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Existence and multiplicity of solutions for nonlocal $p(x)$-Laplacian problems in $\mathbb{R}^N$

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Abstract

In this paper, we study the nonlocal $p(x)$-Laplacian problem of the following form

$$\begin{cases} M(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx)(-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u) \\ \quad = f(x,u) \text{ in } \mathbb{R}^N, \quad u \in W^{1,p(\cdot)}(\mathbb{R}^N). \end{cases}$$

By using the method of weight function and the theory of the variable exponent Sobolev space, under appropriate assumptions on f and M, we obtain some results on the existence and multiplicity of solutions of this problem. Moreover, we get much better results with f in a special form.

MSC: 35B38; 35D05; 35J20

Keywords: critical points; $p(x)$-Laplacian; nonlocal problem; variable exponent Sobolev spaces

1 Introduction

In this paper, we consider the following problem:

$$(P) \quad \begin{cases} M(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx)(-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u) \\ \quad = f(x,u) \text{ in } \mathbb{R}^N, \quad u \in W^{1,p(\cdot)}(\mathbb{R}^N), \end{cases}$$

where $p(x)$ is a function defined on $\mathbb{R}^N$, $M(t)$ is a continuous function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition.

The operator $\triangle_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than p-Laplacian; for example, p-Laplacian is $(p-1)$-homogeneous, that is, $\triangle_p(\lambda u) = \lambda^{p-1}\triangle_p(u)$ for every $\lambda > 0$; but the $p(x)$-Laplacian operator, when $p(x)$ is not a constant, is not homogeneous. These problems with variable exponent are interesting in applications and raise many difficult mathematical problems. Some of the models leading to these problems of this type are the models of motion of electrorheological fluids, the mathematical models of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium. We refer the reader to [1–7] for the study of $p(x)$-Laplacian equations and the corresponding variational problems.

Kirchhoff has investigated the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

which is called the Kirchhoff equation. This equation is an extension of the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during vibrations. A distinguishing feature of the Kirchhoff equation is that the equation contains a nonlocal coefficient $(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx)$ which depends on the average $\frac{1}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx$ of the kinetic energy $\frac{1}{2}|\frac{\partial u}{\partial x}|^2$ on $[0,L]$. Various equations of Kirchhoff type have been studied by many authors, especially after the work of Lions [8], where a functional analysis framework for the problem was proposed; see, *e.g.*, [9–24] for some interesting results and further references. And now the study of a nonlocal elliptic problem has already been extended to the case involving the p-Laplacian; see, *e.g.*, [25, 26]. Corrêa and Figueiredo in [16] present several sufficient conditions for the existence of positive solutions to a class of nonlocal boundary value problems of the p-Kirchhoff type equation. Recently, the Kirchhoff type equation involving the $p(x)$-Laplacian of the form

$$u_{tt}-M\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\triangle_{p(x)}u+Q(t,x,u,u_t)+f(x,u)=0$$

has been investigated by Autuori, Pucci and Salvatori [27]. In [28] Fan studied $p(x)$-Kirchhoff type equations with Dirichlet boundary value problems. Many papers are about these problems in bounded domains. According to the information I have, for Kirchhoff-type problems in $\mathbb{R}^N$, the results are seldom, in [29] Jin and Wu obtained three existence results of infinitely many radial solutions for Kirchhoff-type problems in $\mathbb{R}^N$, and in [30] Ji established the existence of infinitely many radially symmetric solutions of Kirchhoff-type $p(x)$-Laplacian equations in $\mathbb{R}^n$. The main difficulty here arises from the lack of compactness. Jin [29] and Ji [30] investigated these problems in radial symmetric spaces. In this paper, to deal with problem (P), we overcome the difficulty caused by the absence of compactness through the method of weight function. We establish conditions ensuring the existence and multiplicity of solutions for the problem.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, we obtain the solutions with negative energy by the coercivity of functionals, and in Section 4, we obtain the solutions with positive energy by the Mountain Pass Theorem. Finally in Section 5, we obtain the infinity of solutions by the Fountain Theorem and the Dual Fountain Theorem when f satisfies a special form.

2 Preliminaries

In order to discuss problem (P), we need some theories on space $W^{1,p(\cdot)}(\Omega)$ which we call variable exponent Sobolev space. Firstly, we state some basic properties of space $W^{1,p(\cdot)}(\Omega)$ which will be used later (for details, see [6, 31, 32]).

Let Ω be an open domain of $\mathbb{R}^N$, denote by $S(\Omega)$ the set of all measurable real functions defined on Ω, elements in $S(\Omega)$ which are equal to each other and almost everywhere are considered as one element, and denote

$$C_+(\overline{\Omega}) = \{p | p \in C(\overline{\Omega}), p(x) > 1, \forall x \in \overline{\Omega}\},$$
$$p^+ = \sup_{x\in\overline{\Omega}} p(x), \qquad p^- = \inf_{x\in\overline{\Omega}} p(x), \quad \forall p \in C(\overline{\Omega}),$$
$$L^{p(\cdot)}(\Omega) = \left\{u | u \text{ is a measurable real-valued function on } \Omega, \int_\Omega |u|^{p(x)}\,dx < \infty\right\},$$

we can introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

and $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ becomes a Banach space. We call it a variable exponent Lebesgue space.

The space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) | |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(\cdot)}(\Omega),$$

where $|\nabla u|_{p(x)} = \|\nabla u\|_{p(x)}$; and we denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, $p^* = \frac{Np(x)}{N-p(x)}$, $p_* = \frac{(N-1)p(x)}{N-p(x)}$, when $p(x) < N$, and $p^* = p_* = \infty$, when $p(x) > N$.

