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Solvability of right focal boundary value problems with superlinear growth conditions

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Abstract

In this paper, we consider nth-order two-point right focal boundary value problems

$$u^{(n)}(t) = f\big(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\big), \quad \text{a.e. } t \in (0,1),$$
$$u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, m-1,$$
$$u^{(i)}(1) = 0, \quad i = m, m+1, \ldots, n-1,$$

where $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^p-Carathéodory ($1 \le p < \infty$) function and satisfies superlinear growth conditions. The existence and uniqueness of solutions for the above right focal boundary value problems are obtained by Leray-Schauder continuation theorem and analytical technique. Meanwhile, as an application of our results, examples are given.

MSC: 34B15

Keywords: right focal boundary value problem; Leray-Schauder continuation theorem; existence; uniqueness

1 Introduction

In this paper, we shall discuss the existence and uniqueness of solutions of right focal boundary value problems for nth-order nonlinear differential equation

$$u^{(n)}(t) = f\big(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\big), \quad \text{a.e. } t \in (0,1) \tag{1.1}$$

subject to the boundary conditions ($1 \le m \le n-1$)

$$\begin{cases} u^{(i)}(0) = 0, & i = 0, 1, \ldots, m-1, \\ u^{(i)}(1) = 0, & i = m, m+1, \ldots, n-1, \end{cases} \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^n \to \mathbb{R} = (-\infty, +\infty)$ satisfies the L^p-Carathéodory ($1 \le p < \infty$) conditions, that is,

(i) for each $(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}^n$, the function $t \in [0,1] \mapsto f(t, u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}$ is measurable on $[0,1]$;

(ii) for a.e. $t \in [0,1]$, the function $(u_0, u_1, \ldots, u_{n-1}) \mapsto f(t, u_0, u_1, \ldots, u_{n-1})$ is continuous on $\mathbb{R}^n$;

(iii) for each $r > 0$, there exists an $\alpha_r \in L^p[0,1]$ such that $|f(t, u_0, u_1, \ldots, u_{n-1})| \le \alpha_r$ for a.e. $t \in [0,1]$ and all $(u_0, u_1, \ldots, u_{n-1}) \in \mathbb{R}^n$ with $\sum_{j=0}^{n-1} u_j^2 \le r^2$.

As it is well known, the right focal boundary value problems have attracted many scholars' attention. Among a substantial number of works dealing with right focal boundary value problems, we mention [1–16, 18–25].

Recently, using the Leray-Schauder continuation theorem, Hopkins and Kosmatov [16] have obtained sufficient conditions for the existence of at least one sign-changing solution for third-order right focal boundary value problems such as

$$\begin{cases} u'''(t) = f\bigl(t, u(t), u'(t), u''(t)\bigr), & \text{a.e. } t \in (0,1), \\ u(0) = u'(0) = u''(1) = 0 \end{cases}$$

and

$$\begin{cases} u'''(t) = f\bigl(t, u(t), u'(t), u''(t)\bigr), & \text{a.e. } t \in (0,1), \\ u(0) = u'(1) = u''(1) = 0, \end{cases}$$

where $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the L^p-Carathéodory ($1 \le p < \infty$) conditions and the linear growth conditions.

Motivated by [16], in this paper we study the solvability for general nth-order right focal boundary value problems (1.1), (1.2). The existence and uniqueness of sign-changing solutions for the problems are obtained by Leray-Schauder continuation theorem and analytical technique. We note that the nonlinearity of f in our problem allows up to the superlinear growth conditions.

The rest of this paper is organized as follows. In Section 2, we give some lemmas which help to simplify the proofs of our main results. In Section 3, we discuss the existence and uniqueness of sign-changing solutions for nth-order right focal boundary value problems (1.1), (1.2) by Leray-Schauder continuation theorem and analytical technique, and give two examples to demonstrate our results. Our results improve and generalize the corresponding results in [16].

2 Preliminary

In this section, we give some lemmas which help to simplify the presentation of our main results.

Let $AC[0,1]$ denote the space of absolutely continuous functions on $[0,1]$, and $C^{n-1}[0,1]$ denote the Banach space of $(n-1)$ times continuously differentiable functions defined on $[0,1]$ with the norm $\|u\|_{C^{n-1}} = \max\{\|u^{(i)}\|_\infty, i = 0,1,\ldots,n-1\}$, where $\|u^{(i)}\|_\infty = \sup_{t\in[0,1]} |u^{(i)}(t)|$. Let $L^p[0,1]$ be the usual Lebesgue space on $[0,1]$ with norm $\|\cdot\|_p$, $1 \le p < \infty$.

For $1 \le p < \infty$, we introduce the Sobolev space

$$W^{n,p}(0,1) = \left\{ u : [0,1] \to \mathbb{R} \;\middle|\; \begin{array}{ll} u^{(i)} \in AC[0,1], & i = 0,1,\ldots,n-1, \\ u^{(n)} \in L^p[0,1] & \end{array} \right\}$$

with the norm $\|u\| = \|u\|_{C^{n-1}} + \|u^{(n)}\|_p$. Let us consider a special subspace

$$W_r^{n,p}(0,1) = \bigl\{ u \in W^{n,p}(0,1) : u \text{ satisfies (1.2)} \bigr\}.$$

Then it is clear that $W_r^{n,p}(0,1)$ is closed in $W^{n,p}(0,1)$ and hence is itself a Banach space with the norm $\|u\| = \|u\|_{C^{n-1}} + \|u^{(n)}\|_p$.

