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Infinitely many solutions for class of Neumann quasilinear elliptic systems

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Abstract

We investigate the existence of infinitely many weak solutions for a class of Neumann quasilinear elliptic systems driven by a $(p_1, ..., p_n)$-Laplacian operator. The technical approach is fully based on a recent three critical points theorem.

AMS subject classification: 35J65; 34A15.

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1 Introduction

The purpose of this article is to establish the existence of infinitely many weak solutions for the following Neumann quasilinear elliptic system

$$\begin{cases} -\Delta_{p_i} u_i + a_i(x)|u_i|^{p_i-2}u = \lambda F_{u_i}(x, u_1, \ldots, u_n) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

for $i = 1, ..., n$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with a smooth boundary $\partial\Omega$, $p_i > N$ for $i = 1, ..., n$, $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator, $a_i \in L^\infty(\Omega)$ with ess $\inf_\Omega a_i > 0$ for $i = 1, ..., n$, $\lambda > 0$, and $F: \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2,..., t_n) \to F(x, t_1, t_2,..., t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in \Omega$, F_{t_i} is continuous in $\Omega \times \mathbb{R}^n$ for $i = 1,..., n$, and $F(x, 0,..., 0) = 0$ for all $x \in \Omega$ and ν is the outward unit normal to $\partial\Omega$. Here, F_{t_i} denotes the partial derivative of F with respect to t_i.

Precisely, under appropriate hypotheses on the behavior of the nonlinear term F at infinity, the existence of an interval Λ such that, for each $\lambda \in \Lambda$, the system (1) admits a sequence of pairwise distinct weak solutions is proved; (see Theorem 3.1). We use a variational argument due to Ricceri which provides certain alternatives in order to find sequences of distinct critical points of parameter-depending functionals. We emphasize that no symmetry assumption is required on the nonlinear term F (thus, the symmetry version of the Mountain Pass theorem cannot be applied). Instead of such a symmetry, we assume a suitable oscillatory behavior at infinity on the function F.

We recall that a weak solution of the system (1) is any $u = (u_1, ..., u_n) \in W^{1,p_1}(\Omega) \times ... \times W^{1,p_n}(\Omega)$, such that

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$$\int_\Omega \sum_{i=1}^{n} \left(\left|\nabla u_i(x)\right|^{p_i-2} \nabla u_i(x) \nabla v_i(x) + a_i(x) \left|u_i(x)\right|^{p_i-2} u_i(x) v_i(x) \right) dx$$
$$-\lambda \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots u_n(x)) v_i(x) dx = 0$$

for all $v = (v_1, \ldots, v_n) \in W^{1,p_1}(\Omega) \times \ldots \times W^{1,p_n}(\Omega)$.

For a discussion about the existence of infinitely many solutions for differential equations, using Ricceri's variational principle [1]and its variants [2,3] we refer the reader to the articles [4-16].

For other basic definitions and notations we refer the reader to the articles [17-22]. Here, our motivation comes from the recent article [8]. We point out that strategy of the proof of the main result and Example 3.1 are strictly related to the results and example contained in [8].

2 Preliminaries

Our main tool to ensure the existence of infinitely many classical solutions for Dirichlet quasilinear two-point boundary value systems is the celebrated Ricceri's variational principle [[1], Theorem 2.5] that we now recall as follows:

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi: X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(*a*) *for every $r > \inf_X \Phi$ and every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(*b*) *If $\gamma < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds:*

either

(b_1) *I_λ possesses a global minimum,*

or

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(*c*) *If $\delta < +\infty$ then, for each $\lambda \in \left]0, \frac{1}{\delta}\right[$, the following alternative holds:*

either

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ,
or
(c_2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ *that converges weakly to a global minimum of* Φ.