Proposition 2.1 (see [6] and [31])

(1) *If $p \in C_+(\overline{\Omega})$, the space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable, uniform convex Banach space, and its dual space is $L^{q(\cdot)}(\Omega)$, where $1/q(x) + 1/p(x) = 1$. For anyu $\in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have*

$$\int_\Omega |uv|\,dx \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)};$$

(2) *If $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{r(x)} = 1$, then for any $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{q(\cdot)}(\Omega)$, and $w \in L^{r(\cdot)}(\Omega)$,*

$$\int_\Omega |uvw|\,dx \le \left(\frac{1}{p^-} + \frac{1}{q^-} + \frac{1}{r^-}\right)|u|_{p(x)}|v|_{q(x)}|w|_{r(x)} \le 3|u|_{p(x)}|v|_{q(x)}|w|_{r(x)}.$$

Proposition 2.2 (see [6]) *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies*

$$|f(x,s)| \le a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \quad \text{for any } x \in \Omega, s \in \mathbb{R},$$

where $p_1, p_2 \in C_+(\overline{\Omega})$, $a \in L^{p_2(\cdot)}(\Omega)$, $a(x) \ge 0$ and $b \ge 0$ is a constant, then the superposition operator from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$ defined by $(N_f(u))(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.3 (see [6]) *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \forall u \in L^{p(\cdot)}(\Omega),$$

then for $u, u_n \in L^{p(\cdot)}(\Omega)$

(1) $|u(x)|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;
(2) $|u(x)|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$;
$|u(x)|_{p(x)} < 1 \Rightarrow |u|^{p^-}_{p(x)} \ge \rho(u) \ge |u|^{p^+}_{p(x)}$;
(3) $|u_n(x)|_{p(x)} \to 0 \Leftrightarrow \rho(u_n) \to 0$ *as* $n \to \infty$;
$|u_n(x)|_{p(x)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$ *as* $n \to \infty$.

Proposition 2.4 (see [6]) *If* $u, u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \ldots,$ *then the following statements are equivalent to each other*

(1) $\lim_{k\to\infty} |u_k - u|_{p(x)} = 0$;
(2) $\lim_{k\to\infty} \rho(u_k - u) = 0$;
(3) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to\infty} \rho(u_k) = \rho(u)$.

Proposition 2.5 (see [6]) *(1) If* $p \in C_+(\overline{\Omega})$, *then* $W_0^{1,p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are separable reflexive Banach spaces.*

Proposition 2.6 *If* $p : \Omega \to \mathbb{R}$ *is Lipschitz continuous and* $p^+ < N$, *then for* $q \in C_+(\overline{\Omega})$ *with* $p(x) \le q(x) \le p^*(x)$, *there is a continuous embedding* $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$.

For any measurable functions α, β, use the symbol $\alpha \ll \beta$ to denote

$$\operatorname*{ess\,inf}_{x\in\overline{\Omega}} \bigl(\beta(x) - \alpha(x)\bigr) > 0.$$

Proposition 2.7 *Let* Ω *be a bounded domain in* $\mathbb{R}^N$, $p \in C_+(\overline{\Omega})$, $p^+ < N$. *Then for any* $q \in L_+^\infty(\Omega)$ *with* $q \ll p^*$, *there is a compact embedding* $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$.

Proposition 2.8 (Poincare inequality) *There is a constant* $C > 0$, *such that*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

So, $|\nabla u|_{p(x)}$ *is a norm equivalent to the norm* $\|u\|$ *in the space* $W_0^{1,p(\cdot)}(\Omega)$.

3 Solutions with negative energy

In the following sections, we consider problem (P), the nonlocal $p(x)$-Laplacian problem with variational form, where M is a real function satisfying the following condition:

(M_1) $M : (0, +\infty) \to (0, +\infty)$ is continuous and bounded.

And we assume that $N \ge 2$, $p : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous, $1 < p^- \le p^+ < N$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory conditions.

For simplicity, we write $X = W^{1,p(\cdot)}(\mathbb{R}^N)$. Denote by C a general positive constant (the exact value may change from line to line).

Let $t \geq 0$, $u \in X$, define

$$\widehat{M}(t) = \int_0^t M(s)\,ds,$$
$$I_1(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)\,dx,$$
$$J(u) = \widehat{M}\big(I_1(u)\big) = \widehat{M}\bigg(\int_{\mathbb{R}^N} \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)\,dx\bigg),$$
$$\Phi(u) = \int_{\mathbb{R}^N} F(x,u)\,dx,$$
$$E(u) = J(u) - \Phi(u),$$

where $F(x,u) = \int_0^u f(x,t)\,dt$.

Before giving our main results, we first give several lemmas that will be used later.

Lemma 3.1 (see [2] and [28]) *Let (M_1) hold. Then the following statements hold:*

(1) $\widehat{M} \in C^0([0,\infty)) \cap C^1((0,\infty))$, $\widehat{M}(0) = 0$, $\widehat{M}'(t) = M(t) > 0$ *for* $t > 0$.

(2) $J \in C^0(X)$, $J(0) = 0$, $J \in C^1(X\backslash\{0\})$, *and*

$$J'(u)v = M\bigg(\int_{\mathbb{R}^N} \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)\,dx\bigg)\int_{\mathbb{R}^N}\big(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\big)\,dx,$$

for $u, v \in X$.

Lemma 3.2 (see [2]) *Suppose*

$$\big|f(x,t)\big| \leq \sum_{i=1}^m b_i(x)|t|^{q_i(x)-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $b_i(x) \geq 0$, $b_i(x) \not\equiv 0$, $b_i \in L^{r_i}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $r_i, q_i \in L_+^\infty(\mathbb{R}^N)$, $q_i \ll p^*$, *and there are* $s_i \in L_+^\infty(\mathbb{R}^N)$ *such that*

$$p(x) \leq s_i(x) \leq p^*(x), \qquad \frac{1}{r_i(x)} + \frac{q_i(x)}{s_i(x)} = 1.$$

Then $\Phi \in C^1(X,\mathbb{R})$ *and* Φ, Φ' *are weakly-strongly continuous, i.e.,* $u_n \rightharpoonup u$ *implies* $\Phi(u_n) \to \Phi(u)$ *and* $\Phi'(u_n) \to \Phi'(u)$.

