost characteristics are convex, we can get the following result:

Theorem 1.2. Under the assumptions of Theorem 1.1, there exists a unique piecewised C^1 vector-valued function $\Phi(x) = (\Phi_1(x), ..., \Phi_n(x))^T$ such that

$$u(t,x) \to \sum_{i=1}^{n} \Phi_i(x - \lambda_i(0)t)e_i, \qquad t \to +\infty \tag{15}$$

uniformly as t tends to infinity, where $e_i = (0, ..., 0, 1^i, 0, ..., 0)^T$.

Remark 1.1. If the system (1) is non-strictly hyperbolic but each characteristic has constant multiplicity, then the result is similar as Theorems above.

2 Global existence of C^1 solutions

In this section, we will obtain some uniform a *priori* estimate which also play an important role in the proof of Theorem 1.1. In order to proving the global existence of classical solutions of the Goursat problem (1), (5), and (6), we will prove that $||u||_{C^1(D)}$ is bounded. For any fixed $T \geq 0$, we denote $D_T = \{(t, x)|0 \leq t \leq T, x_1(t) \leq x \leq x_n(t)\}$ and introduce

$$w_i(t,x) = \frac{\partial u_i(t,x)}{\partial x}\ (i = 1, \ldots, n)\,, \quad W_1(T) = \sup_{0\leq t\leq T} \int_{x_1(t)}^{x_n(t)} |w(t,x)|dx \tag{16}$$

$$\tilde{W}_1(T) = \max_{i\neq j} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |w_i(t,x)|dt, \quad \tilde{U}_1(T) = \max_{i\neq j} \sup_{\tilde{C}_j} \int_{\tilde{C}_j} |u_i(t,x)|dt \tag{17}$$

$$\bar{W}_1(T) = \max_{i \neq j} \sup_{L_j} \int_{L_j} |w_i(t,x)|dt, \ \ \bar{U}_1(T) = \max_{i \neq j} \sup_{L_j} \int_{L_j} |u_i(t,x)|dt \tag{18}$$

$$W_\infty(T) = \sup_{(t,x)\in D_T} \left\{|w(t,x)\right\}, \ \ U_\infty(T) = \sup_{(t,x)\in D_T} \left\{|u(t,x)\right\} \tag{19}$$

where $\tilde{C}_j$ stands for any given jth characteristic $\frac{dx}{dt} = \lambda_j(u)$, L_j stands for any given radial that has the slope $\lambda_j(0)$ on the domain D_T.

Lemma 2.1. Under the assumptions of Theorem 1.1, there exists a positive constant C such that, the following estimates hold

$$\tilde{W}_1(T), \bar{W}_1(T), W_1(T) \leq CN_2 \tag{20}$$

$$\tilde{U}_1(T), \bar{U}_1(T) \leq CN_1 e^{CN_2} \tag{21}$$

$$W_\infty(T) \leq CMe^{CN_2} \tag{22}$$

$$U_\infty(T) \leq C \tag{23}$$

Remark 2.1. The positive constant C is only depend on δ, M_0 and independent of M, N_1, N_2, T. In the following, the meaning of C is similar but may change from line to line.

Proof. For any fixed point $(t, x) \in D_T$, we draw the ith characteristic $\tilde{C}_i : x = x_i(t)$ through this point and intersecting $x_1(t)$ or $x_n(t)$ at a point $(t_*, x_1(t_*))$ or $(t_*, x_n(t_*))$. Noting system (1), $u_i(t, x)$ is a constant along the ith characteristic, then we have $u_i(t, x) = \varphi_i(t_*)$ or $u_i(t, x) = \psi_i(t_*)$. Then

$$|u_i(t,x)| \leq \sup_{t\in R^+} \left\{|\phi_i(t_*)| + |\psi(t_*)|\right\} \leq M_0 \leq C \tag{24}$$

Then, we obtain the estimate (23).