We let X be the Cartesian product of n Sobolev spaces $W^{1,p_1}(\Omega)$, $W^{1,p_2}(\Omega)$,... and $W^{1,p_n}(\Omega)$, i.e., $X = \prod_{i=1}^n W^{1,p_i}(\Omega)$, equipped with the norm

$$\|(u_1, u_2, \dots, u_n)\| = \sum_{i=1}^n \|u_i\|_{p_i},$$

where

$$\|u_i\|_{p_i} = \left(\int_\Omega |\nabla u_i(x)|^{p_i} + a_i(x)|u_i(x)|^{p_i} dx\right)^{\frac{1}{p_i}}, \; i = 1, \dots, n.$$
$$C = \max\left\{\sup_{u_i \in W^{1,p_i}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega}|u(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}}; \; i = 1, \dots, n\right\}. \tag{2}$$

Since $p_i > N$ for $1 \le i \le n$, one has $C < +\infty$. In addition, if Ω is convex, it is known [23] that

$$\sup_{u_i \in W^{1,p_i}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega}|u_i(x)|}{\|u_i\|_{p_i}} \le 2^{\frac{p_i-1}{p_i}} \max\left\{\left(\frac{1}{\|a_i\|_1}\right)^{\frac{1}{p_i}}; \frac{\operatorname{diam}(\Omega)}{N^{\frac{1}{p_i}}}\left(\frac{p_i-1}{p_i-N} m(\Omega)\right)^{\frac{p_i-1}{p_i}} \frac{\|a_i\|_\infty}{\|a_i\|_1}\right\}$$

for $1 \le i \le n$, where $||\cdot||_1 = \int_\Omega |\cdot(x)|\,dx$, $||\cdot||_\infty = \sup_{x\in\Omega} |\cdot(x)|$ and $m(\Omega)$ is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball.

In the sequel, let $\underline{p} = \min\{p_i; 1 \le i \le n\}$.

For all $\gamma > 0$ we define

$$K(\gamma) = \left\{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \le \gamma\right\}. \tag{3}$$

3 Main results

We state our main result as follows:

Theorem 3.1. *Assume that*

(A1)

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)} F(x, t_1, \dots, t_n)dx}{\xi^{\underline{p}}}$$
$$< \left(\sum_{i=1}^n (p_i C)^{\frac{1}{p_i}}\right)^{\underline{p}} \limsup_{\substack{(t_1,\dots,t_n)\to\infty \\ (t_1,\dots,t_n)\in\mathbb{R}_+^n}} \frac{\int_\Omega F(x, t_1, \dots, t_n)dx}{\sum_{i=1}^n \frac{\|a_i\|_1 |t_i|^{p_i}}{p_i}}$$

where $K(\xi) = \{(t_1, \dots, t_n) | \sum_{i=1}^n |t_i| \le \xi\}$ *(see (3)).*

Then, for each

$$\lambda \in \Lambda := \left]\frac{1}{\limsup_{\substack{(t_1,\ldots,t_n)\to\infty \\ (t_1,\ldots,t_n)\in\mathbb{R}_+^n}} \frac{\int_\Omega F(x,t_1,\ldots,t_n)dx}{\sum_{i=1}^n \frac{\|a_i\|_1|t_i|}{p_i}}}, \frac{\left(\sum_{i=1}^n (p_iC)^{\frac{1}{p_i}}\right)^{\underline{p}}}{\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\xi)} F(x,t_1,\ldots,t_n)dx}{\xi^{\underline{p}}}}\right[$$

the system (1) has an unbounded sequence of weak solutions in X.

Proof. Define the functionals Φ, Ψ: $X \to \mathbb{R}$ for each $u = (u_1, ..., u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}$$

and

$$\Psi(u) = \int_\Omega F(x, u_1(x), \ldots, u_n(x))dx.$$

It is well known that Ψ is a Gâteaux differentiable functional and sequentially weakly lower semicontinuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x), \ldots, u_n(x))v_i(x)dx$$

for every $v = (v_1, ..., v_n) \in X$, and Ψ': $X \to X^*$ is a compact operator. Moreover, Φ is a sequentially weakly lower semicontinuous and Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u_1,\ldots,u_n)(v_1,\ldots,v_n) \int_\Omega \sum_{i=1}^n \left(|\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x) + a_i(x)|u_i(x)|^{p_i-2}u_i(x)v_i(x)\right)dx$$

for every $v = (v_1, ..., v_n) \in X$. Furthermore, $(\Phi')^{-1}$: $X^* \to X$ exists and is continuous.

