Zhang and Liu *Boundary Value Problems* 2012, **2012**:53
http://www.boundaryvalueproblems.com/content/2012/1/53

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence of nontrivial solutions to perturbed p-Laplacian system in $\mathbb{R}^N$ involving critical nonlinearity

Huixing Zhang* and Wenbin Liu

* Correspondence:
zhx20110906@cumt.edu.cn
Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, People's Republic of China

Abstract

We consider a p-Laplacian system with critical nonlinearity in $\mathbb{R}^N$. Under the proper assumptions, we obtain the existence of nontrivial solutions to perturbed p-Laplacian system by using the variational approach.

MR Subject Classification: 35B33; 35J60; 35J65.

Keywords: p-Laplacian system, critical nonlinearity, variational methods.

1 Introduction

This article is concerned with the existence of solutions to the following nonlinear perturbed p-Laplacian system

$$\begin{cases} -\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = K(x)|u|^{p^*-2}u + H_u(u,v), & x \in \mathbb{R}^N, \\ -\varepsilon^p \Delta_p v + V(x)|v|^{p-2}v = K(x)|v|^{p^*-2}v + H_v(u,v), & x \in \mathbb{R}^N, \\ u(x), \quad v(x) > 0, \\ u(x), \quad v(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$ and $p^* = Np/(N - p)$ is the critical exponent.

Throughout the article, we will assume that:

(V_0) $V \in C(\mathbb{R}^N)$, $V(0) = \inf V(x) = 0$ and there exists $b > 0$ such that the set $v^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;
(K_0) $K(x) \in C(\mathbb{R}^N)$, $0 < \inf K \le \sup K < \infty$;
(H_1) $H \in C^1(\mathbb{R}^2)$ and $H_s, H_t = o(|s|^{p-1} + |t|^{p-1})$ as $|s| + |t| \to 0$;
(H_2) there exist $c > 0$ and $p < q < p^*$ such that

$$|H_s(s,t)|, \ |H_t(s,t)| \le c(1 + |s|^{q-1} + |t|^{q-1});$$

(H_3) There are $a_0 > 0$, $\theta \in (p, p^*)$ and $\alpha, \beta > p$ such that $H(s,t) \ge a_0(|s|^\alpha + |t|^\beta)$ and $0 < \theta H(s,t) \le sH_s + tH_t$.

Under the above mentioned conditions, we will get the following result.

Theorem 1. If (V_0), (K_0) and (H_1)-(H_3) hold, then for any $\sigma > 0$, there is $\varepsilon_\sigma > 0$ such that if $\varepsilon < \varepsilon_\sigma$, the problem (1.1) has at least one positive solution $(u_\varepsilon, v_\varepsilon)$ which satisfy

Springer

© 2012 Zhang and Liu; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

$$\frac{\theta - p}{p\theta} \int_{\mathbb{R}^N} \left(\varepsilon^p |\nabla u_\varepsilon|^p + \varepsilon^p |\nabla v_\varepsilon|^p + V(x)|u_\varepsilon|^p + V(x)|v_\varepsilon|^p\right) \leq \sigma \varepsilon^N.$$

The scalar form of the problem (1.1) is as follows

$$-\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = K(x)|u|^{p^*-2}u + h(x,\ u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

The Equation (1.2) has been studied in many articles. The case $p = 2$ was investigated extensively under various hypotheses on the potential and the nonlinearity by many authors including Brézis and Nirenberg [1], Ambrosetti [2] and Guedda and Veron [3] (see also their references) in bounded domains. As far as unbounded domains are concerned, we recall the work by Benci and Cerami [4], Floer and Weistein [5], Oh [6], Clapp [7], Del Pino and Felmer [8], Cingolani and Lazzo [9], Ding and Lin [10]. Especially, in [10], the authors studied the Equation (1.2) in the case $p = 2$. In that article, they made the following assumptions:

(A_1) $V \in C(\mathbb{R}^N)$, min $V = 0$ and there is $b > 0$ such that the set $v^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;
(A_2) $K(x) \in C(\mathbb{R}^N)$, $0 < \inf K \leq \sup K < \infty$
(B_1) $h \in C(\mathbb{R}^N \times \mathbb{R})$ and $h(x, u) = o(|u|)$ uniformly in x as $|u| \to 0$;
(B_2) there are $c_0 > 0$, $q < 2^*$ such that $|h(x, u)| \leq c_0(1 + |u|^{q-1})$ for all (x, u);
(B_3) there are $a_0 > 0$, $p > 2$ and $\mu > 2$ such that $H(x, u) = a_0|u|^p$ and $\mu H(x, u) \leq h(x, u)u$ for all (x, u), where $H(x,u) = \int_0^u h(x,s)ds$.

That article obtained the existence of at least one positive solution u_ε of least energy if the assumptions (A_1)-(A_2) and (B_1)- (B_3) hold.

For the Equation (1.2) in the case $p \neq 2$, we recall some works. Garcia Azorero and Peral Alonso [11] considered (1.2) with $\varepsilon \leq 1$, $V(x) = \mu$, $K(x) = 1$, $h(x, u) = 0$ and proved that (1.2) has a solution if $p^2 \leq N$ and $\mu \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of the p-Laplacian. In [12], Alves and Ding studied the same problem of [11] and obtained the multiplicity of positive solutions in bounded domain $\Omega \subset \mathbb{R}^N$. Moreover, Liu and Zheng [13] investigated (1.2) in $\mathbb{R}^N$ with $\varepsilon = 1$ and $K(x) = 0$. Under the sign-changing potential and subcritical p-superlinear nonlinearity, the authors got the existence result.