Differentiating (1) with respect to x we can get

$$\frac{\partial w_i}{\partial t} + \frac{\partial\left(\lambda_i(u)w_i\right)}{\partial x} = 0 \tag{25}$$

We rewrite (25) as

$$d(w_i(t,\ x)(dx - \lambda_i(u)dt)) = 0$$

Multiplying the system above by sign(w_i), we have

$$d(|w_i(t,\ x)|(dx - \lambda_i(u)dt)) = 0 \tag{26}$$

There are only the following cases(as shown in Figure 1):

Case 1. For any fixed $t_0 \in R^+$, let $\tilde{C}_j : x = x_j(t), j > 1$ stands for any given jth characteristic, passing through any point $A(t_0, x_1(t_0))$ on the boundary $x = x_1(t)$ and intersects $t = T$ at point P. We draw an ith characteristic $\tilde{C}_i : x = x_i(t)$ from P downward, intersecting $x = x_1(t)$ at a point $B(t_1, x_1(t_1))$. Without loss of generality, we assume $t_0 < t_1$, then $j > i$. Integrating (26) in the region APB to get

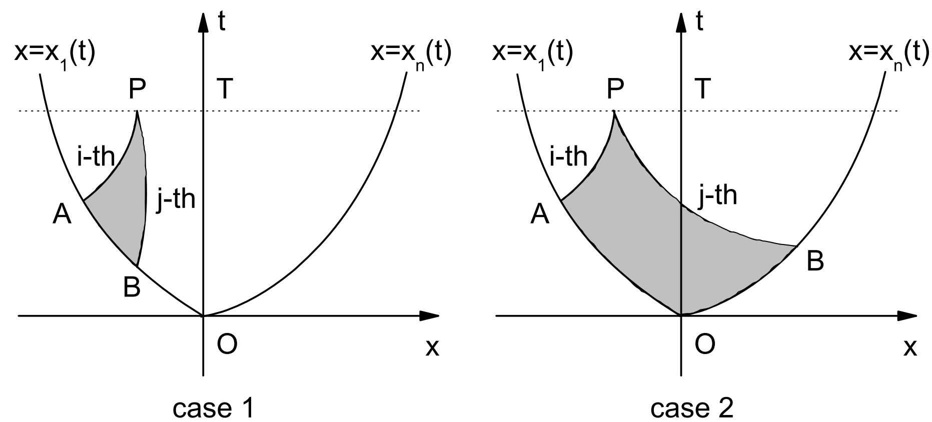

Figure 1 **Cases in estimating** $\tilde{W}_1(T)$.

$$\int_{\tilde{C}_j} (\lambda_j(u) - \lambda_i(u))|w_i(t,x)|dt = \int_{t_0}^{t_1} (\lambda_i(u) - \lambda_i(u)|w_i(t,x_1(t))|dt \tag{27}$$

Along the 1th characteristic, $\frac{dx}{dt} = \lambda_1(u)$, then $\frac{dt}{dx} = \frac{1}{\lambda_1(u)}$

$$\frac{\partial u_i(t,x_1(t))}{\partial x} = \frac{\partial u_i(t,x_1(t))}{\partial t}\frac{\partial t}{\partial x} = \frac{\phi_i'(t)}{\lambda_1(u)}$$

Noting (4), (13), and (23), we can get

$$\int_{\tilde{C}_j} |w_i(t,\ x)|dt \le C\int_0^{\infty} |\phi_i'(t)|dt \le CN_2 \tag{28}$$