Lemma 3.3

(1) The functional $J : X \to \mathbb{R}$ *is sequentially weakly lower semi-continuous,* $\Phi : X \to \mathbb{R}$ *is sequentially weakly continuous, and thus* E *is sequentially weakly lower semi-continuous.*

(2) For any open set $D \subset X\backslash\{0\}$ *with* $\overline{D} \subset X\backslash\{0\}$, *the mappings* J' *and* $E' : \overline{D} \to X^*$ *are bounded, and are of type* (S_+), *namely,*

$$u_n \rightharpoonup u \quad \text{and} \quad \overline{\lim_{n\to\infty}} J'(u_n)(u_n - u) \leq 0, \quad \text{implies } u_n \to u.$$

Proof Since the function $\widehat{M}(t)$ is increasing and the functional I_1 is sequentially weakly lower semi-continuous, we conclude that the functional $J : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous. From Lemma 3.2, we know that $\Phi(u)$and $\Phi'(u)$ are sequentially weakly-strongly continuous. Now let $\overline{D} \subset X\backslash\{0\}$. It is clear that the mapping J' and $E' : \overline{D} \to X^*$ are bounded. To prove that $J' : \overline{D} \to X^*$ is of type (S_+), assuming that $\{u_n\} \subset \overline{D}$, $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} J'(u_n)(u_n - u) \le 0$, then there exist positive constants c_1 and c_2 such that $c_1 \le M(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx) \le c_2$. Noting that $J'(u_n) = M(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx)L_{p(\cdot)}(u_n)$. It follows from $\limsup_{n\to\infty} J'(u_n)(u_n - u) \le 0$ that $\limsup_{n\to\infty} L_{p(\cdot)}(u_n)(u_n - u) \le 0$, where $L_{p(\cdot)}(u)v = \int_{\mathbb{R}^N}(|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv)\,dx$. Since $L_{p(\cdot)}$ is of type (S_+). Moreover, since $\Phi'(u)$ is sequentially weakly-strongly continuous, the mapping $E' : \overline{D} \to X^*$ is of type (S_+). □

Definition 3.1 Let $c \in \mathbb{R}$. A C^1-functional $E : X \to \mathbb{R}$ satisfies $(P.S)_c$ condition if and only if every sequence $\{u_j\}$ in X such that $\lim_j E(u_j) = c$, and $\lim_j E'(u_j) = 0$ in X^* has a convergent subsequence.

Lemma 3.4 (see [28]) *Suppose f satisfies the hypotheses in Lemma 3.2, and let (M_1) hold. Then, for any $c \neq 0$, every bounded $(P.S)_c$ sequence for E, i.e., a bounded sequence $\{u_n\} \subset X\backslash\{0\}$ such that $E(u_n) \to c$ and $E'(u_n) \to 0$, has a strongly convergent subsequence.*

As X is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^\infty \subset X$ and $\{f_n\}_{n=1}^\infty \subset X^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if n } \neq m, \end{cases}$$

$$X = \overline{\operatorname{span}}\{e_n : n = 1, 2, \ldots\}, \qquad X^* = \overline{\operatorname{span}}^{W^*}\{f_n : n = 1, 2, \ldots\}.$$

For $k = 1, 2, \ldots$, denote

$$X_k = \overline{\operatorname{span}}\{e_k\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Lemma 3.5 (see [2]) *Assume that $\Phi : X \to \mathbb{R}$ is weakly-strongly continuous and $\Phi(0) = 0$, $\gamma > 0$ is a given positive number. Set*

$$\beta_k = \sup_{u_k \in Z_k, \|u\| \le \gamma} \left|\Phi(u)\right|,$$

then $\beta_k \to 0$ as $k \to \infty$.

Theorem 3.1 *Suppose f satisfies the hypotheses in Lemma 3.2, let (M_1) hold and the following conditions hold:*

(M_2) There are positive constants α_1, M and C such that $\widehat{M}(t) \ge Ct^{\alpha_1}$ for $t \ge M$.
(H_1) $q^+ < \alpha_1 p_-$.

Then the functional E is coercive and attains its infimum in X at some $u_0 \in X$. Therefore, u_0 is a solution of (P) if E is differentiable at u_0, and in particular, if $u_0 \neq 0$.

Proof We have concluded that E is weakly lower semi-continuous. Let us prove that E is coercive on X, *i.e.*, $E(u) \to +\infty$ as $\|u\| \to +\infty$. For simplicity, we assume that $m = 1$ and denote $b_1 = b$, $q_1 = q$, $s_1 = s$, $r_1 = r$. We have that

$$\begin{aligned}
J(u) &= \widehat{M}\big(I_1(u)\big) = \widehat{M}\bigg(\int_{\mathbb{R}^N} \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + |u|^{p(x)}\big)\,dx\bigg) \\
&\geq \widehat{M}\big(C_1\|u\|^{p_-}\big) \geq C_2\|u\|^{\alpha_1 p_-}, \\
\bigg|\int_{\mathbb{R}^N} F(x,u)\,dx\bigg| &\leq \int_{\mathbb{R}^N} \big|F(x,u)\big|\,dx \leq \int_{\mathbb{R}^N} \frac{b(x)}{q(x)}|u|^{q(x)}\,dx \\
&\leq \frac{2}{q^-}|b|_{r(x)}\big||u|^{q(x)}\big|_{s(x)/q(x)} \leq \frac{2}{q^-}|b|_{r(x)}\big(|u|_{s(x)}\big)^{q^+} \\
&\leq C_3\|u\|^{q^+}.
\end{aligned}$$

When $\|u\|$ is large enough, we have

$$E(u) = J(u) - \Phi(u) \geq C_2\|u\|^{\alpha_1 p^-} - C_4\|u\|^{q^+},$$

and hence E is coercive. Since E is sequentially weakly lower semi-continuous and X is reflexive, E attains its infimum in X at some $u_0 \in X$. In the case where E is differentiable at u_0, u_0 is a solution of (P). □

Theorem 3.2 *Suppose f satisfies the hypotheses in Lemma 3.2. Let (M_1), (M_2), (H_1) and the following conditions hold:*

(M_3) There is a positive constant α_2 such that $\limsup_{t\to 0^+} \frac{\widehat{M}(t)}{t^{\alpha_2}} < +\infty$.