Lemma 2.1 ([21]) *Let $G(t,s)$ be the Green's function of the differential equation* $(-1)^{n-m} \times u^{(n)}(t) = 0$ *subject to the boundary conditions (1.2). Then*

$$G(t,s) = \frac{(-1)^{n-m}}{(n-1)!}\begin{cases} \sum_{i=0}^{m-1}\binom{n-1}{i}t^i(-s)^{n-i-1}, & 0 \le s \le t \le 1, \\ -\sum_{i=m}^{n-1}\binom{n-1}{i}t^i(-s)^{n-i-1}, & 0 \le t \le s \le 1 \end{cases}$$

and

$$\frac{\partial^i}{\partial t^i}G(t,s) \ge 0, \quad \forall (t,s) \in [0,1] \times [0,1], i = 0,1,\ldots,m.$$

Lemma 2.2 *Let $g \in L^p[0,1]$. Then the solution of the differential equation*

$$u^{(n)}(t) = g(t), \quad a.e.\ t \in (0,1)$$

subject to the boundary conditions (1.2) satisfies

$$\|u^{(j)}\|_\infty \le A_j\|g\|_p, \quad j = 0,1,\ldots,n-1, \tag{2.1}$$

where for $p > 1$ ($\frac{1}{p} + \frac{1}{q} = 1$),

$$A_j = \begin{cases} \frac{(-1)^{(n-m)}}{(n-j-1)!}\left[\int_0^1\left(\sum_{i=0}^{m-j-1}\binom{n-j-1}{i}(-s)^{n-j-1-i}\right)^q \mathrm{d}s\right]^{\frac{1}{q}}, & j = 0,1,\ldots,m-1, \\ \frac{1}{(n-j-1)![q(n-j-1)+1]^{\frac{1}{q}}}, & j = m,m+1,\ldots,n-1 \end{cases} \tag{2.2}$$

and for $p = 1$,

$$A_j = \begin{cases} \frac{(-1)^{n-m}}{(n-j-1)!}\sum_{i=0}^{m-j-1}\binom{n-j-1}{i}(-1)^{n-j-1-i}, & j = 0,1,\ldots,m-1, \\ \frac{1}{(n-j-1)!}, & j = m,m+1,\ldots,n-1. \end{cases} \tag{2.3}$$

Proof Firstly, let us show the lemma for case $p > 1$. Since

$$u(t) = (-1)^{n-m}\int_0^1 G(t,s)g(s)\,\mathrm{d}s,$$

we have that for $j = 0,1,\ldots,n-1$,

$$u^{(j)}(t) = (-1)^{n-m}\int_0^1 \frac{\partial^j}{\partial t^j}G(t,s)g(s)\,\mathrm{d}s =: (-1)^{n-m}\int_0^1 G_j(t,s)g(s)\,\mathrm{d}s,$$

where, for $j = 0, 1, \ldots, m-1$,

$$G_j(t,s) = \frac{(-1)^{n-m}}{(n-1)!} \begin{cases} \sum_{i=j}^{m-1} \binom{n-1}{i} \frac{i!}{(i-j)!} t^{i-j}(-s)^{n-i-1}, & 0 \le s \le t \le 1, \\ -\sum_{i=m}^{n-1} \binom{n-1}{i} \frac{i!}{(i-j)!} t^{i-j}(-s)^{n-i-1}, & 0 \le t \le s \le 1 \end{cases}$$

$$= \frac{(-1)^{n-m}}{(n-j-1)!} \begin{cases} \sum_{i=0}^{m-j-1} \binom{n-j-1}{i} t^{i}(-s)^{n-j-1-i}, & 0 \le s \le t \le 1, \\ -\sum_{i=m-j}^{n-j-1} \binom{n-j-1}{i} t^{i}(-s)^{n-j-1-i}, & 0 \le t \le s \le 1 \end{cases}$$

and for $j = m, m+1, \ldots, n-1$,

$$G_j(t,s) = \frac{(-1)^{n-m}}{(n-1)!} \begin{cases} 0, & 0 \le s \le t \le 1, \\ -\sum_{i=j}^{n-1} \binom{n-1}{i} \frac{i!}{(i-j)!} t^{i-j}(-s)^{n-i-1}, & 0 \le t \le s \le 1 \end{cases}$$

$$= \frac{(-1)^{n-m}}{(n-j-1)!} \begin{cases} 0, & 0 \le s \le t \le 1, \\ -(t-s)^{n-j-1}, & 0 \le t \le s \le 1. \end{cases}$$

It follows by Hölder's inequality that, for each $j = 0, 1, \ldots, n-1$,

$$\begin{aligned} \left|u^{(j)}(t)\right| &\le \int_0^1 \left|G_j(t,s)\right| \left|g(s)\right| \mathrm{d}s \\ &\le \|g\|_p \left\|G_j(t,\cdot)\right\|_q \\ &\le \|g\|_p \max_{t\in[0,1]} \left\|G_j(t,\cdot)\right\|_q, \quad t \in [0,1] \end{aligned}$$

and consequently, for each $j = 0, 1, \ldots, n-1$,

$$\left\|u^{(j)}\right\|_\infty \le \|g\|_p \max_{t\in[0,1]} \left\|G_j(t,\cdot)\right\|_q, \quad t \in [0,1]. \tag{2.4}$$