Case 2. For any fixed $t_0 \in R^+$, passing through the point $A(t_0, x_1(t_0))$, we draw $\tilde{C}_j : x = x_j(t)$ and intersecting $t = T$ at point P. We draw the ith characteristic $\tilde{C}_i : x = x_i(t)$ from P downward, intersecting $x = x_n(t)$ at $B(t_1, x_n(t_1))$. Then, we integrate (26) in the region $PAOB$ to get

$$\begin{aligned}\int_{\tilde{C}_j} (\lambda_j(u) - \lambda_i(u))|w_i(t,x)|dt &= \int_0^{t_0} (\lambda_1(u) - (\lambda_i(u)) \left|\frac{\partial u_i(t,x_1(t))}{\partial t}\right| \left|\frac{\partial t}{\partial x}\right| dt \\ &+ \int_0^{t_1} (\lambda_n(u) - \lambda_i(u)) \left|\frac{\partial u_i(t,x_n(t))}{\partial t}\right| \left|\frac{\partial t}{\partial x}\right| dt\end{aligned}$$

Noting $i = 1, ..., m$ $i < j$, Equations (4), (13) and using the above procedure, we have

$$\int_{\tilde{C}_j} |w_i(t,x)|dt \le C\left(\int_0^{\infty} |\phi_i'(t)|dt + \int_0^{\infty} |\psi_i'(t)|dt\right) \le CN_2 \tag{29}$$

If the point A is on the characteristic boundary $x = x_n(t)$, using the same method as above cases we can get the desired estimates. Then, we can obtain

$$\tilde{W}_1(T) \tag{30}$$

In the similar way, substituting L_j for $\tilde{C}_j$ in the above cases we also can get

$$\bar{W}_1(T) \le CN_2 \tag{31}$$

In the following, we will get the estimate $W_1(T) \le CN_2$. Integrating Equation (26) with respect to x from $x_1(t)$ to $x_n(t)$ for any $t \in [0, T]$ leads to

$$\int_{x_1(t)}^{x_n(t)} \partial_t |w_i(t,x)| dx = \lambda_i(u(t, x_n(t))) \left| \frac{\partial u_i(t, x_n(t))}{\partial x} \right| - \lambda_i(u(t, x_1(t))) \left| \frac{\partial u_i(t, x_1(t))}{\partial x} \right| \tag{32}$$

$$\partial t \int_{x_1(t)}^{x_n(t)} |w_i(t,x)| dx = (\lambda_n - \lambda_i)(\psi(t)) \left| \frac{\partial u_i(t, x_n(t))}{\partial x} \right| - (\lambda_1 - \lambda_i)(\psi(t)) \left| \frac{\partial u_i(t, x_1(t))}{\partial x} \right| \tag{33}$$

Using the same procedure as (28), (29), we can get

$$\int_{x_1(t)}^{x_n(t)} |w_i(t,x)| dx \le CN_2 \tag{34}$$

Then, we can get the desired estimate.

Rewriting the Equation (25), we get

$$\frac{\partial w_i}{\partial t} + \lambda_i(u) \frac{\partial w_i}{\partial x} = -\frac{\partial \lambda_i(u)}{\partial x} w_i \tag{35}$$

Case 1. When $i = 1, ..., m$, we draw the ith characteristic intersecting $x = x_n(t)$ with the point $(t_0, x_n(t_0))$. Solving the ODE system we can get

$$w_i(t, x_i(t)) = w_i\left(t_0, x_n(t_0)\right) e^{\int_{\tilde{C}_i} -\frac{\partial \lambda_i(u)}{\partial x} dt}$$

Noting $w_i(t_0, x_n(t_0)) = \frac{\psi_i'(t_0)}{\lambda_n(u)}$ and the estimate (30), then

$$|w_i(t, x_i(t))| \le CMe^{C \int_{\tilde{C}_i} \sum_{j \ne i} |w_j| dt} \le CMe^{C\tilde{W}_1(T)} \le CMe^{CN_2} \tag{36}$$

Case 2. When $i = m + 1, ..., n$ and the ith characteristic intersecting $x = x_1(t)$ with a point $(t_0, x_1(t_0))$, then

$$w_i(t, x_i(t)) = w_i(t_0, x_1(t_0)) e^{\int_{\tilde{C}_i} -\frac{\partial \lambda_i(u)}{\lambda \mathrm{x}} dt}$$

Then, we can get

$$|w_i(t, x_i(t))| \le CMe^{CN_2}$$

Then, we can obtain the estimate

$$W_\infty(T) \le CMe^{CN_2} \tag{37}$$

In the following we estimate $\tilde{U}_1(T)$.