(f_1) There exists a positive constant $\delta > 0$,

$$f(x,t) \geq b_0(x)t^{q_0(x)-1} \quad \text{for } x \in \mathbb{R}^N \text{ and } 0 < t \leq \delta,$$

where $b_0 \geq 0$, $b_0(x) \in C(\mathbb{R}^N,\mathbb{R})$, $b_0 \not\equiv 0$, $q_0(x) \in L_+^\infty(\mathbb{R}^N)$, $q_0^+ < p^-$.

(H_2) $q_0^+ < \alpha_2 p^-$.

Then (P) has at least one nontrivial solution which is a global minimizer of the energy functional E.

Proof From Theorem 3.1 we know that E has a global minimizer u_0. It is clear that $F(x,0) = 0$ and consequently $E(0) = 0$. As $b_0 \geq 0$ and $b_0 \not\equiv 0$, we can find a bounded open set $\Omega \subset \mathbb{R}^N$ such that $b_0(x) > 0$ for $x \in \Omega$. The space $W_0^{k,p(\cdot)}(\Omega)$ is a subspace of X. Take $w \in C_0^\infty(\Omega)\backslash\{0\}$. Then, by ($f_1$), ($M_3$) and ($H_2$), for sufficiently small $\lambda > 0$, we have that

$$\begin{aligned}
E(\lambda w) &= \widehat{M}\bigg(\int_{\mathbb{R}^N} \frac{\lambda^{p(x)}}{p(x)}\big(|\nabla w|^{p(x)} + |w|^{p(x)}\big)\,dx\bigg) - \int_{\mathbb{R}^N} F(x,\lambda w)\,dx \\
&\leq C_1\bigg(\int_{\mathbb{R}^N} \frac{\lambda^{p(x)}}{p(x)}\big(|\nabla w|^{p(x)} + |w|^{p(x)}\big)\,dx\bigg)^{\alpha_2} - \int_\Omega F(x,\lambda w)\,dx \\
&\leq C_4\lambda^{\alpha_2 p^-} - C_5\lambda^{q_0^+} < 0.
\end{aligned}$$

Hence $E(u_0) < 0$ which shows $u_0 \not\equiv 0$. □

Theorem 3.3 *Let the hypotheses of Theorem 3.2 hold, and f satisfy the following condition:*

(f_2) $f(x,-t) = -f(x,t)$ for $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

Then (P) has a sequence of solutions $\{\pm u_k\}$ such that $E(\pm u_k) < 0$, and $E(\pm u_k) \to 0$ as $k \to \infty$.

Proof Denote by $\gamma(A)$ the genus of A. Denote

$$\Sigma = \{A \subset X \backslash \{0\} : A \text{ is compact and } A = -A\},$$
$$\Sigma_k = \{A \in \Sigma : \gamma(A) \geq k\},$$
$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} E(u), \quad k = 1, 2, \ldots,$$

we have $-\infty < c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \cdots$.

From the condition on $b_0(x)$, there exists a bounded open set $\Omega \subset \mathbb{R}^N$ such that $b_0(x) > 0$ for $x \in \Omega$. The space $W_0^{k,p(\cdot)}(\Omega)$ is a subspace of X. For any k, we can choose a k-dimensional linear subspace E_k of $W_0^{k,p(\cdot)}(\Omega)$ such that $E_k \subset C_0^\infty(\Omega)$. As the norms on E_k are equivalent to each other, there exists $\rho_k \in (0,1)$ such that $u \in E_k$ with $\|u\| \leq \rho_k$ implies $|u|_{L^\infty} \leq \delta$. $S_{\rho_k}^{(k)} = \{u \in E_k : \|u\| = \rho_k\}$ is compact, and then there exists a constant d_k such that

$$\int_\Omega \frac{b_0(x)}{q_0(x)} |u|^{q_0(x)}\, dx \geq d_k, \quad \forall u \in S_{\rho_k}^{(k)}.$$

For $u \in S_{\rho_k}^{(k)}$ and $t \in (0,1)$, we have

$$\begin{aligned} E(tu) &\leq \frac{t^{\alpha_2 p^-}}{p^-} \rho_k^{p^-} - \int_\Omega \frac{b_0(x)}{q_0(x)} t^{q_0(x)} |u|^{q_0(x)}\, dx \\ &\leq \frac{t^{\alpha_2 p^-}}{p^-} \rho_k^{p^-} - t^{q_0^+} d_k. \end{aligned}$$

As $q_0^+ < \alpha_2 p^-$, we can find $t_k \in (0,1)$ and $\varepsilon_k > 0$ such that $E(t_k u) \leq -\varepsilon_k < 0$, $\forall u \in S_{\rho_k}^{(k)}$, which implies $E(u) \leq -\varepsilon_k < 0$, $\forall u \in S_{t_k\rho_k}^{(k)}$. Since $\gamma(S_{t_k\rho_k}^{(k)}) = k$, we get the conclusion $c_k \leq -\varepsilon_k < 0$.

By the genus theory, each c_k is a critical value of E, hence there is a sequence of solutions $\{\pm u_k : k = 1, 2, \ldots,\}$ of problem (P) such that $E(\pm u_k) = c_k < 0$.

At last, we will prove $c_k \to 0$ as $k \to \infty$. By the coercive of E, there exists a constant $\gamma > 0$ such that $E(u) > 0$ when $\|u\| \geq \gamma$. For any $A \in \Sigma_k$, let Y_k and Z_k be the subspace of X as mentioned above. According to the properties of genus, we know that $A \cap Z_k \neq \emptyset$. Set

$$\beta_k = \sup_{u \in Z_k, \|u\| \leq \gamma} |\Phi(u)|,$$

we know $\beta_k \to 0$ as $k \to \infty$. When $u \in Z_k$ and $\|u\| \leq \gamma$, we have $E(u) \geq -\beta_k$, and then $c_k \geq -\beta_k$, which concludes $c_k \to 0$ as $k \to \infty$. □

Theorem 3.4 *Let the hypotheses of Lemma 3.2, (f_1), (M_1), (M_2), (M_3), (H_1), (H_2) and the following condition hold,*

(f_+) $f(x,t) \geq 0$ *for* $x \in \mathbb{R}^N$ *and* $t \geq 0$.

Then (P) has at least one nontrivial nonnegative solution with negative energy.