Motivated by some results found in [10,11,13], a natural question arises whether existence of nontrivial solutions continues to hold for the p-Laplacian system with the critical nonlinearity in $\mathbb{R}^N$.

The main difficulty in the case above mentioned is the lack of compactness of the energy functional associated to the system (1.1) because of unbounded domain $\mathbb{R}^N$ and critical nonlinearity. To overcome this difficulty, we make careful estimates and prove that there is a Palais-Smale sequence that has a strongly convergent sequence. The method or idea here is similar to the one of [10]. We can prove that the functional associated to (1.1) possesses $(PS)_c$ condition at some energy level c. Furthermore, we prove the existence result by using the mountain pass theorem due to Rabinowitz [14].

The main result in the present article concentrates on the existence of positive solutions to the system (1.1) and can be seen as a complement of the results developed in [10,11,13].

This article is organized as follows. In Section 2, we give the necessary notations and preliminaries. Section 3 is devoted to the behavior of $(PS)_c$ sequence and the mountain geometry structure. Finally, in Section 4, we prove the existence of nontrivial solution.

2 Notations and preliminaries

Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and $D^{1,p}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ under

$$||u||^p = \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

We introduce the space

$$E(\mathbb{R}^N,\ V) = \{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^p < \infty\}$$

equipped with the norm

$$||u||_E = \left(\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)\right)^{\frac{1}{p}}$$

and the space

$$E_\lambda(\mathbb{R}^N,\ V) = \left\{u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x)|u|^p < \infty, \quad \lambda > 0\right\}$$

under

$$||u||_\lambda = \left(\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V(x)|u|^p)\right)^{\frac{1}{p}}.$$

Observe that $\|\cdot\|_E$ is equivalent to the one $\|\cdot\|_\lambda$ for each λ >0. It follows from (V_0) that $E(\mathbb{R}^N, V)$ continuously embeds in $W^{1,p}(\mathbb{R}^N)$.

Set $B = E_\lambda \times E_\lambda$ and $||(u,v)||_\lambda = ||u||_\lambda^p + ||v||_\lambda^p$ for any $(u, v) \in B$. Let $\lambda = \varepsilon^{-p}$ in the system (1.1), then (1.1) is changed into

$$\begin{cases} -\Delta_p u + \lambda V(x)|u|^{p-2}u = \lambda K(x)|u|^{p^*-2}u + \lambda H_u(u,v), & \in \mathbb{R}^N, \\ -\Delta_p v + \lambda V(x)|v|^{p-2}v = \lambda K(x)|v|^{p^*-2}v + \lambda H_v(u,v), & x \in \mathbb{R}^N, \\ u(x), \quad v(x) > 0, \\ u(x), \quad v(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{2.1}$$

In order to prove Theorem 1, we only need to prove the following result.

Theorem 2. Let (V_0), (K_0) and (H_1)-(H_3) be satisfied. Then for any σ >0, there exists Λ_σ >0 such that if $\lambda \geq \Lambda_\sigma$, the system (2.1) has at least one least energy solution (u_λ, v_λ) satisfying

$$\frac{\theta - p}{p\theta} \int_{\mathbb{R}^N} (|\nabla u_\lambda|^p + |\nabla v_\lambda|^p + \lambda V(x)(|u_\lambda|^p + |v_\lambda|^p)) \leq \sigma \lambda^{1-\frac{N}{p}}. \tag{2.2}$$

The energy functional associated with (2.1) is defined by

$$\begin{aligned}I_\lambda(u,\ v) &= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + \lambda V(x)|u|^p + |\nabla v|^p + \lambda V(x)|v|^p)\\ &\quad - \frac{\lambda}{p^*}\int_{\mathbb{R}^N} K(x)(|u|^{p^*} + |v|^{p^*}) - \lambda\int_{\mathbb{R}^N} H(u,\ v)\\ &= \frac{1}{p}||(u,\ v)||_\lambda^p - \lambda\int_{\mathbb{R}^N} G(u,\ v),\end{aligned}$$

where $G(u,v) = \frac{1}{p^*}K(x)(|u|^{p*} + |v|^{p*}) + H(u,v)$.

From the assumptions of Theorem 2, standard arguments [14] show that $I_\lambda \in C^1(B, \mathbb{R})$ and its critical points are the weak solutions of (2.1).

3 Technical lemmas

In this section, we will recall and prove some lemmas which are crucial in the proof of the main result.

Lemma 3.1. Let the assumptions of Theorem 2 be satisfied. If the sequence $\{(u_n, v_n)\} \subset B$ is a $(PS)_c$ sequence for I_λ, then we get that $c \geq 0$ and $\{(u_n, v_n)\}$ is bounded in the space B.

Proof. One has

$$\begin{aligned}&I_\lambda(u_n, v_n) - \frac{1}{\theta}I'_\lambda(u_n, v_n)(u_n, v_n)\\ &= \frac{1}{p}||(u_n,\ v_n)||_\lambda^p - \frac{\lambda}{p^*}\int_{\mathbb{R}^N} K(x)(|u_n|^{p^*} + |v_n|^{p^*}) - \lambda\int_{\mathbb{R}^N} H(u_n,\ v_n)\\ &\quad - \frac{1}{\theta}\left[||(u_n,\ v_n)||_\lambda^p - \lambda\int_{\mathbb{R}^N} K(x)(|u_n|^{p^*} + |v_n|^{p^*}) - \lambda\int_{\mathbb{R}^N}(u_nH_s(u_n,\ v_n) + v_nH_t(u_n,\ v_n))\right]\\ &= \left(\frac{1}{p} - \frac{1}{\theta}\right)||(u_n,\ v_n)||_\lambda^p + \left(\frac{1}{\theta} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N} K(x)(|u_n|^{p^*} + |v_n|^{p^*})\\ &\quad + \lambda\int_{\mathbb{R}^N}\left(\frac{1}{\theta}(u_nH_s(u_n,\ v_n) + v_nH_t(u_n,\ v_n)) - H(u_n,\ v_n)\right)\end{aligned}$$