But for $j = m, m+1, \ldots, n-1$,

$$\begin{aligned} \max_{t\in[0,1]} \left\|G_j(t,\cdot)\right\|_q^q &= \max_{t\in[0,1]} \int_0^1 \left|G_j(t,s)\right|^q \mathrm{d}s \\ &= \max_{t\in[0,1]} \int_0^t \left|G_j(t,s)\right|^q \mathrm{d}s + \max_{t\in[0,1]} \int_t^1 \left|G_j(t,s)\right|^q \mathrm{d}s \\ &= \max_{t\in[0,1]} \int_t^1 \left|\frac{(-1)^{n-m}}{(n-j-1)!}\left[-(t-s)^{n-j-1}\right]\right|^q \mathrm{d}s \\ &= \frac{1}{[(n-j-1)!]^q} \max_{t\in[0,1]} \int_t^1 (s-t)^{q(n-j-1)} \mathrm{d}s \\ &= \frac{1}{[(n-j-1)!]^q} \max_{t\in[0,1]} \frac{(1-t)^{q(n-j-1)+1}}{q(n-j-1)+1} \end{aligned}$$

$$\begin{aligned}&= \frac{1}{[(n-j-1)!]^q[q(n-j-1)+1]}\\&= A_j^q.\end{aligned}$$

It follows by (2.4) that for $j = m, m+1, \dots, n-1$,

$$\|u^{(j)}\|_\infty \le A_j\|g\|_p.$$

For $j = 0, 1, \dots, m-1$, by Lemma 2.1, $G_j(t,s)$ is nondecreasing in t, and thus

$$\begin{aligned}\max_{t\in[0,1]}\|G_j(t,\cdot)\|_q^q &= \max_{t\in[0,1]}\int_0^1 \bigl[G_j(t,s)\bigr]^q\,\mathrm{d}s\\&\le \int_0^1\Bigl[\max_{t\in[0,1]} G_j(t,s)\Bigr]^q\,\mathrm{d}s\\&= \int_0^1\bigl[G_j(1,s)\bigr]^q\,\mathrm{d}s\\&= \int_0^1\left[\frac{(-1)^{n-m}}{(n-j-1)!}\sum_{i=0}^{m-j-1}\binom{n-j-1}{i}(-s)^{n-j-1-i}\right]^q\mathrm{d}s\\&= A_j^q.\end{aligned}$$

Hence, by (2.4) we have for $j = 0, 1, \dots, m-1$,

$$\|u^{(j)}\|_\infty \le A_j\|g\|_p.$$

In summary,

$$\|u^{(j)}\|_\infty \le A_j\|g\|_p, \quad j = 0, 1, \dots, n-1.$$

Next, we show the lemma for the case $p = 1$. It is easy to see that for $j = m, m+1, \dots, n-1$,

$$\begin{aligned}\bigl|u^{(j)}(t)\bigr| &\le \int_0^1\bigl|G_j(t,s)\bigr|\bigl|g(s)\bigr|\,\mathrm{d}s\\&= \int_t^1\left|\frac{(-1)^{n-m}}{(n-j-1)!}\bigl[-(t-s)^{n-j-1}\bigr]\right|\bigl|g(s)\bigr|\,\mathrm{d}s\\&= \frac{1}{(n-j-1)!}\int_t^1 (s-t)^{n-j-1}\bigl|g(s)\bigr|\,\mathrm{d}s\\&\le \frac{(1-t)^{n-j-1}}{(n-j-1)!}\int_t^1\bigl|g(s)\bigr|\,\mathrm{d}s\\&\le \frac{1}{(n-j-1)!}\|g\|_1\\&= A_j\|g\|_1, \quad t\in[0,1]\end{aligned}$$

and thus for $j = m, m+1, \dots, n-1$,

$$\|u^{(j)}\|_\infty \le A_j\|g\|_1.$$

Also by Lemma 2.1, we have for $j = 0, 1, \dots, m$,

$$G_j(t,s) \geq 0, \quad \forall (t,s) \in [0,1] \times [0,1],$$

so that for each $j = 0, 1, \dots, m-1$, $G_j(t,s)$ is nondecreasing in t, it follows that

$$\begin{aligned}
|u^{(j)}(t)| &\leq \int_0^1 |G_j(t,s)||g(s)|\,\mathrm{d}s \\
&\leq \int_0^1 \max_{t\in[0,1]} G_j(t,s)|g(s)|\,\mathrm{d}s \\
&= \int_0^1 G_j(1,s)|g(s)|\,\mathrm{d}s \\
&= \int_0^1 \left[\frac{(-1)^{n-m}}{(n-j-1)!} \sum_{i=0}^{m-j-1} \binom{n-j-1}{i} (-s)^{n-j-1-i} \right] |g(s)|\,\mathrm{d}s.
\end{aligned} \tag{2.5}$$

Let

$$\phi(t) = \frac{(-1)^{n-m}}{(n-j-1)!} \sum_{i=0}^{m-j-1} \binom{n-j-1}{i} (-t)^{n-j-1-i}, \quad t \in [0,1].$$