Proof Define

$$\widetilde{f}(x,t) = \begin{cases} f(x,t) & \text{if } t \geq 0, \\ f(x,0) & \text{if } t < 0, \end{cases}$$

$$\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)\,ds, \quad x \in \mathbb{R}^N, t \in \mathbb{R},$$

$$\widetilde{E}(u) = \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right) - \int_{\mathbb{R}^N} \widetilde{F}(x,u)\,dx, \quad u \in X.$$

Then, like in the proof of Theorem 3.2, using truncation functions above, similarly to the proof of Theorem 3.4 in [28], we can prove that $\widetilde{E}$ has a nontrivial global minimizer u_0 and u_0 is a nontrivial nonnegative solution of (P). □

4 Solution with positive energy

In this section we will find the Mountain Pass type critical points of the energy functional E associated with problem (P).

Lemma 4.1 *Let (f_1), (M_1) and the following conditions hold:*

(M_2)′ $\exists\alpha_1 > 0$, $M > 0$, *and* $C > 0$ *such that*

$$\widehat{M}(t) \geq Ct^{\alpha_1} \quad \textit{for } t \geq M$$

with $\alpha_1 p_- > 1$ *hold.*

(M_4) $\exists\lambda > 0$, $M > 0$ *such that*

$$\lambda\widehat{M}(t) \geq M(t)t \quad \textit{for } t \geq M.$$

(f_3) $\exists\mu > 0$, $M > 0$ *such that*

$$0 \leq \mu F(x,t) \leq f(x,t)t, \quad \textit{for } |t| \geq M \textit{ and } x \in \mathbb{R}^N.$$

(H_3) $\lambda p_+ < \mu$.

Then E satisfies condition $(P.S)_c$ *for any* $c \neq 0$.

Proof By (M_4), for $\|u\|$ large enough, we have

$$\begin{aligned}\lambda p_+ J(u) &\geq p_+ M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right)\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx \\ &\geq M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx\right)\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)dx \\ &= J'(u)u.\end{aligned}$$

By (f_3) we conclude that there exists $C_1 > 0$ such that

$$-C_1 \leq \mu \int_{\mathbb{R}^N} F(x,u)\,dx \leq \int_{\mathbb{R}^N} f(x,u)u\,dx + C_1, \quad \forall u \in X,$$

and thus, given any $\varepsilon \in (0,\mu)$, there exists $M_\varepsilon \geq M > 0$ such that

$$(\mu - \varepsilon)\int_{\mathbb{R}^N} F(x,u)\,dx \leq \int_{\mathbb{R}^N} f(x,u)u\,dx, \quad \text{if } \int_{\mathbb{R}^N} F(x,u)\,dx \geq M_\varepsilon,$$

we claim that there exists $C_\varepsilon > 0$ such that

$$\Phi'(u)u - (\mu - \varepsilon)\Phi(u) \geq -C_\varepsilon \quad \text{for } u \in X,$$

the notation of this conclusion can be seen in [28].

Now let $\{u_n\} \subset X\backslash\{0\}$, $E(u_n) \to c \neq 0$ and $E'(u_n) \to 0$. By (H_3), there exists $\varepsilon > 0$ small enough such that $\lambda p_+ < (\mu - \varepsilon)$. Then, since $\{u_n\}$ is a $(P.S)_c$ sequence, for sufficiently large n, we have

$$\begin{aligned}
&(\mu - \varepsilon)c + 1 + \|u\| \\
&\quad \geq (\mu - \varepsilon)E(u_n) - E'(u_n)u_n \\
&\quad \geq \big((\mu - \varepsilon) - \lambda p_+\big)J(u_n) + \big(\lambda p_+ J(u_n) - J'(u_n)u_n\big) + \big(\Phi'(u_n)u_n - (\mu - \varepsilon)\Phi(u_n)\big) \\
&\quad \geq C_2\|u_n\|^{\alpha_1 p_-} - C_3 - C_\varepsilon,
\end{aligned}$$

we conclude that $\{\|u_n\|\}$ is bounded, since $\alpha_1 p_- > 1$. By Lemma 3.4, E satisfies condition $(P.S)_c$ for $c \neq 0$. □

Lemma 4.2 *Under the hypotheses of Lemma 4.1, for any* $w \in X\backslash\{0\}$, $E(sw) \to -\infty$ *as* $s \to +\infty$.

Proof Let $w \in X\backslash\{0\}$ be given. From (M_4) for sufficiently large $t > 0$ we have

$$\widehat{M}(t) \leq C_1 t^\lambda,$$

and then it follows that

$$J(sw) \leq d_1 s^{\lambda p_+} \quad \text{for } s \text{ large enough},$$

where d_1 is a positive constant depending on w. From (f_4) for $|t|$ large enough we have

$$F(x,t) \geq C_2|t|^\mu,$$

which implies that

$$\Phi(sw) = \int_{\mathbb{R}^N} F(x,sw)\,dx \geq d_2 s^\mu \quad \text{for } s \text{ large enough},$$

where d_2 is a positive constant depending on w. Hence for s large enough, we have

$$E(sw) \leq d_1 s^{\lambda p_+} - d_2 s^{\mu},$$

and then $E(sw) \to -\infty$ as $s \to +\infty$. □

Lemma 4.3 *Under the hypotheses of Lemma 3.2, (M_1) holds and the following conditions hold:*

(M_5) There is a positive constant α_3 such that $\limsup_{t\to 0^+} \frac{\widehat{M}(t)}{t^{\alpha_3}} > 0$.
(f_4) There exists $r_1(x) \in C^0(\mathbb{R}^N)$ such that $1 < r_1(x) < p^*(x)$ for $x \in \mathbb{R}^N$ and

$$\liminf_{t\to 0} \frac{|F(x,t)|}{|t|^{r_1(x)}} < +\infty$$

uniformly in $x \in \mathbb{R}^N$.
(H_4) $\alpha_3 p_+ < r_{1^-}$.

Then there exist positive constants ρ and δ such that $E(u) \geq \delta$ for $\|u\| = \rho$.