Noting system (1), we denote the multiplier $H_i \in C^1$, $(i = 1, ..., n)$ such that,

$$\frac{\partial(H_i u_i)}{\partial t} + \frac{\partial\left(\lambda_i(u)H_i u_i\right)}{\partial x} = 0 \tag{38}$$

Noting (1), then

$$\frac{\partial H_i}{\partial t} + \lambda_i(u)\frac{\partial H_i}{\partial x} = -\frac{\partial\left(\lambda_i(u)\right)}{\partial x} \cdot H_i$$

Let $H_i(t, x_1(t)) = 1$ and $H_i(t, x_n(t)) = 1$ we can get

$$H_i = e^{-\int_{\tilde{C}_i} \frac{\partial \lambda_i(u(\tau, x))}{\partial x} d\tau}$$

We note the system (1) is linearly degenerate, then

$$\begin{aligned} \left|\frac{\partial \lambda_i(u)}{\partial x}\right| = \left|\sum_{j\neq i}^{n} \frac{\partial \lambda_i(u)}{\partial u_j} w_j\right| &\le C\sum_{j\neq i}|w_j| \\ e^{-C\sum_{j\neq i}\int_{\tilde{C}_i}|w_j|dt} \le\ H_i \le\ & e^{C\sum\limits_{j\neq i}\int_{\tilde{C}_i}|w_j|dt} \\ e^{-C\tilde{W}_1(T)} \le H_i \le\ & e^{C\tilde{W}_1(T)} \\ e^{-CN_2} \le H_i \le\ & e^{CN_2} \end{aligned} \tag{39}$$

Noting Equation (38), we can rewrite it as

$$d(H_i|u_i|(dx - \lambda_i(u)dt)) = 0 \tag{40}$$

There are only the following cases like we estimate $\tilde{W}_1(T)$:

Case 1. We can integrate (40) in the region *APB* which is same to the case 1 of proof of $\tilde{W}_1(T)$ to get

$$\int_{\tilde{C}_j} H_i|u_i|(t, x)dx \le C\int_{t_0}^{t_1} H_i|u_i|(t, x_1(t))dt$$

Then

$$\int_{\tilde{C}_j} |u_i|(t, x)dx \le Ce^{CN_2}\int_0^{+\infty} |\phi(t)|dt \le CN_1 e^{CN_2} \tag{41}$$

Case 2. Integrating the Equation (40) in the region *PAOB* which is same to the case 2 of proof of $\tilde{W}_1(T)$, we can get

$$\begin{aligned} \int_{c_j} (\lambda_i(u) - \lambda_j(u))H_i|u_i|(t,\ x)dx = &\int_0^{t_0} (\lambda_1(u) - \lambda_i(u))H_i|u_i(t,\ x_1(t))|dt \\ &+ \int_0^{t_1} (\lambda_n(u) - \lambda_i(u))H_i\lambda_i(u)|u_i(t,\ x_n(t))|dt \end{aligned}$$

Using the above procedures, we can get

$$\int_{\tilde{C}_j} |u_i(t,x)|dx \leq Ce^{CN_2}\left(\int_0^{+\infty} |\phi(t)|dt + \int_0^{+\infty} |\psi(t)|dt\right) \leq CN_1 e^{CN_2} \tag{42}$$

Then

$$\tilde{U}_1(T) \leq CN_1 e^{CN_2} \tag{43}$$

In the similar way, substituting L_j for $\tilde{C}_j$ we can get

$$\begin{aligned} &\int_{L_j} |u_i(t,x)|dx \leq CN_1 e^{CN_2} \\ &\quad \bar{U}_1(T) \leq CN_1 e^{CN_2} \end{aligned} \tag{44}$$

Combining (24), (30), (31), (34), (37), (43), and (44) together we can obtain the conclusion of Lemma 2.1.