Then

$$\begin{aligned}
\phi^{(n-m)}(t) &= \frac{(-1)^{n-m}}{(n-j-1)!} \sum_{i=0}^{m-j-1} \binom{n-j-1}{i} (-1)^{n-j-1-i} (n-j-1-i) \\
&\quad \cdot (n-j-1-i-1) \cdots (n-j-1-i-n+m+1) t^{m-j-1-i} \\
&= \frac{(-1)^{n-m}}{(n-j-1)!} \sum_{i=0}^{m-j-1} \frac{(n-j-1)!}{i!(m-j-1-i)!} (-1)^{n-j-1-i} t^{m-j-1-i} \\
&= \frac{1}{(m-j-1)!} \sum_{i=0}^{m-j-1} \frac{(m-j-1)!}{i!(m-j-1-i)!} (-t)^{m-j-1-i} \\
&= \frac{1}{(m-j-1)!} (1-t)^{m-j-1} \geq 0, \quad t \in [0,1].
\end{aligned}$$

Since

$$\phi^{(k)}(0) = 0, \quad k = n-m-1, n-m-2, \dots, 2, 1,$$

we have for each $k = n-m-1, n-m-2, \dots, 2, 1$,

$$\phi^{(k)}(t) \geq 0, \quad t \in [0,1],$$

in particular

$$\phi'(t) \geq 0, \quad t \in [0,1],$$

so that $\phi(t)$ is nondecreasing on $[0,1]$. Hence by (2.5), we have

$$\begin{aligned}\left|u^{(j)}(t)\right| &\le \int_0^1 \left[\frac{(-1)^{n-m}}{(n-j-1)!}\sum_{i=0}^{m-j-1}\binom{n-j-1}{i}(-1)^{n-j-1-i}\right]\left|g(s)\right|\,\mathrm{d}s \\ &= \frac{(-1)^{n-m}}{(n-j-1)!}\sum_{i=0}^{m-j-1}\binom{n-j-1}{i}(-1)^{n-j-1-i}\int_0^1\left|g(s)\right|\,\mathrm{d}s \\ &= A_j\|g\|_1.\end{aligned}$$

Thus for $j = 0,1,\ldots,m-1$,

$$\left\|u^{(j)}\right\|_\infty \le A_j\|g\|_1.$$

In summary,

$$\left\|u^{(j)}\right\|_\infty \le A_j\|g\|_1, \quad j = 0,1,\ldots,n-1. \qquad \square$$

Lemma 2.3 ([17] Leray-Schauder continuation theorem) *Let X be a real Banach space and let Ω be a bounded open neighbourhood of* 0 *in X. Let $T : \bar{\Omega} \to X$ be a completely continuous operator such that for all $\lambda \in (0,1)$, and $u \in \partial\Omega$, $u \neq \lambda Tu$. Then the operator equation*

$$u = Tu$$

has a solution $u \in \bar{\Omega}$.

3 Main results

Now we are ready to establish our existence theorems of solutions for nth-order right focal boundary value problems (1.1), (1.2). The Leray-Schauder continuation theorem plays key roles in the proofs.

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfy L^p-Carathéodory's conditions. Suppose that*

(i) there exist functions $\alpha_j(t), \beta_j(t), \gamma(t) \in L^p[0,1]$, $j = 0,1,\ldots,n-1$, and a constant $\sigma > 1$ such that

$$\left|f(t,u_0,u_1,\ldots,u_{n-1})\right| \le \sum_{j=0}^{n-1}\alpha_j(t)|u_j| + \sum_{j=0}^{n-1}\beta_j(t)|u_j|^\sigma + \gamma(t) \tag{3.1}$$

for a.e. $t \in [0,1]$ and all $(u_0,u_1,\ldots,u_{n-1}) \in \mathbb{R}^n$;

(ii)

$$a := 1 - \sum_{j=0}^{n-1}A_j\|\alpha_j\|_p > 0, \tag{3.2}$$

where the constants A_j, $j = 0,1,\ldots,n-1$ are given in Lemma 2.2;

(iii)

$$a^{\frac{\sigma}{\sigma-1}}\left(\sigma^{\frac{\sigma}{1-\sigma}}-\sigma^{\frac{1}{1-\sigma}}\right)+b^{\frac{1}{\sigma-1}}\|\gamma\|_p<0, \tag{3.3}$$

where $b:=\sum_{j=0}^{n-1}A_j^{\sigma}\|\beta_j\|_p$.

Then BVP (1.1), (1.2) has at least one solution in $W^{n,p}(0,1)$.

Proof We define a linear mapping $L:W_r^{n,p}(0,1)\subset W^{n,p}(0,1)\to L^p[0,1]$, by setting for $u\in W_r^{n,p}(0,1)$,

$$(Lu)(t)=u^{(n)}(t).$$

We also define a nonlinear mapping $N:W_r^{n,p}(0,1)\to L^p[0,1]$ by setting for $y\in W_r^{n,p}(0,1)$,

$$(Nu)(t)=f\left(t,u(t),u'(t),\dots,u^{(n-1)}(t)\right).$$

Then, we note that N is a bounded continuous mapping by Lebesgue's dominated convergence theorem. It is easy to see that the linear mapping $L:W_r^{n,p}(0,1)\to L^p[0,1]$ is a one-to-one mapping. Also, let the linear mapping $K:L^p[0,1]\to W_r^{n,p}(0,1)$ for $u\in L^p[0,1]$ be defined by

$$(Ku)(t)=(-1)^{n-m}\int_0^1 G(t,s)u(s)\,\mathrm{d}s,$$

where $G(t,s)$ is the Green's function of BVP in Lemma 2.1.