Proof It follows from (M_5) that

$$J(u) \geq C_1 \|u\|^{\alpha_3 p_+} \quad \text{for } \|u\| \text{ small enough.}$$

It follows from the hypotheses of Lemma 3.2 and (f_4) that

$$\left|\Phi(u)\right| \leq C_2 \|u\|^{r_{1^-}} \quad \text{for } \|u\| \text{ small enough.}$$

Thus by (H_4), we obtain the assertion of Lemma 4.3. □

By the famous Mountain Pass lemma, from Lemmas 4.1-4.3, we have the following:

Theorem 4.1 *Let all hypotheses of Lemmas 4.1-4.3 hold. Then (P) has a nontrivial solution with positive energy.*

5 The case of concave-convex nonlinearity

In this section, we will obtain much better results with f in a special form. We have the following theorem:

Theorem 5.1 *Let $f(x,u) = a(x)|u|^{\alpha(x)-2}u + b(x)|u|^{q(x)-2}u$, where*

$$\alpha, q \in L_+^{\infty}(\mathbb{R}^N), \qquad 1 < \alpha^- \leq \alpha^+ < p^- \leq p^+ < q^-, \qquad q \ll p^*,$$
$$a(x) > 0, \qquad a \in L^{\infty}(\mathbb{R}^N) \cap L^{r_1(\cdot)}(\mathbb{R}^N), \qquad \frac{1}{r_1(x)} + \frac{\alpha(x)}{s_1(x)} = 1,$$
$$b(x) > 0, \qquad b \in L^{\infty}(\mathbb{R}^N) \cap L^{r_2(\cdot)}(\mathbb{R}^N), \qquad \frac{1}{r_2(x)} + \frac{\alpha(x)}{s_2(x)} = 1,$$
$$p(x) \leq s_1(x) \leq p^*(x), \qquad p(x) \leq s_2(x) \leq p^*(x).$$

Then we have

(1) *If* (M_1), $(M_2)'$, (M_4), (H_3) *hold and we also assume that* $\alpha^+ < \alpha_1 p^-$ *and* $\lambda p^+ < q^-$, *then problem (P) has solutions* $\{\pm u_k\}_{k=1}^{\infty}$ *such that* $E(\pm u_k) \to +\infty$ *as* $k \to \infty$.

(2) *If* (M_1), (M_4), (M_5), (H_3) *hold and we also assume that* $\alpha^- < \alpha_3 p^+$ *and* $\alpha^+ < \lambda p^-$, *then problem (P) has solutions* $\{\pm v_k\}_{k=1}^{\infty}$ *such that* $E(\pm v_k) < 0$, $E(\pm v_k) \to 0$ *as* $k \to \infty$.

We will use the following 'Fountain Theorem' and the 'Dual Fountain Theorem' to prove Theorem 5.1.

Proposition 5.1 (Fountain Theorem, see [11]) *Assume*

(A_1) *X is a Banach space,* $E \in C^1(X, \mathbb{R})$ *is an even functional, the subspaces* X_k, Y_k *and* Z_k *are defined by (3.2).*

If for each $k = 1, 2, \ldots$, *there exists* $\rho_k > r_k > 0$ *such that*

(A_2) $\inf_{u \in Z_k, \|u\| = r_k} E(u) \to +\infty$ *as* $k \to \infty$.

(A_3) $\max_{u \in Y_k, \|u\| = \rho_k} E(u) \le 0$.

(A_4) *E satisfies the* $(PS)_c$ *condition for every* $c > 0$. *Then E has a sequence of critical values tending to* $+\infty$.

Proposition 5.2 (Dual Fountain Theorem, see [11]) *Assume* (A_1) *is satisfied and there is a* $k_0 > 0$ *so as to for each* $k \ge k_0$, *there exists* $\rho_k > r_k > 0$ *such that*

(B_1) $\inf_{u \in Z_k, \|u\| = \rho_k} E(u) \ge 0$.

(B_2) $b_k := \max_{u \in Y_k, \|u\| = r_k} E(u) < 0$.

(B_3) $d_k := \inf_{u \in Z_k, \|u\| \le \rho_k} E(u) \to 0$ *as* $k \to \infty$.

(B_4) *E satisfies* $(PS)_c^*$ *condition for every* $c \in [d_{k_0}, 0)$. *Then E has a sequence of negative critical values converging to* 0.

Definition 5.1 We say that E satisfies the $(PS)_c^*$ condition (with respect to (Y_k)), if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $E(u_{n_j}) \to c$ and $(E\,|_{Y_{n_j}})'(u_{n_j}) \to 0$, contains a subsequence converging to a critical point of E.

Proof of Theorem 5.1 Firstly, we verify the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. Suppose $\{u_{n_j}\} \subset X$, $n_j \to \infty$, $E(u_{n_j}) \to c$ and $(E\,|_{Y_{n_j}})'(u_{n_j}) \to 0$. It is easy to obtain that $f(x)$ satisfies condition (f_3), when it has this special form. So similar to the method in Lemma 4.1, we have that

$$
\begin{aligned}
&(\mu - \varepsilon)c + 1 + \|u_{n_j}\| \\
&\quad \ge (\mu - \varepsilon)E(u_n) - E'(u_{n_j})u_{n_j} \\
&\quad \ge \big((\mu - \varepsilon) - \lambda p_+\big)J(u_{n_j}) + \big(\lambda p_+ J(u_{n_j}) - J'(u_{n_j})u_{n_j}\big) + \big(\Phi'(u_{n_j})u_{n_j} - (\mu - \varepsilon)\Phi(u_{n_j})\big) \\
&\quad \ge C_2 \|u_{n_j}\|^{\alpha_1 p_-} - C_3 - C_\varepsilon,
\end{aligned}
$$

hence, we can get that $\{\|u_{n_j}\|\}$ is bounded. Going if necessary to a subspace, we can assume that $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$
\lim_{n_j \to \infty} E'(u_{n_j})(u_{n_j} - u) = \lim_{n_j \to \infty} E'(u_{n_j})(u_{n_j} - v_{n_j})
$$

$$
\begin{aligned}
&+ \lim_{n_j\to\infty} E'(u_{n_j})(v_{n_j} - u) \\
&= \lim_{n_j\to\infty} (E\,|_{Y_{n_j}})'(u_{n_j})(u_{n_j} - v_{n_j}) = 0.
\end{aligned}
$$

As E' is of (S^+) type, we can conclude $u_{n_j} \to u$; furthermore, we have $E'(u_{n_j}) \to E'(u)$.

