Boujary and Afrouzi *Boundary Value Problems* 2012, **2012**:44
http://www.boundaryvalueproblems.com/content/2012/1/44

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

A note on some nonlinear principal eigenvalue problems

Mohsen Boujary[1*] and Ghasem Alizadeh Afrouzi[2]

* Correspondence: m.bouujary@gmail.com
[1]Department of Mathematics, Science and Research Branch, Islamic, Azad University (Iau), Tehran, Iran
Full list of author information is available at the end of the article

Abstract

We investigate the existence of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) the boundary value problem

$$\begin{cases} -\Delta_p u(x) = \lambda g(x)|u(x)|^{p-2}u(x), & \text{in } \Omega, \\ \text{Ru} = |\nabla u(x)|^{p-2}\dfrac{\partial u}{\partial \nu}(x) + \alpha|u(x)|^{p-2}u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $1 < p < \infty$ and α is a real number.
AMS Subject Classification: 35J60; 35B30; 35B40.

Keywords: *p*-Laplacian, principal eigenvalue

1. Introduction

Mathematical models described by nonlinear partial differential equations have become more common recently. In particular, the *p*-Laplacian operator appears in subjects such as filtration problem, power-low materials, non-Newtonian fluids, reaction-diffusion problems, nonlinear elasticity, petroleum extraction, etc., see,[1]. The nonlinear boundary condition describes the flux through the boundary $\partial\Omega$ which depends on the solution itself.

The purpose of this study is to discuss the existence of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem

$$\begin{cases} -\Delta_p u(x) = \lambda g(x)|u(x)|^{p-2}u(x), & \text{in } \Omega, \\ \text{Ru} = |\nabla u(x)|^{p-2}\dfrac{\partial u}{\partial \nu}(x) + \alpha|u(x)|^{p-2}u(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $1 < p < \infty$ and α is a real number. Attention has been confined mainly to the cases of Dirichlet and Neumann boundary conditions but we have the Robin boundary in (1.1).

We discuss about to exist principal eigenvalue for (1.1). In the case $0 < \alpha < \infty$, We shall show that there has exactly two principal eigenvalues, one positive and one negative.

2. Main result

Our analysis is based on a method used by Afrouzi and Brown [2]. Consider, for fixed λ, the eigenvalue problem

$$\begin{cases} -\Delta_p u(x) - \lambda g(x)\left|u(x)\right|^{p-2}u(x), = \mu\left|u(x)\right|^{p-2}u(x), \text{ in } \Omega, \\ \mathrm{Ru} = \left|\nabla u(x)\right|^{p-2}\dfrac{\partial u}{\partial \nu}(x) + \alpha\left|u(x)\right|^{p-2}u(x) = 0, \qquad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We denote the lowest eigenvalue of (2.1) by $\mu(\alpha, \lambda)$. Let

$$S_{\alpha,\lambda} = \left\{ \int_{\Omega} |\nabla\phi|^p dx + \alpha \int_{\partial\Omega} |\phi|^p dS_x - \lambda \int_{\Omega} g|\phi|^p dx : \phi \in W^{1,p}(\Omega), \int_{\Omega} |\phi|^p = 1 \right\}$$

When $\alpha \geq 0$, it is clear that $S_{\alpha,\lambda}$ is bounded below. It is shown by variational arguments that $\mu(\alpha, \lambda) = \inf S_{\alpha,\lambda}$ and that an eigenfunction corresponding to $\mu(\alpha, \lambda)$ does not change sign on Ω [3]. Thus, clearly, λ is a principal eigenvalue of (1.1) if and only if $\mu(\alpha, \lambda) = 0$.

When $\alpha < 0$, the boundedness below of $S_{\alpha,\lambda}$ is not obvious, but is a consequence of the following lemma.

Lemma 2.1. *For every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that*

$$\int_{\partial\Omega} |\phi|^p dS_x \leq \varepsilon \int_{\Omega} |\nabla\phi|^p dx + C(\varepsilon) \int_{\Omega} |\phi|^p dx$$

for all $\varphi \in W^{1,p}(\Omega)$.