Then K satisfies that for $u\in L^p[0,1]$, $Ku\in W_r^{n,p}(0,1)$ and $LKu=u$, and also for $u\in W_r^{n,p}(0,1)$, $KLu=u$. Furthermore, it follows easily by using Arzelà-Ascoli theorem that $KN:W_r^{n,p}(0,1)\to W_r^{n,p}(0,1)$ is a completely continuous operator.

Here we also note that $u\in W_r^{n,p}(0,1)$ is a solution of BVP (1.1), (1.2) if and only if $u\in W_r^{n,p}(0,1)$ is a solution of the operator equation

$$Lu=Nu$$

which is equivalent to the operator equation

$$u=KNu.$$

We now apply the Leray-Schauder continuation theorem to the operator equation $u=KNu$. To do this, it is sufficient to verify that the set of all possible solutions of the family of equations

$$u^{(n)}(t)=\lambda f\left(t,u(t),u'(t),\dots,u^{(n-1)}(t)\right),\quad 0<t<1 \tag{3.4}$$

with boundary conditions

$$\begin{cases}u^{(i)}(0)=0, & i=0,1,\dots,m-1,\\ u^{(i)}(1)=0, & i=m,m+1,\dots,n-1\end{cases} \tag{3.5}$$

is, a priori, bounded in $W_r^{n,p}(0,1)$ by a constant independent of $\lambda \in (0,1)$.

Suppose $u(t) \in W_r^{n,p}(0,1)$ is a solution of BVP (3.4), (3.5) for some $\lambda \in (0,1)$. Then from (3.4), (3.1) and (2.2) in Lemma 2.2, we obtain

$$\begin{aligned}
\|u^{(n)}\|_p &= \lambda \|f(t,u(t),u'(t),\dots,u^{(n-1)}(t))\|_p \\
&\le \|f(t,u(t),u'(t),\dots,u^{(n-1)}(t))\|_p \\
&\le \sum_{j=0}^{n-1} \|\alpha_j u^{(j)}\|_p + \sum_{j=0}^{n-1} \|\beta_j (u^{(j)})^\sigma\|_p + \|\gamma\|_p \\
&\le \sum_{j=0}^{n-1} \|\alpha_j\|_p \|u^{(j)}\|_\infty + \sum_{j=0}^{n-1} \|\beta_j\|_p \|u^{(j)}\|_\infty^\sigma + \|\gamma\|_p \\
&\le \sum_{j=0}^{n-1} A_j \|\alpha_j\|_p \|u^{(n)}\|_p + \sum_{j=0}^{n-1} A_j^\sigma \|\beta_j\|_p \|u^{(n)}\|_p^\sigma + \|\gamma\|_p \\
&= (1-a)\|u^{(n)}\|_p + b\|u^{(n)}\|_p^\sigma + \|\gamma\|_p.
\end{aligned}$$

Consequently we obtain

$$b\|u^{(n)}\|_p^\sigma - a\|u^{(n)}\|_p + \|\gamma\|_p \ge 0. \tag{3.6}$$

Now we have two cases to consider:

Case 1. $b = 0$. In this case (3.6) becomes $-a\|u^{(n)}\|_p + \|\gamma\|_p \ge 0$, i.e. $\|u^{(n)}\|_p \le \frac{\|\gamma\|_p}{a}$. Thus from (2.1) in Lemma 2.2, we have that there exists a constant $M > 0$ which is independent of $\lambda \in (0,1)$ such that

$$\begin{aligned}
\|u\| &= \max\{\|u^{(j)}\|_\infty, j = 0,1,\dots,n-1\} + \|u^{(n)}\|_p \\
&\le \max\{A_j, j = 0,1,\dots,n-1\}\|u^{(n)}\|_p + \|u^{(n)}\|_p \\
&\le (1 + \max\{A_j, j = 0,1,\dots,n-1\}) \frac{\|\gamma\|_p}{a} \\
&=: M.
\end{aligned} \tag{3.7}$$

Now, let

$$\Omega = \{u \in W_r^{n,p}(0,1) : \|u\| < M + 1\}.$$

Then estimate (3.7) show that λKN has no fixed point on $\partial\Omega$. Hence KN has a fixed point in $\bar{\Omega}$ by the Leray-Schauder continuation theorem.