Proof of Theorem 1.1.

Noting the conclusion of Lemma 2.1, we can get

$$\tilde{W}_1(\infty), \bar{W}_1(\infty), W_1(\infty) \leq CN_2 \tag{45}$$

$$\tilde{U}_1(\infty), \bar{U}_1(\infty), \leq \; CN_1 e^{CN_2} \tag{46}$$

$$W_\infty(\infty) \leq CMe^{CN_2} \tag{47}$$

$$U_\infty(\infty) \leq C \tag{48}$$

Therefore, we can obtain that the system (1), (5), and (6) have global classical solutions on the domain $D = \{(t, x)| \; t \geq 0, x_1(t) \leq x \leq x_n(t)\}$.

3 Asymptotic behavior of global classical solutions

In this section, under the assumption of the leftmost and rightmost characteristics, we will study the asymptotic behavior of the global classical solutions of system (1), (5), and (6) and give the proof of Theorem 1.2.

Let

$$\frac{D}{D_i t} = \frac{\partial}{\partial t} + \lambda_i(0)\frac{\partial}{\partial x} \tag{49}$$

Obviously,

$$\frac{D}{D_i t} = \frac{d}{d_i t} + \left(\lambda_i(0) - \lambda_i(u)\right)\frac{\partial}{\partial x}$$

where $\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x}$. Thus, noting system (1)

$$\frac{Du_i}{D_i t} = \left(\lambda_i(0) - \lambda_i(u)\right)\frac{\partial u_i}{\partial x} \tag{50}$$

Using the Hadamard's Lemma, we can obtain

$$\frac{Du_i}{D_i t} = -\sum_{j\neq i}\{\Lambda_{ij}(u)u_j w_i\} \tag{51}$$

where $\Lambda_{ij}(u) = \int_0^1 \frac{\partial\lambda_i(su_1,\ldots,su_i-1,u_i,su_i+1,\ldots,su_n)}{\partial u_j}ds$.

For any fixed point $(t, x) \in D$, Passing through (t, x), we draw down the characteristic $x = x_r(t)$, which intersect with the characteristic boundary in the point $(x_r^{-1}(\alpha),\alpha)$. Then $\alpha = x - \lambda_i(0)(t - x_r^{-1}(\alpha))$ (where $r = 1$, when $i = m + 1, \ldots, n$ or $r = n$, when $i = 1, \ldots, m$).

Then, it follows from Equation (51) that

$$\begin{aligned} u_i(t,x) &= u_i\left(t,\alpha+\lambda_i(0)\left(t-x_r^{-1}(\alpha)\right)\right) \\ &= u_i(x_r^{-1}(\alpha),\alpha) - \int_{x_r^{-1}(\alpha)}^{t}\sum_{j\neq i}\left\{\Lambda_{ij}(u)u_j w_i(s,\alpha+\lambda_i(0)(s-x_r^{-1}(\alpha)))\right\}ds \end{aligned} \tag{52}$$

Noting (46) and (47), we can get

$$\left|\int_{x_r^{-1}(\alpha)}^{t}\sum_{j\neq i}\left\{\Lambda_{ij}(u)u_j w_i\right\}ds\right| \le CW_\infty(\infty)\bar{U}_1(\infty) \le CMN_1e^{CN_2} \tag{53}$$

This implies that the integral $\int_{x_r^{-1}(\alpha)}^{t}\sum_{j\neq i}\left\{\Lambda_{ij}(u)u_j w_i(s,\alpha+\lambda_i(0)(s-x_r^{-1}(\alpha)))\right\}ds$ converges uniformly for $\alpha \in R$. Then, there exists a unique function $\Phi_i(\alpha)$ such that,

$$u_i(t,x) \to \Phi_i(\alpha), \quad t \to +\infty \tag{54}$$

Moreover, noting (13) and Equations (52), (53), we have

$$|\Phi_i(\alpha)| \le C(1 + MN_1)e^{CN_2}$$

In what follows, we will study the regularity of $\Phi(\alpha)$. Noting Equation (52), we can get $\Phi(\alpha) \in C^0(R)$. From any fixed point $A(t,x) = (t, a + \lambda_i(0)(t - x_r^{-1}(\alpha)))$, we draw a characteristic $\tilde{C}_j$ intersecting the boundary $x = x_1(t)$ or $x = x_n(t)$ at $(x_r^{-1}(\theta_i(t,\alpha)),\theta_i(t,\alpha))$.