By the assumptions (K_0) and (H_3), we have

$$I_\lambda(u_n,\ v_n) - \frac{1}{\theta}I'_\lambda(u_n,\ v_n)(u_n,\ v_n) \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)||(u_n,\ v_n)||_\lambda^p.$$

Together with $I_\lambda(u_n, v_n) \to c$ and $I'_\lambda(u_n, v_n) \to 0$ as $n \to \infty$, we easily obtain that the $(PS)_c$ sequence is bounded in B and the energy level $c \geq 0$. □

From Lemma 3.1, there exists $(u, v) \in B$ such that $(u_n, v_n) \rightharpoonup (u, v)$ in B. Furthermore, passing to a subsequence, we have $u_n \to u$ and $v_n \to v$ in $L^d_{loc}(\mathbb{R}^N)$ for any $d \in [p, p^*)$ and $u_n \to u$, $v_n \to v$ *a.e.* in $\mathbb{R}^N$.

Lemma 3.2. Let $d \in [p, p^*)$. There exists a subsequence $\{(u_{n_j}, v_{n_j})\}$ such that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ with

$$\limsup_{i\to\infty}\int_{B_i\setminus B_r}(|u_{n_i}|^d + |v_{n_i}|^d) \leq \varepsilon$$

for any $r \geq r_\varepsilon$, where $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$.

Proof. The proof of Lemma 3.2 is similar to the one of Lemma 3.2 of [10], so we omit it. □

Let $\eta \in C^{\infty}(\mathbb{R}^{+})$ be a smooth function satisfying $0 \le \eta(t) \le 1$, $\eta(t) = 1$ if $t \le 1$ and $\eta(t) = 0$ if $t \ge 2$. Define $\tilde{u}_j(x) = \eta(2|x|/j)u(x)$, $\tilde{v}_j(x) = \eta(2|x|/j)v(x)$. It is obvious that

$$||u - \tilde{u}_j||_{\lambda} \to 0 \text{ and} ||v - \tilde{v}_j||_{\lambda} \to 0 \text{ as } j \to \infty. \tag{3.1}$$

Lemma 3.3. One has

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} (H_s(u_{n_j}, v_{n_j}) - H_s(u_{n_j} - \tilde{u}_j, v_{n_j} - \tilde{v}_j) - H_s(\tilde{u}_j, \tilde{v}_j))\varphi = 0$$

and

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} (H_t(u_{n_j}, v_{n_j}) - H_t(u_{n_j} - \tilde{u}_j, v_{n_j} - v_j) - H_t(\tilde{u}_j, \tilde{v}_j))\psi = 0$$

uniformly in $(\phi, \psi) \in B$ with $\|(\phi, \psi\|_B \le 1$.

Proof. From the assumptions (H_1)-(H_2) and Lemma 3.2, we have

$$\begin{aligned}
&\limsup_{j\to\infty} \int_{\mathbb{R}^N} (H_s(u_{n_j}, v_{n_j}) - H_s(u_{n_j} - \tilde{u}_j, v_{n_j} - \tilde{v}_j) - H_s(\tilde{u}_j, \tilde{v}_j))\varphi \\
&= \limsup_{j\to\infty} \int_{B_j} (H_s(u_{n_j}, v_{n_j}) - H_s(u_{n_j} - \tilde{u}_j, v_{n_j} - \tilde{v}_j) - H_s(\tilde{u}_j, \tilde{v}_j))\varphi \\
&= \limsup_{j\to\infty} \int_{B_j\setminus B_r} (H_s(u_{n_j}, v_{n_j}) - H_s(u_{n_j} - \tilde{u}_j, v_{n_j} - \tilde{v}_j) - H_s(\tilde{u}_j, \tilde{v}_j))\varphi \\
&\le c \limsup_{j\to\infty} \int_{B_j\setminus B_r} (|u_{n_j}|^{p-1} + |v_{n_j}|^{p-1} + |u_{n_j}|^{q-1} + |v_{n_j}|^{q-1} + |\tilde{u}_j|^{p-1} + |\tilde{v}_j|^{p-1} \\
&\quad + |\tilde{u}_j|^{q-1} + |\tilde{v}_j|^{q-1} + |u_{n_j} - \tilde{u}_j|^{p-1} + |v_{n_j} - \tilde{v}_j|^{p-1} + |u_{n_j} - \tilde{u}_j|^{q-1} + |v_{n_j} - \tilde{v}_j|^{q-1})\varphi \\
&\le c_1 \limsup_{j\to\infty} \int_{B_j\setminus B_r} (|u_{n_j}|^{p-1} + |v_{n_j}|^{p-1} + |\tilde{u}_j|^{p-1} + |\tilde{v}_j|^{p-1})\varphi \\
&\quad + c_2 \limsup_{j\to\infty} \int_{B_j\setminus B_r} (|u_{n_j}|^{q-1} + |v_{n_j}|^{q-1} + |\tilde{u}_j|^{q-1} + |\tilde{v}_j|^{q-1})\varphi
\end{aligned} \tag{3.2}$$