Put I_λ: $= \Phi - \lambda\Psi$. Clearly, the weak solutions of the system (1) are exactly the solutions of the equation $I'_\lambda(u_1, \ldots, u_n) = 0$. Now, we want to show that

$$\gamma < +\infty.$$

Let $\{\xi_m\}$ be a real sequence such that $\xi_m \to +\infty$ as $m \to \infty$ and

$$\lim_{m\to\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\xi_m)} F(x,t_1,\ldots,t_n)dx}{\xi_m^{\underline{p}}} = \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\xi)} F(x,t_1,\ldots,t_n)dx}{\xi^{\underline{p}}}.$$

Put $r_m = \frac{\xi_m^{\underline{p}}}{\left(\sum_{i=1}^n (p_iC)^{\frac{1}{p_i}}\right)^{\underline{p}}}$ for all $m \in \mathbb{N}$. Since

$$\sup_{x\in\Omega} |u_i(x)|^{p_i} \leq C\|u_i\|_{p_i}^{p_i}$$

for each $u_i \in W^{1,p_i}(\Omega)$ for $1 \le i \le n$, we have

$$\sup_{x\in\Omega}\sum_{i=1}^{n}\frac{|u_i(x)|^{p_i}}{p_i} \le C\sum_{i=1}^{n}\frac{\|u_i\|_{p_i}^{p_i}}{p_i}. \tag{4}$$

for each $u = (u_1, u_2, ..., u_n) \in X$. This, for each $r > 0$, together with (4), ensures that

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{u \in X; \sup\sum_{i=1}^{n}\frac{|u_i(x)|^{p_i}}{p_i} \le Cr \text{ for each } x \in \Omega\right\}.$$

Hence, an easy computation shows that $\sum_{i=1}^{n}|u_i| \le \xi_m$ whenever $u = (u_1, ..., u_n) \in \Phi^{-1}(] -\infty, r_m])$. Hence, one has

$$\begin{aligned}\varphi(r_m) &= \inf_{u\in\Phi^{-1}(]-\infty,r_m[)}\frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_m[)}\Psi(v)) - \Phi(u)}{r_m - \Phi(u)}\\ &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_m[)}\Psi(v)}{r_m}\\ &\le \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi_m)}F(x,t_1,\dots,t_n)dx}{\dfrac{\xi_m^{\underline{p}}}{\left(\sum_{i=1}^{n}(p_iC)^{\frac{1}{p_i}}\right)^{\underline{p}}}}.\end{aligned}$$

Therefore, since from Assumption (A1) one has

$$\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)}F(x,t_1,\dots,t_n)dx}{\xi^{\underline{p}}} < \infty,$$

we deduce

$$\begin{aligned}\gamma &\le \liminf_{m\to+\infty}\varphi(r_m)\\ &\le \left(\sum_{i=1}^{n}(p_iC)^{\frac{1}{p_i}}\right)^{\underline{p}}\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(\xi)}F(x,t_1,\dots,t_n)dx}{\xi^{\underline{p}}} < +\infty.\end{aligned} \tag{5}$$

Assumption (A1) along with (5), implies

$$\Lambda \subseteq \left]0, \frac{1}{\gamma}\right[.$$

Fix $\lambda \in \Lambda$. The inequality (5) concludes that the condition (*b*) of Theorem 2.1 can be applied and either I_λ has a global minimum or there exists a sequence $\{u_m\}$ where $u_m = (u_{1m}, ..., u_{nm})$ of weak solutions of the system (1) such that $\lim_{m\to\infty}||(u_{1m}, ..., u_{nm})|| = +\infty$.