Proof. Suppose that the result does not hold. Then $\varepsilon_0 > 0$ and sequence $\{u_n\} \subseteq W^{1,p}(\Omega)$ such that $\int_\Omega |\nabla u_n|^p = 1$ and

$$\int_{\partial\Omega} |u_n|^p dS_x \geq \varepsilon_0 + n \int_{\Omega} |u_n|^p dx. \tag{2.2}$$

Suppose first that $\{\int_\Omega |u_n|^p dx\}$ is unbounded. Let $v_n = \dfrac{u_n}{\|u_n\|_{L^p(\Omega)}}$. Clearly, $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$, and so in $L^p(\partial\Omega)$. But $\int_{\partial\Omega} |v_n|^p dS_x \geq n \int_\Omega |v_n|^p dx = n$, which is impossible.

Suppose now that $\{\int_\Omega |u_n|^p dx\}$ is bounded, then $\{u_n\}$ is bounded in $W^{1,p}$ and so has a subsequence, which we again denote by $\{u_n\}$, converging weakly to u in $W^{1,p}$. Since $W^{1,p}$is compactly embedded in $L^p(\partial\Omega)$ and in $L^p(\Omega)$, it follows that $\{u_n\}$ converges to some function u in $L^p(\partial\Omega)$ and in $L^p(\Omega)$. Thus $\{\int_{\partial\Omega} |u_n|^p dx\}$ is bounded, and so it follows from (2.2) that $\lim_{n\to\infty} \int\Omega |u_n|^p dx = 0$, i.e.,, $\{u_n\}$ converges to zero in $L^p(\Omega)$. Hence $\{u_n\}$ converges to zero in $L^p(\partial\Omega)$, and this is impossible because (2.2).

Choosing $\varepsilon < \frac{1}{\alpha}$, it is easy to deduce from the above result the $S_{\alpha,\lambda}$ is bounded below, and it follows exactly as in [3] that $\mu(\alpha, \lambda) = \inf S_{\alpha,\lambda}$ and that an eigenfunction corresponding to $\mu(\alpha, \lambda)$ does not change sign on Ω. Thus it is again λ is a principal eigenvalue of (1.1) if and only if $\mu(\alpha, \lambda) = 0$.

For fixed $\varphi \in W^{1,p}(\Omega)$, $\lambda \to \int_\Omega |\nabla\varphi|^p dx + \alpha \int_{\partial\Omega} |\varphi|^p dS_x - \lambda \int_\Omega g|\varphi|^p dx$ is an affine and so concave function. As the infimum of any collection of concave functions is concave, it follows that $\lambda \to \mu(\alpha, \lambda)$ is concave. Also, by considering test functions $\varphi_1, \varphi_2 \in W^{1,p}(\Omega)$ such that $\int_\Omega g|\varphi_1|^p dx > 0$ and $\int_\Omega g|\varphi_2|^p dx < 0$, it is easy to see that $\mu(\alpha, \lambda) \to -\infty$ as $\lambda \to \pm\infty$. Thus $\lambda \to \mu(\alpha, \lambda)$ is an increasing function until it attains its maximum, and is an decreasing function thereafter.

It is natural that the flux across the boundary should be outwards if there is a positive concentration at the boundary. This motivates the fact that the sign of $\alpha > 0$. For a physical motivation of such conditions, see for example [4]. Suppose that $0 < \alpha < \infty$, i.e., we have the Robin boundary condition. Then, as can be seen from the the variational characterization of $\mu(\alpha, \lambda)$ or $-\Delta_p$ has a positive principal eigenvalue, $\mu(\alpha, 0) > 0$ and so $\lambda \to \mu(\alpha, \lambda)$ must has exactly two zero. Thus in this case (1.1) exactly two principal eigenvalues, one positive and one negative.

Our results may be summarized in the following theorem.

Theorem 2.2. *If* $0 < \alpha < \infty$, *then (1.1) exactly two principal eigenvalues, one positive and one negative.*

However, for $\alpha < 0$ we have $\mu(\alpha, 0) \le 0$. For $p = 2$, if u_0 is eigenfunction of (2.1) corresponding to principal eigenvalue $\mu(\alpha, \lambda)$, then

$$\frac{d\mu}{d\lambda}(\alpha, \lambda) = -\frac{\int_\Omega g u_0^2 dx}{\int_\Omega u_0^2 dx}. \tag{2.3}$$

Therefore, $\lambda \to \mu(\alpha, \lambda)$ is an increasing (decreasing) function, if we have $\frac{\int_\Omega g u_0^2 dx}{\int_\Omega u_0^2 dx} < 0 (> 0)$ and at critical points we must have $\frac{\int_\Omega g u_0^2 dx}{\int_\Omega u_0^2 dx} = 0$ (see, [2], Lemma 2]).