By Hölder inequality and Lemma 3.2, it follows that

$$\begin{aligned}
\limsup_{j\to\infty} \int_{B_j\setminus B_r} |u_{n_j}|^{p-1}|\varphi| &\le \limsup_{j\to\infty} \left(\int_{B_j\setminus B_r} |u_{n_j}|^p\right)^{\frac{p-1}{p}} \left(\int_{B_j\setminus B_r} |\varphi|^p\right)^{\frac{1}{p}} \\
&\le \limsup_{j\to\infty} \left(\int_{B_j\setminus B_r} |u_{n_j}|^p\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |\varphi|^p\right)^{\frac{1}{p}} \\
&\le \limsup_{j\to\infty} \left(\int_{B_j\setminus B_r} |u_{n_j}|^p\right)^{\frac{p-1}{p}} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}\limsup_{j\to\infty}\int_{B_j\setminus B_r}|u_{n_j}|^{p-1}|\varphi| &\le \limsup_{j\to\infty}\left(\int_{B_j\setminus B_r}|u_{n_j}|^{p}\right)^{\frac{q-1}{p}}\left(\int_{B_j\setminus B_r}|\varphi|^{q}\right)^{\frac{1}{q}}\\ &\le \limsup_{j\to\infty}\left(\int_{B_j\setminus B_r}|u_{n_j}|^{q}\right)^{\frac{q-1}{q}}\left(\int_{\mathbb{R}^N}|\varphi|^{q}\right)^{\frac{1}{q}}\\ &\le \limsup_{j\to\infty}\left(\int_{B_j\setminus B_r}|u_{n_j}|^{q}\right)^{\frac{q-1}{q}}\\ &= 0\end{aligned}$$

Similarly, we get

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}(|v_{n_j}|^{p-1}|+|\tilde{u}_j|^{p-1}+|\tilde{v}_j|^{p-1})\varphi = 0$$

and

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}(|v_{n_j}|^{q-1}|+|\tilde{u}_j|^{q-1}+|\tilde{v}_j|^{q-1})\varphi = 0.$$

Thus

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}(H_s(u_{n_j},\ v_{n_j})-H_s(u_{n_j}-\tilde{u}_j,\ v_{n_j}-\tilde{v}_j)-H_s(\tilde{u}_j,\ \tilde{v}_j))\varphi = 0.$$

From the similar argument, we also get

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}(H_t(u_{n_j},\ v_{n_j})-H_t(u_{n_j}-\tilde{u}_j,\ v_{n_j}-\tilde{v}_j)-H_t(\tilde{u}_j,\ \tilde{v}'_j))\psi = 0.$$

□

Lemma 3.4. One has along a subsequence

$$I_\lambda(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)\to c-I_\lambda(u,\ v)$$

and

$$I'_\lambda(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)\to 0 \text{ in } B^{-1}(\text{the dual space of } B).$$

Proof. From the Lemma 2.1 of [15] and the argument of [16], we have

$$\begin{aligned}&I_\lambda(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)\\ &=\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u_n-\nabla\tilde{u}_n|^p+\lambda V(x)|u_n-\tilde{u}_n|^p+|\nabla v_n-\nabla\tilde{v}_n|^p+\lambda V(x)|v_n-\tilde{v}_n|^p)\\ &\quad-\frac{\lambda}{p^*}\int_{\mathbb{R}^N}K(x)(|u_n-\tilde{u}_n|^{p^*}+|v_n-\tilde{v}_n|^{p^*})-\lambda\int_{\mathbb{R}^N}H(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)\\ &=I_\lambda(u_n,\ v_n)-I_\lambda(\tilde{u}_n,\ \tilde{v}_n)\\ &\quad+\frac{\lambda}{p^*}\int_{\mathbb{R}^N}K(x)((|u_n|^{p^*}-|u_n-\tilde{u}_n|^{p^*}-|\tilde{u}_n|^{p^*})+(|v_n|^{p^*}-|v_n-\tilde{v}_n|^{p^*}-|\tilde{v}_n|^{p^*}))\\ &\quad+\lambda\int_{\mathbb{R}^N}(H(u_n,\ v_n)-H(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)-H(\tilde{u}_n,\ \tilde{v}_n))+o(1).\end{aligned}$$

By (3.1) and the similar idea of proving the Brézis-Lieb Lemma [17], it is easy to get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)((|u_n|^{p^*}-|u_n-\tilde{u}_n|^{p^*}-|\tilde{u}_n|^{p^*})+(|v_n|^{p^*}-|v_n-\tilde{v}_n|^{p^*}-|\tilde{v}_n|^{p^*}))=0$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (H(u_n,\ v_n)-H(u_n-\tilde{u}_n,\ v_n\tilde{v}_n)-H(\tilde{u}_n,\ \tilde{v}_n))=0.$$

In connection with the fact $I_\lambda\ (u_n, v_n)\to c$ and $I_\lambda(\tilde{u}_n,\tilde{v}_n)\to I_\lambda(u,v)$, we obtain

$$I_\lambda(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)\to c-I_\lambda(u,\ v).$$

In the following, we will verify the fact $I'_\lambda(u_n-\tilde{u}_n, v_n-\tilde{v}_n)\to 0$.