It only remains to prove $E'(u) = 0$. For any $w_k \in Y_k$ and $n_j \geq k$ we have

$$
\begin{aligned}
E'(u)w_k &= \bigl(E'(u) - E'(u_{n_j})\bigr)w_k + E'(u_{n_j})w_k \\
&= \bigl(E'(u) - E'(u_{n_j})\bigr)w_k + (E\,|_{Y_{n_j}})'(u_{n_j})w_k.
\end{aligned}
$$

Going to the limit on the right side of the above equation reaches

$$
E'(u)w_k = 0, \quad \forall w_k \in Y_k,
$$

so $E'(u) = 0$, this shows that E satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. Obviously, E also satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.

(1) We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_2) and (A_3) are satisfied. (A_2) For $k = 1, 2, \ldots$, denote

$$
\theta_k = \sup_{v\in Z_k,\|v\|\leq 1} \int_{\mathbb{R}^N} \frac{a(x)}{\alpha(x)} |v|^{\alpha(x)}\,dx, \qquad \beta_k = \sup_{v\in Z_k,\|v\|\leq 1} \int_{\mathbb{R}^N} \frac{b(x)}{q(x)} |v|^{q(x)}\,dx,
$$

then $\theta_k > 0$, $\beta_k > 0$, and $\theta_k \to 0$, $\beta_k \to 0$, as $k \to \infty$. When $u \in Z_k$, $\|u\| \geq M$,

$$
E(u) \geq \frac{1}{p^+} \|u\|^{\alpha_1 p^-} - \|u\|^{\alpha^+}\theta_k - \|u\|^{q^+}\beta_k.
$$

For sufficiently large k, we have $\theta_k < \frac{1}{2p^+}$. As $\alpha^+ < \alpha_1 p^-$, we get

$$
E(u) \geq \frac{1}{2p^+} \|u\|^{\alpha_1 p^-} - \|u\|^{q^+}\beta_k.
$$

Choose $r_k = (\frac{p^-}{2p^+q^+\beta_k})^{\frac{1}{q^+-\alpha_1 p^-}}$, we have

$$
E(u) \geq \left(\frac{p^-}{2p^+q^+}\right)^{\frac{q^+}{q^+-\alpha_1 p^-}} \frac{q^+ - p^-}{p^-} \left(\frac{1}{\beta_k}\right)^{\frac{p^-}{q^--\alpha_1 p^-}}.
$$

Since $\beta_k \to 0$, we have $\inf_{u\in Z_k,\|u\|=r_k} E(u) \to +\infty$ as $k \to \infty$. (A_2) is satisfied.

(A_3) For $k = 1, 2, \ldots$, denote

$$
e_k = \inf_{v\in Y_k,\|v\|=1} \int_{\mathbb{R}^N} \frac{b(x)}{q(x)} |v|^{q(x)}\,dx.
$$

Then $e_k > 0$. For any $v \in Y_k$, with $\|v\| = 1$ and t large enough, since $\dim Y_k < \infty$, all norms are equivalent in Y_k, we have

$$
E(tv) \leq \frac{1}{p^-} t^{\lambda p^+} - e_k t^{q^-}.
$$

As $q^- > \lambda p^+$, there exists $\rho_k > r_k$ such that $t = \rho_k$ concludes $E(tv) \le 0$ and then

$$\max_{u \in Y_k, \|u\| = \rho_k} E(u) \le 0,$$

so (A_2) is satisfied.

Conclusion (1) is reached by the Fountain Theorem.

(2) We use the Dual Fountain Theorem to prove conclusion (2), and now it remains for us to prove that there exist $\rho_k > r_k > 0$ such that if k is large enough (B_1), (B_2) and (B_3) are satisfied.

(B_1) Let θ_k and β_k be defined as above, when $v \in Z_k$, $\|v\| = 1$ and t small enough we have

$$E(tv) \ge \frac{1}{p^+} t^{\alpha_3 p^+} - t^{\alpha^-}\theta_k - t^{q^-}\beta_k \ge \frac{1}{p^+} t^{\alpha_3 p^+} - t^{\alpha^-}\theta_k - t^{p^+}\beta_k.$$

For sufficiently large k we have $\beta_k < \frac{1}{2p^+}$, thus

$$E(tv) \ge \frac{1}{2p^+} t^{\alpha_3 p^+} - t^{\alpha^-}\theta_k.$$

Choose $\rho_k = (2p^+\beta_k)^{\frac{1}{\alpha_3 p^+ - \alpha^-}}$, then for sufficiently large k, $\rho_k < 1$. When $t = \rho_k$, $v \in Z_k$ with $\|v\| = 1$, we have $E(tv) \ge 0$, which implies

$$\inf_{u \in Z_k, \|u\| = \rho_k} E(u) \ge 0.$$

Hence (B_1) is satisfied.

(B_2) For $k = 1, 2, \ldots$, denote

$$\delta_k = \inf_{v \in Y_k, \|v\| = 1} \int_{\mathbb{R}^N} \frac{a(x)}{\alpha(x)} |v|^{\alpha(x)}\, dx,$$

then $\delta_k > 0$. For $v \in Y_k$, $\|v\| = 1$ and t small enough, we have

$$E(tv) \le \frac{1}{p^-} t^{\lambda p^-} - \delta_k t^{\alpha^+},$$

since $\dim Y_k < \infty$ and $\alpha^+ < \lambda p^-$, we get

$$b_k := \max_{u \in Y_k, \|u\| = r_k} E(u) < 0,$$

with $r_k \in (0, \rho_k)$ small enough. Hence (B_2) is satisfied.