Now fix $\lambda \in \Lambda$ and let us verify that the functional I_λ is unbounded from below. Arguing as in [8], consider n positive real sequences $\{d_{i,m}\}_{i=1}^{n}$ such that $\sqrt{\sum_{i=1}^{n}d_{i,m}^2} \to +\infty$ as $m \to \infty$

and

$$\lim_{m\to+\infty}\frac{\int_\Omega F(x,d_{1,m},\dots,d_{n,m})dx}{\sum_{i=1}^{n}\frac{d_{i,m}^{p_i}}{p_i}} = \limsup_{\substack{(t_1,\dots,t_n)\to\infty\\(t_1,\dots,t_n)\in\mathbb{R}_+^n}}\frac{\int_\Omega F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^{n}\frac{\|a_i\|_1|t_i|^{p_i}}{p_i}}. \tag{6}$$

For all $m \in \mathbb{N}$ define $w_m(x) = (d_{1,m}, ..., d_{n,m})$. For any fixed $m \in \mathbb{N}$, $w_m \in X$ and, in particular, one has

$$\Phi(w_m) = \sum_{i=1}^{n} \frac{d_{i,m}^{p_i} \|a_i\|_1}{p_i}.$$

Then, for all $m \in \mathbb{N}$,

$$I_\lambda(w_m) = \Phi(w_m) - \lambda\Psi(w_m) = \sum_{i=1}^{n} \frac{d_{i,m}^{p_i} \|a_i\|_1}{p_i} - \lambda \int_\Omega F(x, d_{1,m}, \dots, d_{n,m})dx.$$

Now, if

$$\limsup_{\substack{(t_1,\dots,t_n)\to\infty \\ (t_1,\dots,t_n)\in\mathbb{R}_+^n}} \frac{\int_\Omega F(x, t_1, \dots, t_n)dx}{\sum_{i=1}^n \frac{\|a_i\|_1 |t_i|^{p_i}}{p_i}} < \infty,$$

we fix $\epsilon \in \left] \frac{1}{\lambda \limsup_{\substack{(t_1,\dots,t_n)\to\infty \\ (t_1,\dots,t_n)\in\mathbb{R}_+^n}} \frac{\int_\Omega F(x,t_1,\dots,t_n)dx}{\sum_{i=1}^n \frac{\|a_i\|_1|t_i|^{p_i}}{p_i}}}, 1 \right[$. From (6) there exists τ_ε such that

$$\int_\Omega F(x, d_{1,m}, \dots, d_{n,m})dx > \epsilon \limsup_{\substack{(t_1,\dots,t_n)\to\infty \\ (t_1,\dots,t_n)\in\mathbb{R}_+^n}} \frac{\int_\Omega F(x, t_1, \dots, t_n)dx}{\sum_{i=1}^n \frac{\|a_i\|_1 |t_i|^{p_i}}{p_i}} \left(\sum_{i=1}^{n} \frac{d_{i,m}^{p_i} \|a_i\|_1}{p_i} \right) \quad \forall m > \tau_\epsilon,$$

therefore

$$I_\lambda(w_m) \le \left(1 - \lambda\epsilon \limsup_{\substack{(t_1,\dots,t_n)\to\infty \\ (t_1,\dots,t_n)\in\mathbb{R}_+^n}} \frac{\int_\Omega F(x, t_1, \dots, t_n)dx}{\sum_{i=1}^n \frac{\|a_i\|_1 |t_i|^{p_i}}{p_i}} \right) \sum_{i=1}^{n} \frac{d_{i,m}^{p_i} \|a_i\|_1}{p_i} \quad \forall m > \tau_\epsilon,$$

and by the choice of ε, one has

$$\lim_{m\to+\infty} [\Phi(w_m) - \lambda\Psi(w_m)] = -\infty.$$

If

$$\limsup_{\xi\to+\infty} \frac{\int_\Omega F(x, t_1, \dots, t_n)dx}{\sum_{i=1}^n \frac{\|a_i\|_1 |t_i|^{p_i}}{p_i}} = \infty,$$

let us consider $K > \frac{1}{\lambda}$. From (6) there exists τ_K such that

$$\int_\Omega F(x, d_{1,m}, \dots, d_{n,m})dx > K \sum_{i=1}^{n} \frac{d_{i,m}^{p_i} \|a_i\|_1}{p_i} \quad \forall m > \tau_K,$$