Case 2. $b > 0$. When $\|\gamma\|_p = 0$ in (3.1), it is easy to see that BVP (1.1), (1.2) has the trivial solution $u \equiv 0$. Thus assume $\|\gamma\|_p > 0$ and let $h(t) = bt^\sigma - at + \|\gamma\|_p$, $t \ge 0$. Then from (3.6), $h(\|u^{(n)}\|_p) \ge 0$. It is easy to see that $h'(t) = 0$ has a unique positive solution $(\frac{a}{b\sigma})^{\frac{1}{\sigma-1}}$, say ρ^*. By (3.3), we have $h(\rho^*) < 0$ and thus $h(t) = 0$ has a minimum positive solution, say $\bar{\rho}$ which is less than ρ^* and independent of $\lambda \in (0,1)$. Hence it follows that if $\|u^{(n)}\|_p \le \rho^*$, then

$$\|u^{(n)}\|_p \le \bar{\rho} < \rho^*. \tag{3.8}$$

From (2.1) in Lemma 2.2, we get

$$\begin{aligned}\|u\| &= \max\left\{\left\|u^{(j)}\right\|_{\infty}, j = 0,1,\ldots,n-1\right\} + \left\|u^{(n)}\right\|_{p} \\ &\le \left(1 + \max\{A_j, j = 0,1,\ldots,n-1\}\right)\left\|u^{(n)}\right\|_{p}.\end{aligned} \tag{3.9}$$

Now, we let

$$\Omega = \left\{u \in W_r^{n,p}(0,1) : \|u\| < M+1, \left\|u^{(n)}\right\|_p < \rho^*\right\},$$

where $M = (1 + \max\{A_j, j = 0,1,\ldots,n-1\})\rho^*$. Then estimates (3.8) and (3.9) show that λKN has no fixed point on $\partial\Omega$. Consequently, KN has a fixed point in $\bar{\Omega}$ by the Leray-Schauder continuation theorem. This completes the proof of the theorem. □

Corollary 3.1 *Let conditions (i) and (ii) of Theorem 3.1 hold. If* $b = 0$ *or* $b > 0$ *is small enough, then BVP (1.1), (1.2) has at least one solution in* $W^{n,p}(0,1)$.

Corollary 3.2 *Let conditions (i) and (ii) of Theorem 3.1 hold. If* $\|\gamma\|_p > 0$ *is small enough, then BVP (1.1), (1.2) has at least one solution in* $W^{n,p}(0,1)$.

Remark 3.1 Theorem 3.1-3.4 in [16] are special cases of above Theorem 3.1.

Next, we give some results on the uniqueness of solutions for BVP (1.1), (1.2).

Theorem 3.2 *Let* $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ *satisfy* L^p*-Carathéodory's conditions. Suppose that*

(i) there exist functions $\alpha_j(t), \beta_j(t) \in L^p[0,1]$, $j = 0,1,\ldots,n-1$, *and a constant* $\sigma > 1$ *such that*

$$\begin{aligned}&\left|f(t,u_0,u_1,\ldots,u_{n-1}) - f(t,v_0,v_1,\ldots,v_{n-1})\right| \\ &\quad\le \sum_{j=0}^{n-1}\alpha_j(t)|u_j - v_j| + \sum_{j=0}^{n-1}\beta_j(t)|u_j - v_j|^{\sigma}\end{aligned} \tag{3.10}$$

for a.e. $t \in [0,1]$ *and all* $(u_0,u_1,\ldots,u_{n-1}), (v_0,v_1,\ldots,v_{n-1}) \in \mathbb{R}^n$;

(ii)

$$a := 1 - \sum_{j=0}^{n-1} A_j\|\alpha_j\|_p > 0, \tag{3.11}$$

where the constants A_j, $j = 0,1,\ldots,n-1$ *are given in Lemma 2.2;*

(iii)

$$a^{\frac{\sigma}{\sigma-1}}\left(\sigma^{\frac{\sigma}{1-\sigma}} - \sigma^{\frac{1}{1-\sigma}}\right) + b^{\frac{1}{\sigma-1}}\left\|f(t,0,\ldots,0)\right\|_p < 0, \tag{3.12}$$

where $b := \sum_{j=0}^{n-1} A_j^{\sigma}\|\beta_j\|_p$.

Then BVP (1.1), (1.2) has at least one solution $u(t) \in W^{n,p}(0,1)$ *and in particular has at most one solution* $u(t) \in W^{n,p}(0,1)$ *with* $\|u^{(n)}\|_p < \frac{1}{2}(\frac{a}{b})^{\frac{1}{\sigma-1}}$.

Proof We note that assumption (3.10) implies

$$\left|f(t,u_0,u_1,\ldots,u_{n-1})\right| \le \sum_{j=0}^{n-1}\alpha_j(x)|u_j| + \sum_{j=0}^{n-1}\beta_j(x)|u_j|^\sigma + \left|f(t,0,\ldots,0)\right|$$

for a.e. $x\in[0,1]$ and all $(u_0,u_1,\ldots,u_{n-1})\in\mathbb{R}^n$. Accordingly from Theorem 3.1, BVP (1.1), (1.2) has at least one solution in $W^{n,p}(0,1)$.

Now, suppose that $u_1(t)$, $u_2(t)$ are two solutions of BVP (1.1), (1.2) with $\|u_i^{(n-1)}\|_\infty < \frac{1}{2}(\frac{a}{b})^{\frac{1}{\sigma-1}}$, $i=1,2$. Let $w(t)=u_1(t)-u_2(t)$. Then $w(t)$ satisfies the boundary condition (1.2) and

$$\left|w^{(n)}(t)\right| \le \sum_{j=0}^{n-1}\alpha_j(t)\left|w^{(j)}(t)\right| + \sum_{j=0}^{n-1}\beta_j(t)\left|w^{(j)}(t)\right|^\sigma.$$

Similarly to the proof of Theorem 3.1, we can show easily that

$$\left\|w^{(n)}\right\|_p \le (1-a)\left\|w^{(n)}\right\|_p + b\left\|w^{(n)}\right\|_p^\sigma,$$

which gives

$$b\left\|w^{(n)}\right\|_p^\sigma - a\left\|w^{(n)}\right\|_p \ge 0. \tag{3.13}$$

Now consider two cases. If $b=0$, then $\|w^{(n)}\|_p=0$ from (3.13). Since $\|w\|_\infty\le A_0\|w^{(n)}\|_p$, we have $w(t)\equiv 0$ on $[0,1]$, i.e., $u_1(t)\equiv u_2(t)$ on $[0,1]$.