For any $(\phi,\ \psi)\in\ B$, it follows that

$$\begin{aligned}
&I'_\lambda(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)(\varphi,\ \psi)\\
&=I'_\lambda(u_n,\ v_n)(\varphi,\ \psi)-I'_\lambda(\tilde{u}_n,\ \tilde{v}_n)(\varphi,\ \psi)\\
&\quad+\lambda\int_{\mathbb{R}^N} K(x)[(|u_n|^{p^*-2}u_n-|u_n-\tilde{u}_n|^{p^*-2}(u_n-\tilde{u}_n)-|\tilde{u}_n|^{p^*-2}\tilde{u}_n)\varphi\\
&\quad+(|v_n|^{p^*-2}v_n-|v_n-\tilde{v}_n|^{p^*-2}(v_n-\tilde{v}_n)-|\tilde{v}_n|^{p^*-2}\tilde{v}_n)\psi]\\
&\quad+\lambda\int_{\mathbb{R}^N} [(H_s(u_n,\ v_n)-H_s(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)-H_s(\tilde{u}_n,\ \tilde{v}_n))\varphi\\
&\quad+(H_t(u_n,\ v_n)-H_t(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)-H_t(\tilde{u}_n,\ \tilde{v}_n))\psi]+o(1).
\end{aligned}$$

Standard argument shows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|u_n|^{p^*-2}u_n-|u_n-\tilde{u}_n|^{p^*-2}(u_n-\tilde{u}_n)-|\tilde{u}_n|^{p^*-2}\tilde{u}_n)\varphi=0$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|v_n|^{p^*-2}v_n-|v_n-\tilde{v}_n|^{p^*-2}(v_n-\tilde{v}_n)-|\tilde{v}_n|^{p^*-2}\tilde{v}_n)\psi=0$$

uniformly in $\|\phi,\ \psi)\|_B\leq 1$.

By Lemma 3.3, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (H_s(u_n,\ v_n)-H_s(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)-H_s(\tilde{u}_n,\ \tilde{v}_n))\varphi=0$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} (H_t(u_n,\ v_n)-H_t(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)-H_t(\tilde{u}_n,\ \tilde{v}_n))\psi=0$$

uniformly in $\|(\phi,\ \psi)\|_B\leq 1$. From the facts above mentioned, we obtain

$$I'_\lambda(u_n-\tilde{u}_n,\ v_n-\tilde{v}_n)\to 0 \text{ in } B^{-1}.$$

□

Let $u_n^1=u_n-\tilde{u}_n$, $v_n^1=v_n-\tilde{v}_n$, then $u_n-u=u_n^1+(\tilde{u}_n-u)$, $v_n-v=v_n^1+(\tilde{v}_n-v)$. From (3.1), we get $(u_n, v_n)\to(u,\ v)$ in B if and only if $(u_n^1, v_n^1)\to(0,0)$ in B.

Observe that

$$\begin{aligned}
&I_\lambda(u_n^1,\ v_n^1) - \frac{1}{p}I'_\lambda(u_n^1,\ v_n^1)(u_n^1,\ v_n^1) \\
&= \left(\frac{1}{p} - \frac{1}{p^*}\right)\lambda\int_{\mathbb{R}^N} K(x)(|u_n^1|^{p^*} + |v_n^1|^{p^*}) \\
&\quad + \lambda\int_{\mathbb{R}^N}\left(\frac{1}{p}(u_n^1H_s(u_n^1,\ v_n^1) + v_n^1H_t(u_n^1,\ v_n^1)) - H(u_n^1,\ v_n^1)\right) \\
&\geq \frac{\lambda}{N}\int_{\mathbb{R}^N} K(x)(|u_n^1|^{p^*} + |v_n^1|^{p^*}) \\
&\geq \frac{\lambda}{N}K_{\min}\int_{\mathbb{R}^N}(|u_n^1|^{p^*} + |v_n^1|^{p^*}),
\end{aligned}$$

where $K_{\min} = \inf_{x\in\mathbb{R}^N} K(x) > 0$.

Thus by Lemma 3.4, we get

$$||(u_n^1,\ v_n^1)||_{p^*}^{p^*} \leq \frac{N(c - I_\lambda(u, v))}{\lambda K_{\min}} + o(1). \tag{3.3}$$

Now, we consider the energy level of the functional I_λ below which the $(PS)_c$ condition hold.

Let $V_b(x)$:= max$\{V\ (x),\ b\}$, where b is the positive constant in the assumption (V_0). Since the set v_b has finite measure and u_n^1, $v_n^1 \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} V(x)(|u_n^1|^p + |v_n^1|^p) = \int_{\mathbb{R}^N} V_b(x)(|u_n^1|^p + |v_n^1|^p) + o(1). \tag{3.4}$$

From (K_0), (H_1)-(H_3) and Young inequality, there is C_b >0 such that

$$\begin{aligned}
&\int_{\mathbb{R}^N}(K(x)(|u|^{p^*} + |v|^{p^*}) + uH_s(u,\ v) + vH_t(u,\ v)) \\
&\leq b(||u||_p^p + ||v||_p^p) + C_b(||u||_{p^*}^{p^*} + ||v||_{p^*}^{p^*}).
\end{aligned} \tag{3.5}$$

Let S be the best Sobolev constant of the immersion

$$S||u||_{p^*}^p \leq \int_{\mathbb{R}^N}|\nabla u|^p \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N).$$

Lemma 3.5. Let the assumptions of Theorem 2 be satisfied. There exists α_0 >0 independent of λ such that, for any $(PS)_c$ sequence $\{(u_n,\ v_n)\} \subset B$ for I_λ with $(u_n,\ v_n) \rightharpoonup (u,\ v)$, either $(u_n,\ v_n) \to (u,\ v)$ or $c - I_\lambda(u, v) \geq \alpha_0\lambda^{1-\frac{N}{p}}$.