Then, integrating it along the ith characteristic, we obtain

$$\alpha + \lambda_i(0)\left(t - x_r^{-1}(\alpha)\right) = \theta(t,\alpha) + \int_{x_r^{-1}(\theta_i(t,\alpha))}^{t}\lambda_i\left(u(\tau,x_i(\tau,\theta_i(t,\alpha)))\right)d\tau \tag{55}$$

Then, we can get the following Lemma

Lemma 3.1 Under the assumptions of Theorem 1.1, for the $\theta_i(t, \alpha)$ defined above, there exists a unique $\vartheta_i(\alpha)$, such that

$$\lim_{t\to+\infty}\theta_i(t,\alpha) = \vartheta_i(\alpha). \tag{56}$$

Proof. Using the Hardarmad's formula, we can rewrite (55) as following

$$\theta_i(t,\ \alpha) = \alpha + \int_{x_r^{-1}(\theta_i(t,\alpha))}^{t} \Lambda_{ij} u_j(\tau,\ x_i(\tau,\ \theta_i(t,\ \alpha)))d\tau + \lambda_i(0)(x_r^{-1}(\theta_i(t,\ \alpha)) - x_r^{-1}(\alpha)) \tag{57}$$

where

$$\Lambda_{ij} = -\int_0^1 \frac{\partial\lambda_i(su_1, \ldots, su_{i-1}, u_i, su_{i+1}, \ldots, su_n)}{\partial u_j} ds.$$

are C^1 functions. Therefore

$$\theta_i(t,\alpha) - \lambda_i(0)x_r^{-1}(\theta_i(t,\alpha)) = \alpha + \int_{x_r^{-1}(\theta_i(t,\alpha))}^{t} \Lambda_{ij} u_j\left(\tau, x_i(\tau, \theta_i(t,\alpha))\right)d\tau - \lambda_i(0)x_r^{-1}(\alpha) \tag{58}$$

where r is either 1 or n. Then, we know that when t tends to ∞, the right hand of (58) convergence absolutely. For any given α, the right hand of (57) convergence to some function with respect to α. That implies that there exists a unique function $\vartheta(\alpha)$, such that

$$\lim_{t\to+\infty} \theta_i(t,\alpha) = \vartheta_i(\alpha) \tag{59}$$

Lemma 3.2 Under the assumptions of Theorem 1.1, for any given point $(x_r^{-1}(\alpha), \alpha)$ on the boundary, there exists a unique function $\Psi_i(\vartheta(\alpha)) \in C^0$, such that

$$\lim_{t\to+\infty} w_i\left(t, \alpha + \lambda_i(0)(t - x_r^{-1}(\alpha))\right) = \Psi_i(\vartheta_i(\alpha)). \tag{60}$$

uniformly for any $\alpha \lfloor\ R$.