If $b>0$, let $h(t)=bt^\sigma-at$. Then $h(\|w^{(n)}\|_p)\ge 0$ from (3.13). It follows that $h(0)=h((\frac{a}{b})^{\frac{1}{\sigma-1}})=0$ and $h(t)<0$ on $(0,(\frac{a}{b})^{\frac{1}{\sigma-1}})$. Since $\|w^{(n)}\|_p\le\|u_1^{(n)}\|_p+\|u_2^{(n)}\|_p<(\frac{a}{b})^{\frac{1}{\sigma-1}}$, we get $\|w^{(n)}\|_p=0$. Consequently, $u_1(t)\equiv u_2(t)$ on $[0,1]$. This completes the proof of the theorem. □

Corollary 3.3 *Let conditions (i) and (ii) of Theorem 3.2 hold. If $b=0$, then BVP (1.1), (1.2) has exactly one solution in $W^{n,p}(0,1)$.*

Finally, we give two examples to which our results can be applicable.

Example 3.1 Consider the boundary value problem

$$\begin{cases} u''' = \dfrac{1}{16}t^{-\frac{1}{4}} + t^{-\frac{1}{3}}u^{\frac{1}{3}}\left(u'\right)^{\frac{2}{3}} + \dfrac{1}{10}\left(u''\right)^2, & \text{a.e. } t\in(0,1), \\ u(0)=u'(1)=u''(1)=0. \end{cases}$$

Let $f(t,u_0,u_1,u_2)=\frac{1}{16}t^{-\frac{1}{4}}+t^{-\frac{1}{3}}u_0^{\frac{1}{3}}u_1^{\frac{2}{3}}+\frac{1}{10}u_2^2$. Then it is easy to see that f satisfies L^2-Carathéodory's conditions. By the inequality $A^{\frac{1}{p}}B^{\frac{1}{q}}\le\frac{A}{p}+\frac{B}{q}$ for any $A,B>0$ with $p,q>0$ and $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\left|f(t,u_0,u_1,u_2)\right| \le \frac{1}{16}t^{-\frac{1}{4}} + \frac{1}{3}t^{-\frac{1}{3}}|u_0| + \frac{2}{3}t^{-\frac{1}{3}}|u_1| + \frac{1}{10}u_2^2.$$

Let $\alpha_0(t) = \frac{1}{3}t^{-\frac{1}{3}}$, $\alpha_1(t) = \frac{2}{3}t^{-\frac{1}{3}}$, $\alpha_2(t) = 0$, $\beta_0(t) = \beta_1(t) = 0$, $\beta_2(t) = \frac{1}{10}$, $\gamma(t) = \frac{1}{16}t^{-\frac{1}{4}}$, $\sigma = 2$. Then we have

$$\left|f(t,u_0,u_1,u_2)\right| \le \sum_{j=0}^{2}\alpha_j(t)|u_j| + \sum_{j=0}^{2}\beta_j(t)|u_j|^{\sigma} + \gamma(t).$$

It is easy to compute that

$$\begin{aligned}
&\|\alpha_0\|_2 = \frac{\sqrt{3}}{3}, \qquad \|\alpha_1\|_2 = \frac{2\sqrt{3}}{3}, \qquad \|\alpha_2\|_2 = 0, \qquad \|\gamma\|_2 = \frac{\sqrt{2}}{16},\\
&\|\beta_0\|_2 = 0, \qquad \|\beta_1\|_2 = 0, \qquad \|\beta_2\|_2 = \frac{1}{10},\\
&A_0 = \frac{\sqrt{5}}{10}, \qquad A_1 = \frac{\sqrt{3}}{3}, \qquad A_2 = 1.
\end{aligned}$$

Consequently, we have

$$a = 1 - \sum_{j=0}^{2} A_j\|\alpha_j\|_p = \frac{1}{3} - \frac{\sqrt{15}}{30} > 0, \qquad b = \sum_{j=0}^{2} A_j^{\sigma}\|\beta_j\|_2 = \frac{1}{10},$$

and

$$a^{\frac{\sigma}{\sigma-1}}\left(\sigma^{\frac{\sigma}{1-\sigma}} - \sigma^{\frac{1}{1-\sigma}}\right) + b^{\frac{1}{\sigma-1}}\|\gamma\|_2 = -\left(\frac{1}{3} - \frac{\sqrt{15}}{30}\right)^2 \cdot \frac{1}{4} + \frac{\sqrt{2}}{160} < 0.$$

Thus by Theorem 3.1, the above boundary value problem has at least one solution in $W^{3,2}(0,1)$.