Proof. Assume that $(u_n,\ v_n) \nrightarrow (u,\ v)$, then

$$\lim\inf_{n\to\infty} ||(u_n^1,\ v_n^1)||_\lambda > 0$$

and

$$c - J_\lambda(u,\ v) > 0.$$

By the Sobolev inequality, (3.4) and (3.5), we get

$$
\begin{aligned}
&S(||u_n^1||_{p^*}^p + ||v_n^1||_{p^*}^p) \\
&\le \int_{\mathbb{R}^N} (|\nabla u_n^1|^p + |\nabla v_n^1|^p) \\
&= \int_{\mathbb{R}^N} (|\nabla u_n^1|^p + \lambda V(x)|u_n^1|^p + |\nabla v_n^1|^p + \lambda V(x)|v_n^1|^p) - \lambda \int_{\mathbb{R}^N} V(x)(|u_n^1|^p + |v_n^1|^p) \\
&= \lambda \int_{\mathbb{R}^N} K(x)(|u_n^1|^{p^*} + |v_n^1|^{p^*}) + u_n^1 H_s(u_n^1,\ v_n^1) + v_n^1 H_t(u_n^1,\ v_n^1) \\
&\quad - \lambda \int_{\mathbb{R}^N} V(x)(|u_n^1|^p + |v_n^1|^p) + o(1) \\
&\le \lambda b(||u_n^1||_p^p + ||v_n^1||_p^p) + \lambda C_b(||u_n^1||_{p^*}^{p^*} + ||v_n^1||_{p^*}^{p^*}) - \lambda b(||u_n^1||_p^p + ||v_n^1||_p^p) + o(1) \\
&= \lambda C_b(||u_n^1||_{p^*}^{p^*} + ||v_n^1||_{p^*}^{p^*}) + o(1).
\end{aligned}
$$

This, together with $\liminf_{n\to\infty}(||u_n^1||_{p^*}^{p^*} + ||v_n^1||_{p^*}^{p^*}) > 0$ and (3.3), gives

$$
\begin{aligned}
S &\le \lambda C_b(||u_n^1||_{p^*}^{p^*} + ||v_n^1||_{p^*}^{p^*})^{\frac{p^*-p}{p^*}} + o(1) \\
&\le \lambda C_b\left(\frac{N(c - I_\lambda(u, v))}{\lambda K_{\min}}\right)^{\frac{p}{N}} + o(1) \\
&= \lambda^{1-\frac{p}{N}} C_b\left(\frac{N}{K_{\min}}\right)^{\frac{p}{N}} (c - I_\lambda(u,\ v))^{\frac{p}{N}} + o(1).
\end{aligned}
$$

Set $\alpha_0 = S^{\frac{N}{p}} C_b^{-\frac{N}{p}} N^{-1} K_{\min}$, then

$$
\alpha_0 \lambda^{1-\frac{N}{p}} \le c - I_\lambda(u,\ v) + o(1).
$$

This proof is completed. □

Since $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ is not compact, I_λ does not satisfy the $(PS)_c$ condition for all $c > 0$. But Lemma 3.5 shows that I_λ satisfies the following local $(PS)_c$ condition.

Lemma 3.6. From the assumptions of Theorem 2, there exists a constant $\alpha_0 > 0$ independent of λ such that, if a $(PS)_c$ sequence $\{(u_n,\ v_n)\} \subset B$ for I_λ satisfies $c \le \alpha_0 \lambda^{1-\frac{N}{p}}$, the sequence $\{(u_n, v_n)\}$ has a strongly convergent subsequence in B.

Proof. By the fact $c \le \alpha_0 \lambda^{1-\frac{N}{p}}$, we have

$$
c - I_\lambda(u,\ v) \le \alpha_0 \lambda^{1-\frac{N}{p}} - I_\lambda(u,\ v).
$$

This, together with $I_\lambda(u, v) \ge 0$ and Lemma 3.5, gives the desired conclusion. □

Next, we consider $\lambda = 1$. From the following standard argument, we get that I_λ possesses the mountain-pass structure.

Lemma 3.7. Under the assumptions of Theorem 2, there exist $\alpha_\lambda, \rho_\lambda > 0$ such that

$$
I_\lambda(u,\ v) > 0 \text{ if } 0 < ||(u,\ v)||_\lambda < \rho_\lambda \text{ and } I_\lambda(u,\ v) \ge \alpha_\lambda \text{ if} ||(u,\ v)||_\lambda = \rho_\lambda.
$$

Proof. By (3.5), we get that for any $\delta > 0$, there is $C_\delta > 0$ such that

$$\int_{\mathbb{R}^N} G(u, v) \leq \delta(||u||_p^p + ||v||_p^p) + C_\delta(||u||_{p^*}^{p^*} + ||v||_{p^*}^{p^*}).$$

Thus

$$\begin{aligned} I_\lambda(u, v) &= \frac{1}{p}||(u, v)||_\lambda^p - \lambda \int_{\mathbb{R}^N} G(u, v) \\ &\geq \frac{1}{p}||(u, v)||_\lambda^p - \lambda\delta(||u||_p^p + ||v||_p^p) - \lambda C_\delta(||u||_{p^*}^{p^*} + ||v||_{p^*}^{p^*}). \end{aligned}$$

Note that $||u||_p^p + ||v||_p^p \leq C_1||(u, v)||_\lambda^p$. If $\delta \leq (2p\lambda C_1)^{-1}$, then

$$I_\lambda(u, v) \geq \frac{1}{2p}||(u, v)||_\lambda^p - \lambda C_\delta(||u||_{p^*}^{p^*} + ||v||_{p^*}^{p^*}).$$

The fact $p^* > p$ implies the desired conclusion. □

Lemma 3.8. Under the assumptions of Lemma 3.7, for any finite dimensional subspace

$F \subset B$, we have

$$I_\lambda(u, v) \to -\infty \quad \text{as } (u, v) \in F, \quad ||(u, v)||_\lambda \to \infty.$$

Proof. By the assumption (H_3), it follows that

$$I_\lambda(u, v) \leq \frac{1}{p}||(u, v)||_\lambda^p - \lambda a_0(|u|_\alpha^\alpha + |v|_\beta^\beta) \quad \text{for all } (u, v) \in B.$$

Since all norms in a finite-dimensional space are equivalent and $\alpha, \beta > p$, we prove the result of this Lemma. □

By Lemma 3.6, for λ larger enough and c_λ small sufficiently, I_λ satisfies $(PS)_{c\lambda}$ condition.