Proof. Noting

$$w_i(t, \alpha + \lambda_i\,(0)\,(t - x_r^{-1}(\alpha))) = w_i(t, x_i(t, \theta_i(t,\alpha))). \tag{61}$$

In the following, we prove that there exists a unique $\Psi_i(\vartheta(\alpha)) \in C^0$, such that

$$\lim_{t\to+\infty} w_i(t, x_i(t, \theta_i(t,\alpha)))) = \Psi_i(> \vartheta_i(\alpha)). \tag{62}$$

Integrating (35) along the ith characteristic $\tilde{C}_j$ gives

$$\begin{aligned} w_i(t, x_i(t,\alpha))) &= w_i\left(x_r^{-1}(\theta_i(t,\alpha)), \theta_i(t,\alpha)\right) \\ &+ \int_{x_r^{-1}(\theta_i(t,\alpha))}^{t} \sum_{j\neq i} \gamma_{ij}(u) w_j w_i(\tau, x_i(\tau, \theta_i(t,\alpha)))d\tau \end{aligned} \tag{63}$$

where $\gamma_{ij}(u) = \frac{\partial\lambda_i(u)}{\partial u_j}$. Noting (45) and (47), we can get

$$\left| \int_{x_r^{-1}(\theta_i(t,\alpha))}^{t} \sum_{j\neq i} \gamma_{ij}(u) w_j w_i(\tau, x_i(\tau, \theta_i(t,\alpha)))d\tau \right| \leq CMN_2 e^{CN_2}$$

Then, as t tends to $+\infty$, the integrals in the right-hand side of (63) convergence, i.e., there exists a unique function $\Psi_i(\vartheta_i(\alpha))$, such that

$$\lim_{t\to\infty} w_i(t, x_i(t,\alpha))) = \Psi_i(\vartheta_i(\alpha)).$$

Lemma 3.3

$$\frac{d\Phi_i(\alpha)}{d\alpha} = \left(1 - \frac{\lambda_i(0)}{\lambda_i(x_1^{-1}(\alpha),\alpha)}\right)\Psi_i(\vartheta_i(\alpha)) \tag{64}$$

$$\frac{d\Phi_i(\alpha)}{d\alpha} = \left(1 - \frac{\lambda_i(0)}{\lambda_n(x_n^{-1}(\alpha),\alpha)}\right)\Psi_i(\vartheta_i(\alpha)) \tag{65}$$

Proof. For any fixed $\alpha \in R$, we calculate

$$\begin{aligned}
\frac{d\Phi_i(\alpha)}{d\alpha} &= \lim_{\Delta\alpha\to\infty} \frac{\Phi_i(\alpha+\Delta\alpha) - \Phi_i(\alpha)}{\Delta\alpha} \\
&= \lim_{\Delta\alpha\to 0}\lim_{t\to+\infty} \frac{u_i\left(t, \alpha+\Delta\alpha+\lambda_i(0)\left(t - x_r^{-1}(\alpha+\Delta\alpha)\right)\right) - u_i\left(t,\alpha+\lambda_i(0)\left(t-x_r^{-1}(\alpha)\right)\right)}{\Delta\alpha} \\
&= \lim_{t\to+\infty}\lim_{\Delta\alpha\to 0} \frac{u_i\left(t, \alpha+\Delta\alpha+\lambda_i(0)\left(t - x_r^{-1}(\alpha+\Delta\alpha)\right)\right) - u_i\left(t,\alpha+\lambda_i(0)\left(t-x_r^{-1}(\alpha)\right)\right)}{\Delta\alpha} \\
&= \lim_{t\to+\infty}\lim_{\Delta\alpha\to 0}\left[\frac{u_i\left(t,\alpha+\lambda_i(0)\left(t-x_r^{-1}(\alpha)\right)+\Delta\alpha-\lambda_i(0)\frac{x_r^{-1}(\alpha+\Delta\alpha)-x_r^{-1}(x)}{\Delta\alpha}.\Delta\alpha\right)}{\Delta\alpha}\right. \\
&\quad \left. - \frac{u_i\left(t,\alpha+\lambda_i(0)\left(t-x_r^{-1}(\alpha)\right)\right)}{\Delta\alpha}\right]
\end{aligned} \tag{66}$$