Example 3.2 Consider the boundary value problem

$$\begin{cases} u''' = \dfrac{1}{32}t^{-\frac{1}{4}} + \dfrac{\sqrt{3}}{8}t^{-\frac{1}{3}}\sin\left(4u + u'\right) + \dfrac{\sqrt{2}}{8}g\left(u''\right), \quad \text{a.e. } t \in (0,1),\\ u(0) = u'(1) = u''(1) = 0,\end{cases}$$

where

$$g(u_2) = \begin{cases} \sqrt{u_2^2 - 1}, & u_2 \ge \sqrt{2},\\ \dfrac{1}{2}u_2^2, & 0 \le u_2 \le \sqrt{2},\\ -\dfrac{1}{2}u_2^2, & -\sqrt{2} \le u_2 \le 0,\\ -\sqrt{u_2^2 - 1}, & u_2 \le -\sqrt{2}.\end{cases}$$

Let $f(t,u_0,u_1,u_2) = \frac{1}{32}t^{-\frac{1}{4}} + \frac{\sqrt{3}}{8}t^{-\frac{1}{3}}\sin(4u_0 + u_1) + \frac{\sqrt{2}}{8}g(u'')$. Then it is easy to see that f satisfies L^2-Carathéodory's conditions and

$$\begin{aligned}
&\left|f(t,u_0,u_1,u_2) - f(t,v_0,v_1,v_2)\right|\\
&\quad \le \frac{\sqrt{3}}{2}t^{-\frac{1}{3}}|u_0 - v_0| + \frac{\sqrt{3}}{8}t^{-\frac{1}{3}}|u_1 - v_1| + \frac{1}{4}|u_2 - v_2| + \frac{\sqrt{2}}{16}|u_2 - v_2|^2.
\end{aligned}$$

Let $\alpha_0(t) = \frac{\sqrt{3}}{2}t^{-\frac{1}{3}}$, $\alpha_1(t) = \frac{\sqrt{3}}{8}t^{-\frac{1}{3}}$, $\alpha_2(t) = \frac{1}{4}$, $\beta_0(t) = \beta_1(t) = 0$, $\beta_2(t) = \frac{\sqrt{2}}{16}$. Then it is easy to compute that

$$\begin{aligned}
&\|\alpha_0\|_2 = \frac{3}{2}, \qquad \|\alpha_1\|_2 = \frac{3}{8}, \qquad \|\alpha_2\|_2 = \frac{1}{4}, \\
&\|\beta_0\|_2 = 0, \qquad \|\beta_1\|_2 = 0, \qquad \|\beta_2\|_2 = \frac{\sqrt{2}}{16}, \\
&A_0 = \frac{\sqrt{5}}{10}, \qquad A_1 = \frac{\sqrt{3}}{3}, \qquad A_2 = 1.
\end{aligned}$$

Consequently, we have

$$a = 1 - \sum_{j=0}^{2} A_j \|\alpha_j\|_p = 1 - \frac{3\sqrt{5}}{20} - \frac{\sqrt{3}}{8} - \frac{1}{4} > \frac{1}{8} > 0, \qquad b = \frac{\sqrt{2}}{16}.$$

Since $\|f(t,0,0,0)\|_2 = \|\frac{1}{32}t^{-\frac{1}{4}}\|_2 = \frac{\sqrt{2}}{32}$ and $\sigma = 2$, we have

$$a^{\frac{\sigma}{\sigma-1}}\left(\sigma^{\frac{\sigma}{1-\sigma}} - \sigma^{\frac{1}{1-\sigma}}\right) + b^{\frac{1}{\sigma-1}}\left\|f(t,0,0,0)\right\|_2 < \left(\frac{1}{8}\right)^2\left(\frac{1}{4} - \frac{1}{2}\right) + \frac{\sqrt{2}}{16} \cdot \frac{\sqrt{2}}{32} = 0.$$

Thus by Theorem 3.2, the above boundary value problem has at least one solution $u(t) \in W^{3,2}(0,1)$ and in particular has at most one solution $u(t) \in W^{3,2}(0,1)$ with $\|u'''\|_2 < \frac{1}{2}(\frac{a}{b})^{\frac{1}{\sigma-1}} = 4\sqrt{2}a$.

Also, since from the equation of the boundary value problem we have

$$\begin{aligned}
\left\|u'''\right\|_2 &\le \frac{1}{32}\left\|t^{-\frac{1}{3}}\right\|_2 + \frac{\sqrt{3}}{8}\left\|t^{-\frac{1}{3}}\right\|_2 + \frac{\sqrt{2}}{8}\left\|u''\right\|_2 \\
&\le \frac{\sqrt{2}}{32} + \frac{3}{8} + \frac{\sqrt{2}}{8}\left\|u'''\right\|_2,
\end{aligned}$$

it follows that

$$\left\|u'''\right\|_2 \le \frac{\frac{\sqrt{2}}{32} + \frac{3}{8}}{1 - \frac{\sqrt{2}}{8}} \approx 0.518 < \frac{\sqrt{2}}{2} < 4\sqrt{2}a.$$

Hence above boundary value problem has a unique solution $u(t) \in W^{3,2}(0,1)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MP carried out most of calculations and manuscript preparation. SKC carried out literature survey and conceived ideas. YSO participated in discussions and coordination. All authors read and approved the final manuscript.

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