therefore

$$I_\lambda(w_m) \le (1 - \lambda K) \sum_{i=1}^{n} \frac{d_{i,m}^{p_i} \|a_i\|_1}{p_i} \quad \forall m > \tau_K,$$

and by the choice of K, one has

$$\lim_{m\to+\infty}[\Phi(w_m) - \lambda\Psi(w_m)] = -\infty.$$

Hence, our claim is proved. Since all assumptions of Theorem 2.1 are satisfied, the functional I_λ admits a sequence $\{u_m = (u_{1m}, ..., u_{nm})\} \subset X$ of critical points such that

$$\lim_{m\to\infty} \left\|(u_{1m}, \ldots, u_{nm})\right\| = +\infty,$$

and we have the conclusion. □

Here, we give a consequence of Theorem 3.1.

Corollary 3.2. *Assume that*

(A2) $\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\xi)} F(x,t_1,\ldots,t_n)dx}{\xi^{\underline{p}}} < \left(\sum_{i=1}^{n} (p_i C)^{\frac{1}{p_i}}\right)^{\underline{p}}$;

(A3) $\limsup_{\substack{(t_1,\ldots,t_n)\to\infty \\ (t_1,\ldots,t_n)\in\mathbb{R}_+^n}} \frac{\int_\Omega F(x,t_1,\ldots,t_n)dx}{\sum_{i=1}^{n} \frac{\|a_i\|_1 |t_i|^{p_i}}{p_i}} > 1.$

Then, the system

$$\begin{cases} -\Delta_{p_i} u_i + a_i(x)|u_i|^{p_i-2}u = F_{u_i}(x, u_1, \ldots, u_n) & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

for $1 \le i \le n$, *has an unbounded sequence of classical solutions in X.*

Now, we want to present the analogous version of the main result (Theorem 3.1) in the autonomous case.

Theorem 3.3. *Assume that*

(A4)

$$\liminf_{\xi\to+\infty} \frac{\sup_{(t_1,\ldots,t_n)\in K(\xi)} F(t_1,\ldots,t_n)}{\xi^{\underline{p}}} < \left(\sum_{i=1}^{n} (p_i C)^{\frac{1}{p_i}}\right)^{\underline{p}} \limsup_{\substack{(t_1,\ldots,t_n)\to\infty \\ (t_1,\ldots,t_n)\in\mathbb{R}_+^n}} \frac{F(t_1,\ldots,t_n)}{\sum_{i=1}^{n} \frac{\|a_i\|_1 |t_i|^{p_i}}{p_i}}$$

where $K(\xi) = \{(t_1, \ldots, t_n) | \sum_{i=1}^{n} |t_i| \le \xi\}$ *(see (3)).*

Then, for each

$$\lambda \in \Lambda := \left] \frac{1}{\limsup_{\substack{(t_1,\ldots,t_n)\to\infty \\ (t_1,\ldots,t_n)\in\mathbb{R}_+^n}} \frac{F(t_1,\ldots,t_n)}{\sum_{i=1}^{n} \frac{\|a_i\|_1 |t_i|}{p_i}}}, \frac{\left(\sum_{i=1}^{n} (p_i C)^{\frac{1}{p_i}}\right)^{\underline{p}}}{\liminf_{\xi\to+\infty} \frac{\sup_{(t_1,\ldots,t_n)\in K(\xi)} F(t_1,\ldots,t_n)}{\xi^{\underline{p}}}} \right[$$

the system

$$\begin{cases} -\Delta_{p_i} u_i + a_i(x)|u_i|^{p_i-2}u = \lambda F_{u_i}(u_1, \dots, u_n) & in \quad \Omega, \\ \dfrac{\partial u_i}{\partial \nu} = 0 & on\ \partial\Omega \end{cases}$$

has an unbounded sequence of weak solutions in X.