Thus, when $i = 1, ..., s$, i.e., $r = 1$, we can get

$$\begin{aligned}
\frac{d\Phi_i(\alpha)}{d\alpha} &= \left(1 - \frac{\lambda_i(0)}{\lambda_1\left(x_1^{-1}(\alpha),\alpha\right)}\right)\lim_{t\to+\infty} w_i\left(t,\alpha+\lambda_i(0)\left(t-x_1^{-1}(\alpha)\right)\right) \\
&= \left(1 - \frac{\lambda_i(0)}{\lambda_1\left(x_1^{-1}(\alpha),\alpha\right)}\right)\Psi_i(\vartheta_i(\alpha)).
\end{aligned} \tag{67}$$

Using the similar procedure, when $\alpha > 0$, i.e., $i = s + 1, ..., n$ and $r = n$, we can get

$$\frac{d\Phi_i(\alpha)}{d\alpha} = \left(1 - \frac{\lambda_i(0)}{\lambda_n\left(x_n^{-1}(\alpha),\alpha\right)}\right)\Psi_i(\vartheta_i(\alpha)). \tag{68}$$

Noting the above lemmas, we get the conclusion of Theorem 1.2.

4 Applications

Recent years observations of the luminosity of type Ia distant supernovae point towards an accelerated expansion of the universe, which implies that the pressure p and the energy density ρ of the universe should violate the strong energy condition, i.e., $\rho + 3p < 0$. Here, we consider a recently proposed class of simple cosmological models based on the use of peculiar perfect fluids [19]. In the simplest case, we study the model of a universe filled with the so called Chaplygin gas, which is a perfect fluid characterized by the following equation of state $p = -\frac{A}{\rho} = -A\tau$, where A is a positive constant.

In Lagrange coordinate, the 1D gas dynamics equations in isentropic case can be written as

$$\begin{cases} \tau_t - u_x = 0 \\ u_t + p(\tau)_x = 0 \end{cases} \tag{69}$$

Noting (69), in isentropic case we can get the system of one dimensional Chaplygin gas model

$$\begin{cases} \tau_t - u_x = 0 \\ u_t - A\tau_x = 0 \end{cases} \tag{70}$$

Nothing systems (70) is linear systems, it is easy to get the eigenvalues

$$\lambda_+ = \sqrt{A}, \quad \lambda_- = -\sqrt{A}. \tag{71}$$

and left eigenvectors

$$l_+ = \left(\sqrt{A},\ 1\right), \quad l_- = \left(-\sqrt{A},\ 1\right) \tag{72}$$

Introduce the Riemann invariants

$$v_1 = u + \sqrt{A}\,\tau, \quad v_2 = u - \sqrt{A}\tau \tag{73}$$

we can rewrite the system as following

$$\begin{cases} \frac{\partial v_1}{\partial t} - \sqrt{A}\frac{\partial v_1}{\partial x} = 0 \\ \frac{\partial v_2}{\partial t} - \sqrt{A}\frac{\partial v_2}{\partial x} = 0 \end{cases} \tag{74}$$

Consider the Goursat problem for system (69) with following characteristic boundary conditions:

$$x = -\sqrt{A}t : \tau = \tau_-(t), u = u_-(t); \quad x = \sqrt{A}t : \tau = \tau_+(t), u = u_+(t) \tag{75}$$

Then, the above system satisfies the assumptions of Theorems 1.1 and 1.2. More precisely, we can get the following theorems:

Theorem 4.1. The Goursat problem (69) and (75) admits a unique global C^1 solution (τ, u) (t, x) on the domain $D = \left\{(t, x) \mid t \geq 0, -\sqrt{A}t \leq x \leq \sqrt{A}t\right\}$.

Theorem 4.2. There exists a unique piece-wised C^1 vector-value function $\Phi(x) = (\Phi_1(x), ..., \Phi_n(x))^T$ such that

$$(\tau, u)\ (t, x) \to \sum_{i=1}^{n} \Phi_i \left(x - \sqrt{A}t\right) e_i, \qquad t \to +\infty \tag{76}$$

uniformly as t tends to infinity, where $e_i = (0, ..., 0, 1^i, 0, ..., 0)^T$.

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Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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