Thus, we will find special finite-dimensional subspaces by which we establish sufficiently small minimax levels.

Define the functional

$$\Phi_\lambda(u, v) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + \lambda V(x)|u|^p + |\nabla v|^p + \lambda V(x)|v|^p) - \lambda a_0 \int_{\mathbb{R}^N} (|u|^\alpha + |v|^\beta).$$

It is apparent that $\Phi_\lambda \in C^1(B)$ and $I_\lambda(u, v) \leq \Phi_\lambda(u, v)$ for all $(u, v) \in B$.

Observe that

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla\phi|^p : \phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}), \ |\phi|_{L^\alpha(\mathbb{R}^N)} = 1\right\} = 0$$

and

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla\psi|^p : \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}), \ |\psi|_{L^\beta(\mathbb{R}^N)} = 1\right\} = 0.$$

For any $\delta>0$, there are $\phi_\delta,\ \psi_\delta \in C_0^\infty(\mathbb{R}^N,\mathbb{R})$ with $|\phi_\delta|_{L^\alpha(\mathbb{R}^N)} = |\psi_\delta|_{L^\beta(\mathbb{R}^N)} = 1$ and $\mathrm{supp}\phi_\delta$, $\mathrm{supp}\psi_\delta \subset B_{r_\delta}(0)$ such that $|\nabla\phi_\delta|_p^p, |\nabla\psi_\delta|_p^p < \delta$.

Let $w_\lambda(x) = (\phi_\delta(\sqrt[p]{\lambda}x),\ \psi_\delta(\sqrt[p]{\lambda}x))$, then $\mathrm{supp} w_\lambda \subset B_{\lambda^{-\frac{1}{p}}r_\delta}(0)$. For $t \geq 0$, we get

$$\begin{aligned}\Phi_\lambda(tw_\lambda) &= \frac{t^p}{p}\|w_\lambda\|_\lambda^p - a_0\lambda t^\alpha\int_{\mathbb{R}^N}|\phi_\delta(\sqrt[p]{\lambda}x)|^\alpha - a_0\lambda t^\beta\int_{\mathbb{R}^N}|\psi_\delta(\sqrt[p]{\lambda}x)|^\beta \\ &= \lambda^{1-\frac{N}{p}}J_\lambda(t\phi_\delta, t\psi_\delta),\end{aligned}$$

where

$$J_\lambda(u,\ v) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + |\nabla v|^p + V(\lambda^{-\frac{1}{p}}x)(|u|^p + |v|^p)) - a_0\int_{\mathbb{R}^N}(|u|^\alpha + |v|^\beta).$$

We easily prove that

$$\begin{aligned}\max_{t\geq 0}J_\lambda(t\phi_\delta,\ t\psi_\delta) \leq{}& \frac{\alpha-p}{p\alpha(\alpha a_0)^{\frac{p}{\alpha-p}}}\left\{\int_{\mathbb{R}^N}(|\nabla\phi_\delta|^p + V(\lambda^{-\frac{1}{p}}x)|\phi_\delta|^p\right\}^{\frac{\alpha}{\alpha-p}} \\ &+ \frac{\beta-p}{p\beta(\beta a_0)^{\frac{p}{\beta-p}}}\left\{\int_{\mathbb{R}^N}(|\nabla\psi_\delta|^p + V(\lambda^{-\frac{1}{p}}x)|\psi_\delta|^p\right\}^{\frac{\beta}{\beta-p}}.\end{aligned}$$

Together with $V(0) = 0$ and $|\nabla\phi_\delta|_p^p,\ |\nabla\psi_\delta|_p^p < \delta$, this implies that there is $\Lambda_\delta > 0$ such that for all $\lambda \geq \Lambda_\delta$, we have

$$\max_{t\geq 0}I_\lambda(t\phi_\delta,\ t\psi_\delta) \leq \left(\frac{\alpha-p}{p\alpha(\alpha a_0)^{\frac{p}{\alpha-p}}}(2\delta)^{\frac{\alpha}{\alpha-p}} + \frac{\beta-p}{p\beta(\beta a_0)^{\frac{p}{\beta-p}}}(2\delta)^{\frac{\beta}{\beta-p}}\right)\lambda^{1-\frac{N}{p}}. \tag{3.6}$$

It follows from (3.6) that

Lemma 3.9. Under the assumptions of Lemma 3.7, for any $\subset > 0$, there is $\Lambda_\sigma > 0$ such that $\lambda \geq \Lambda_\sigma$, there exists $\bar{w}_\lambda \in B$ with $\|\bar{w}_\lambda\|_\lambda > \rho_\lambda, I_\lambda(\bar{w}_\lambda) \leq 0$ and

$$\max_{t\geq 0}I_\lambda(t\bar{w}_\lambda) \leq \sigma\lambda^{1-\frac{N}{p}},$$

where ρ_λ is defined in Lemma 3.7.

Proof. This proof is similar to the one of Lemma 4.3 in [10], it can be easily proved. □

4 Proof of the main result

In the following, we will give the proof of Theorem 2.

Proof. From Lemma 3.9, for any $\sigma > 0$ with $0 < \sigma < \alpha_0$, there is $\Lambda_\sigma > 0$ such that for $\lambda \geq \Lambda_\sigma$, we obtain

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \le \sigma\lambda^{1-\frac{N}{p}},$$

where $\Gamma_\lambda = \{\gamma \in C([0,1],\ B) : \gamma(0) = 0,\ \gamma(1) = \bar{w}_\lambda\}$.