Proof. Set $F(x, u_1, ..., u_n) = F(u_1, ..., u_n)$ for all $x \in \Omega$ and $(u_1, ..., u_n) \in \mathbb{R}^n$. The conclusion follows from Theorem 3.1. □

Remark 3.1. We observe in Theorem 3.1 we can replace $\xi \to +\infty$ and $(t_1, ..., t_n) \to (+\infty, ..., +\infty)$ with $\xi \to 0^+$ $(t_1, ..., t_n) \to (0^+, ..., 0^+)$, respectively, that by the same way as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.1 instead of (b), the system (1) has a sequence of weak solutions, which strongly converges to 0 in X.

Finally, we give an example to illustrate the result.

Example 3.1. Let $\Omega \subset \mathbb{R}^2$ be a non-empty bounded open set with a smooth boundary $\vartheta\Omega$ and consider the increasing sequence of positive real numbers given by

$$a_n := 2, \quad a_{n+1} := n!(a_n)^{\frac{5}{4}} + 2$$

for every $n \geq 1$. Define the function

$$F(t_1, t_2) = \begin{cases} (a_{n+1})^5 e^{-\dfrac{1}{1 - [(t_1 - a_{n+1})^2 + (t_2 - a_{n+1})^2]}} & (t_1, t_2) \in \bigcup_{n \geq 1} B((a_{n+1}, a_{n+1}), 1), \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where $B((a_{n+1}, a_{n+1}), 1))$ be the open unit ball of center (a_{n+1}, a_{n+1}). We observe that the function F is non-negative, $F(0, 0) = 0$, and $F \in C^1(\mathbb{R}^2)$. We will denote by f and g, respectively, the partial derivative of F respect to t_1 and t_2. For every $n \in \mathbb{N}$, the restriction F on $B((a_{n+1}, a_{n+1}), 1)$ attains its maximum in (a_{n+1}, a_{n+1}) and $F(a_{n+1}, a_{n+1}) = (a_{n+1})^5$,

then

$$\limsup_{n \to +\infty} \frac{F(a_{n+1}, a_{n+1})}{\frac{a_{n+1}^3}{3} + \frac{a_{n+1}^4}{4}} = +\infty$$

So

$$\limsup_{(t_1,t_2) \to (+\infty,+\infty)} \frac{F(t_1, t_2)}{\frac{|t_1|^3}{3} + \frac{|t_2|^4}{4}} = +\infty$$

On the other by setting $y_n = a_{n+1} - 1$ for every $n \in \mathbb{N}$, one has

$$\sup_{(t_1,t_2) \in K(y_n)} F(t_1, t_2) = a_n^5 \quad \forall n \in \mathbb{N}$$

Then

$$\lim_{n \to \infty} \frac{\sup_{(t_1,t_2) \in K(y_n)} F(t_1, t_2)}{(a_{n+1} - 1)^3} = 0,$$

and hence

$$\liminf_{\xi\to\infty}\frac{\sup_{(t_1,t_2)\in K(\xi)}F(t_1,t_2)}{\xi^3}=0.$$

Finally

$$0=\liminf_{\xi\to+\infty}\frac{\sup_{(t_1,t_2)\in K(\xi)}F(t_1,t_2)}{\xi^3}$$
$$<\left((3C)^{\frac{1}{3}}+(4C)^{\frac{1}{4}}\right)^3\limsup_{(t_1,t_2)\to(+\infty,+\infty)_{(t_1,t_2)\in\mathbb{R}_+^n}}\frac{F(t_1,t_2)}{\frac{|t_1|^3}{3}+\frac{|t_2|^4}{4}}=+\infty.$$

So, since all assumptions of Theorem 3.3 is applicable to the system

$$\begin{cases}-\Delta_3 u+|u|\,u=\lambda f(u,v) & \text{in }\Omega,\\ -\Delta_4 v+|v|^2 g=\lambda g(u,v) & \text{in }\Omega,\\ \dfrac{\partial u}{\partial \nu}=\dfrac{\partial v}{\partial \nu}=0 & \text{on}\partial\Omega\end{cases}$$

for every $\lambda\in\ [0,+\infty[$.

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Authors' contributions
DMS has presented the main purpose of the article and has used GAA contribution due to reaching to conclusions. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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