But, we cannot generalize it for $p \ne 2$. Because, if $v(\lambda) = \frac{d\mu}{d\lambda}$, then we have

$$-\frac{d}{d\lambda}\Delta_p u(\lambda) = -(p-1) div(\nabla v |\nabla u|^{p-2}).$$

So, we cannot get a similar result (2.3).

Now our analysis is based by Drabek and Schindler [5]. We define the space V_p as completion of $W^{1,p}(\Omega) \cap C(\bar{\Omega})$ with respect to the norm

$$\|u\|_{V_p} = \left(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} |u|^p ds \right)^{\frac{1}{p}}. \tag{2.4}$$

The spaces equivalent to V_p were introduced in [6]. In particular, V_p is a uniformly convex (and hence a reflexive) Banach space, $V_p \hookrightarrow L^q(\Omega)$ continuously for $1 \le q \le \frac{Np}{N-1}$ and $V_p \hookrightarrow L^q(\Omega)$ compactly for $1 \le q \le \frac{Np}{N-1}$[6].

We say that $u \,\llcorner\, V_p$ is a weak solution to (1.1) if for all $\varphi \,\llcorner\, V_p$ we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u.\nabla \phi dx + \alpha \int_{\partial\Omega} |u|^{p-2} u\phi ds = \int_\Omega \lambda g(x) |u|^{p-2} u \phi dx. \tag{2.5}$$

In fact there are domains Ω for which the embedding $V_p \hookrightarrow L^p(\Omega)$ is not injective. This is to the influence of the wildness of the boundary $\partial\Omega$. The domains for which the above embedding is injective are then called admissible. Ω is called admissible irregular domain for which $W^{1,p}(\Omega)$ is not subset $L^q(\Omega)$ for all $p > q$.

We assume that the domain $\Omega \subset \mathbb{R}^N$ is bounded, $N > 1$, $\alpha > 0$, and $1 < p < N$. We apply variational for (1.1) with $\lambda = 1$. We introduce the C^1-functionals

$$l(u) = \int_\Omega |\nabla u|^p + \alpha \int_{\partial\Omega} |u|^p ds. \tag{2.6}$$

and

$$j(u) = \int_{\Omega} g(x)|u|^{p}. \tag{2.7}$$

If $w \in V_p$ be a global minimizer of l subject to the constraint $j(w) = 1$, then the Lagrange multiplier method yields a $\lambda \in \mathbb{R}$ such that $l'(u) = \lambda j'(u)$, i.e.,

$$p\int_{\Omega} |\nabla w|^{p-2}\nabla w.\nabla\phi dx + p\alpha\int_{\partial\Omega} |w|^{p-2}w\phi ds = \lambda p\int_{\Omega} g(x)|u|^{p-2}u\phi dx$$

holds for any $\varphi \in V_p$. Then w is a weak solution (1.1). The existence of a minimizer follows from the fact that $l(u)$ is bounded from below on the manifold $M = \{u \in V_p : j(u) = 1\}$ and from Palais-Smale condition satisfied by the functional l on M.

Author details
[1]Department of Mathematics, Science and Research Branch, Islamic, Azad University (Iau), Tehran, Iran [2]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

Authors' contributions
Boujary has presented the main purpose of the article and has used Afrouzi contribution due to reaching to conclusions. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 29 November 2011 Accepted: 16 April 2012 Published: 16 April 2012

References
1. Diaz, JI: Nonlinear Partial Differential Equatians and Free Boundaries. Elliptic Equations, LondonI (1985)
2. Afrouzi, GA, Brown, KJ: On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions. Proc Am Math Soc. **127**(1):125–130 (1999)
3. Smoller, J: Shock Waves and Reaction-Diffusion Equations. Springer-Verlag, Berlin (1983)
4. Pao, CV: Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York, London (1992)
5. Drabek, P, Schindler, I: Positive solutions for *p*-Laplacian with Robin boundary conditions on irregular domains. Appl Math Lett. **24**, 588–591 (2011)
6. Maz'ja, VG: Sobolev Spaces in: Springer Series in Soviet Mathematics (translated from the Russian by Shaposhnikova, TO). Springer-Verlag, Berlin (1985)

doi:10.1186/1687-2770-2012-44
Cite this article as: Boujary and Afrouzi: **A note on some nonlinear principal eigenvalue problems.** *Boundary Value Problems* 2012 **2012**:44.