(B_3) From the proof above and $Y_k \cap Z_k \ne \emptyset$, we have

$$d_k := \inf_{u \in Z_k, \|u\| \le \rho_k} E(u) \le b_k := \max_{u \in Y_k, \|u\| = r_k} E(u).$$

For $v \in Z_k$, $\|v\| = 1$ and $u = tv$ small enough, we have

$$E(u) = E(tv) \ge \frac{1}{2p^+} t^{\alpha_3 p^+} - t^{\alpha^-}\theta_k \ge -t^{\alpha^-}\theta_k \ge -\rho_k^{\alpha^-}\theta_k \ge -\theta_k,$$

hence $d_k \to 0$. Hence (B_3) is satisfied.

Conclusion (2) is reached by the Dual Fountain Theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
EG and PZ contributed to each part of this work equally. All the authors read and approved the final manuscript.

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References

1. Fan, XL: On the sub-supersolution method for $p(x)$-Laplacian equations. J. Math. Anal. Appl. **330**, 665-682 (2007)
2. Fan, XL, Han, XY: Existence and multiplicity of solutions for $p(x)$-Laplacian equations in R^n. Nonlinear Anal. **59**, 173-188 (2004)
3. Fan, XL, Shen, JS, Zhao, D: Sobolev embedding theorems for space $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. **262**, 749-760 (2001)
4. Fan, XL, Zhang, QH: Existence of solutions for $p(x)$-Laplacian Dirichlet problems. Nonlinear Anal. **52**, 1843-1852 (2003)
5. Fan, XL, Zhang, QH, Zhao, D: Eigenvalues of $p(x)$-Laplacian Dirichlet problem. J. Math. Anal. Appl. **302**, 306-317 (2005)
6. Fan, XL, Zhao, D: On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**, 424-446 (2001)
7. Fan, XL, Zhao, YZ, Zhang, QH: A strong maximum principle for $p(x)$-Laplacian equations. Chin. Ann. Math., Ser. A **24**, 495-500 (2003) (in Chinese); Chinese Contemp. Math. **24**, 277-282 (2003)
8. Lions, JL: On some questions in boundary value problems of mathematical physical. In: de la Penha, GM, Medeiros, LAJ (eds.) Proceedings of International Symposium on Continuum Mechanics and Partial Differential Equations, Rio de Janerro 1977. Math. Stud., vol. 30, pp. 284-346. North-Holland, Amsterdam (1978)
9. Chipot, M, Lovat, B: Some remarks on non local elliptic and parabolic problems. Nonlinear Anal. **30**, 4619-4627 (1997)
10. Ma, TF: Remarks on an elliptic equation on Kirchhoff type. Nonlinear Anal. **63**, 1967-1977 (2005)
11. Perera, K, Zhang, ZT: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. **221**, 246-255 (2006)
12. Alves, CO, Corrêa, FJSA: On existence of solutions for a class of problem involving a nonlinear operator. Commun. Appl. Nonlinear Anal. **8**, 43-56 (2001)
13. Alves, CO, Corrêa, FJSA, Ma, TF: Positive solutions for a quasilinear elliptic equation of Kirchhoff type. Comput. Math. Appl. **49**, 85-93 (2005)
14. Andrade, D, Ma, TF: An operator equation suggested by a class of stationary problems. Commun. Appl. Nonlinear Anal. **4**, 65-71 (1997)
15. Chipot, M, Valente, V, Vergara Caffarelli, G: Remarks on a nonlocal problem involving the Dirichlet energy. Rend. Semin. Mat. Univ. Padova **110**, 199-220 (2003)
16. Corrêa, FJSA: On positive solutions of nonlocal and nonvariational elliptic problems. Nonlinear Anal. **59**, 1147-1155 (2004)
17. Corrêa, FJSA, Figueiredo, GM: On an elliptic equation of p-Kirchhoff type via variational methods. Bull. Aust. Math. Soc. **74**, 263-277 (2006)
18. Corrêa, FJSA, Figueiredo, GM: On a p-Kirchhoff equation via Krasnoselskii's genus. Appl. Math. Lett. **22**, 819-822 (2009)
19. Corrêa, FJSA, Menezes, SDB, Ferreira, J: On a class of problems involving a nonlocal operator. Appl. Math. Comput. **147**, 475-489 (2004)
20. He, XM, Zou, WM: Infinitely many positive solutions for Kirchhoff-type problems. Nonlinear Anal. **70**, 1407-1414 (2009)
21. Ricceri, B: On an elliptic Kirchhoff-type problem depending on two parameters. J. Glob. Optim. **46**, 543-549 (2010)
22. Arosio, A, Panizzi, S: On the well-posedness of the Kirchhoff string. Trans. Am. Math. Soc. **348**, 305-330 (1996)
23. Cavalcanti, MM, Domingos Cavalcanti, VN, Soriano, JA: Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation. Adv. Differ. Equ. **6**, 701-730 (2001)
24. D'Ancona, P, Spagnolo, S: Global solvability for the degenerate Kirchhoff equation with real analytic date. Invent. Math. **108**, 447-462 (1992)
25. Dreher, M: The Kirchhoff equation for the p-Laplacian. Rend. Semin. Mat. (Torino) **64**, 217-238 (2006)
26. Dreher, M: The wave equation for the p-Laplacian. Hokkaido Math. J. **36**, 21-52 (2007)
27. Autuori, G, Pucci, P, Salvatori, MC: Asymptotic stability for anistropic Kirchhoff systems. J. Math. Anal. Appl. **352**, 149-165 (2009)
28. Fan, XL: On nonlocal $p(x)$-Laplacian Dirichlet problems. Nonlinear Anal. **72**, 3314-3323 (2010)
29. Jin, J, Wu, X: Infinitely many radial solutions for Kirchhoff-type problems in R^N. J. Math. Anal. Appl. **369**, 564-574 (2010)
30. Ji, C: Infinitely many radial solutions for the Kirchhoff-type equation with oscillatory nonlinearities in R^N. J. Math. Anal. Appl. **388**, 727-738 (2012)
31. Edmunds, DE, Rákosník, J: Density of smooth functions in $W^{k,p(x)}(\Omega)$. Proc. R. Soc. Lond. A **437**, 229-236 (1992)
32. Edmunds, DE, Rákosník, J: Sobolev embedding with variable exponent. Stud. Math. **143**, 267-293 (2000)

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