Furthermore, Lemma 3.6 implies that I_λ satisfies $(PS)_{c\lambda}$ condition. Hence, by the mountain-pass theorem, there is $(u_\lambda,\ v_\lambda) \in B$ satisfying $I_\lambda\ (u_\lambda,\ v_\lambda) = c_\lambda$ and $I'_\lambda(u_\lambda,\ v_\lambda) = 0$. This shows $(u_\lambda,\ v_\lambda)$ is a weak solution of (2.1). Similar to the argument in [10], we also get that $(u_\lambda,\ v_\lambda)$ is a positive least energy solution.

Finally, we prove $(u_\lambda,\ v_\lambda)$ satisfies the estimate (2.2). Observe that $I_\lambda(u_\lambda,\ v_\lambda) \le \sigma\lambda^{1-\frac{N}{p}}$ and $I'_\lambda(u_\lambda,\ v_\lambda) = 0$. we have

$$\begin{aligned}
I_\lambda(u_\lambda, v_\lambda) &= I_\lambda(u_\lambda, v_\lambda) - \frac{1}{\theta} I'_\lambda(u_\lambda, v_\lambda)(u_\lambda, v_\lambda) \\
&= \left(\frac{1}{p} - \frac{1}{\theta}\right) \|(u_\lambda, v_\lambda)\|_\lambda^p + \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \lambda \int_{\mathbb{R}^N} K(x)(|u_\lambda|^{p^*} + |v_\lambda|^{p^*}) \\
&\quad + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\theta}(u_\lambda H_s(u_\lambda, v_\lambda) + v_\lambda H_t(u_\lambda, v_\lambda)) - H(u_\lambda, v_\lambda)\right) \\
&\ge \left(\frac{1}{p} - \frac{1}{\theta}\right) \|(u_\lambda, v_\lambda)\|_\lambda^p.
\end{aligned}$$

This shows that $(u_\lambda,\ v_\lambda)$ satisfies the estimate (2.2). The proof is complete. □

Acknowledgements
The authors would like to appreciate the referees for their precious comments and suggestions about the original manuscript. This research is supported by the National Natural Science Foundation of China (10771212) and the Fundamental Research Funds for the Central Universities (2010LKSX09).

Authors' contributions
The authors contributed equally in this article. They read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 29 September 2011 Accepted: 4 May 2012 Published: 4 May 2012

References

1. Brézis, H, Nirenberg, L: Some variational problems with lack of compactness. Proc Symp Pure Math. **45**, 165–201 (1986)
2. Ambrosetti, A, Rabinowitz, PH: Dual variational methods in critical point theory and applications. J Funct Anal. **14**, 349–381 (1973)
3. Guedda, M, Veron, L: Quasilinear elliptic equations involving critical Sobolev exponents. Non Anal. **12**, 879–902 (1989)
4. Benci, V, Cerami, G: Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}$ in ℝ^N. J Funct Anal. **88**, 90–117 (1990)
5. Floer, A, Weinstein, A: Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. J Funct Anal. **69**, 397–408 (1986)
6. Oh, YG: On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential. Commun Math Phys. **131**, 223–253 (1990)
7. Clapp, M, Ding, YH: Minimal nodal solutions of a Schrödinger equation with critical nonlinearity and symmetric potential. Diff Integr Equ. **16**, 981–992 (2003)
8. Del Pino, M, Felmer, P: Semi-classical states for nonlinear Schrödinger equations. Annales Inst H Poincaré. **15**, 127–149 (1998)
9. Cingolani, S, Lazzo, M: Multiple positive solutions to nonlinear Schrödinger equation with competing potential functions. J Diff Equ. **160**, 118–138 (2000)
10. Ding, YH, Lin, FH: Solutions of perturbed Schrödinger equations with critical nonlinearity. Calc Var. **30**, 231–249 (2007)
11. Garcia Azoero, J, Peral Alonso, I: Existence and non-uniqueness for the *p*-Laplacian: Nonlinear eigenvalues. Commun Partial Diff Equ. **12**, 1389–1430 (1987)

12. Alves, CO, Ding, YH: Multiplicity of positive solutions to a *p*-Laplacian equation involving critical nonlinearity. J Math Anal Appl. **279**, 508–521 (2003)
13. Liu, CG, Zheng, YQ: Existence of nontrivial solutions for *p*-Laplacian equations in $\mathbb{R}^N$. J Math Anal Appl. **380**, 669–679 (2011)
14. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Springer-Verlag, New York (1989)
15. Li, YY, Guo, QQ, Niu, PC: Global compactness results for quasilinear elliptic problems with combined critical Sobolev-Hardy terms. Non Anal. **74**, 1445–1464 (2011)
16. Ghoussoub, N, Yuan, C: Multiple solutions for quasi-linear PDES involving the critical Sobolev and Hardy exponents. Trans Am Math Soc. **352**, 5703–5743 (2000)
17. Brézis, H, Lieb, E: A relation between pointwise convergence of functions and convergence of functional. Proc Am Math Soc. **88**, 486–490 (1983)

doi:10.1186/1687-2770-2012-53
Cite this article as: Zhang and Liu: **Existence of nontrivial solutions to perturbed *p*-Laplacian system in $\mathbb{R}^N$ involving critical nonlinearity.** *Boundary Value Problems* 2012 **2012**:53.

Submit your manuscript to a SpringerOpen journal and benefit from:

► Convenient online submission
► Rigorous peer review
► Immediate publication on acceptance
► Open access: articles freely available online
► High visibility within the field
► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com