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On positive solutions for nonhomogeneous m-point boundary value problems with two parameters

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Abstract
This paper is concerned with the existence, multiplicity, and nonexistence of positive solutions for nonhomogeneous m-point boundary value problems with two parameters. The proof is based on the fixed-point theorem, the upper-lower solutions method, and the fixed-point index.
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1 Introduction

Many authors have studied the existence, nonexistence, and multiplicity of positive solutions for multipoint boundary value problems by using the fixed-point theorem, the fixed point index theory, and the lower and upper solutions method. We refer the readers to the references [1–4]. Recently, Hao, Liu and Wu [5] studied the existence, nonexistence, and multiplicity of positive solutions for the following nonhomogeneous boundary value problems:

$$\begin{cases} -u''(t) = a(t)f(t,u(t)), \\ u(0) = 0, \qquad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = b, \end{cases}$$

where $b > 0$, $k_i > 0$ $(i = 1, 2, \ldots, m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} k_i\xi_i < 1$, $a(t)$ may be singular at $t = 0$ and/or $t = 1$. They showed that there exists a positive number $b^* > 0$ such that the problem has at least two positive solutions for $0 < b < b^*$, at least one positive solution for $b = b^*$ and no solution for $b > b^*$ by using the Krasnosel'skii-Guo fixed-point theorem, the upper-lower solutions method, and the topological degree theory.

Inspired by the above references, the purpose of this paper is to study the following more general nonhomogeneous boundary value problems:

$$\begin{cases} -u''(t) = \lambda h(t)f(u(t)), \\ u(0) = 0, \qquad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = \mu \int_0^1 g(u(s))\,ds, \end{cases} \tag{1}$$

where λ, μ are positive parameters, $k_i > 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. The main result of the present paper is summarized as follows.

Theorem 1.1 *Assume the following conditions hold:*

(H1) $(\lambda, \mu) \in R_+^2 \setminus \{(0,0)\}$ *are nonnegative parameters;*

(H2) $h : [0,1] \to [0,+\infty)$ *is continuous,* $h(t)$ *does not vanish identically on any subinterval of* $[0,1]$ *and* $\int_0^1 G(s,s)h(s)\,ds < +\infty$, *where* $G(s,s)$ *is given in Sect. 2;*

(H3) $f,g \in C(R_+, R_+)$ *is nondecreasing with respect to u, respectively, that is,*

$$f(u_1) \le f(u_2) \quad \text{if } u_1 \le u_2,$$
$$g(u_1) \le g(u_2) \quad \text{if } u_1 \le u_2.$$

And either $f(0) > 0$ *or* $g(0) > 0$;

(H4) *There exist constants* $m_1, m_2 > 0$ *such that* $f(u) \ge m_1 u$ *and* $g(u) \ge m_2 u$, *respectively, for all* $u \ge 0$;

(H5) $\lim_{|u|\to+\infty} \frac{f(u)}{u} = +\infty$, $\lim_{|u|\to+\infty} \frac{g(u)}{u} = +\infty$.

If $0 < \sum_{i=1}^{m-2} k_i < 1$, *then there exists a bounded and continuous curve* Γ *separating* $R_+^2 \setminus \{(0,0)\}$ *into two disjoint subsets* Ω_1 *and* Ω_2 *such that (1) has at least two positive solutions for* $(\lambda,\mu) \in \Omega_1$, *one positive solution for* $(\lambda,\mu) \in \Gamma$, *and no solution for* $(\lambda,\mu) \in \Omega_2$. *Moreover, let* $\Gamma_+ \cup \Gamma_0$ *be the parametric representation of* Γ, *where*

$$\Gamma_+ : \mu = \mu(\lambda) > 0, \qquad \Gamma_0 : \mu = \mu(\lambda) = 0.$$

Then on Γ_+, *the function* $\mu = \mu(\lambda)$ *is continuous and nonincreasing, that is, if* $\lambda \le \lambda'$, *we have* $\mu(\lambda) \ge \mu(\lambda')$.

For the proof of Theorem 1.1, we also need the following lemmas.

Lemma 1.2 [6] *Let E be a Banach space, K a cone in E and* Ω *bounded open in E. Let* $0 \in \Omega$ *and* $T : K \cap \overline{\Omega} \to K$ *be condensing. Suppose that* $Tx \ne \lambda x$ *for all* $x \in K \cap \partial\Omega$ *and all* $\lambda \ge 1$. *Then*

$$i(T, K \cap \Omega, K) = 1.$$

Lemma 1.3 [6] *Let E be a Banach space and K a cone in E. For* $r > 0$, *define* $K_r = \{x \in K : \|x\| < r\}$. *Assume that* $T : \overline{K_r} \to K$ *is a compact map such that* $Tx \ne x$ *for* $x \in \partial K_r$. *If* $\|x\| \le \|Tx\|$ *for all* $x \in \partial K_r$, *then*

$$i(T, K_r, K) = 0.$$

2 Preliminaries

Lemma 2.1 [5] *Assume that* $0 < \sum_{i=1}^{m-2} k_i \xi_i < 1$. *If* $y(t) \in C(0,1)$ *with* $\int_0^1 G(s,s)y(s)\,ds < +\infty$, *then the Green function for the homogeneous BVP*

$$\begin{cases} -u''(t) = y(t), \\ u(0) = 0, \qquad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = 0 \end{cases}$$

is given by

$$G(t,s)=\frac{1}{1-\sum_{i=1}^{m-2}k_i\xi_i}\begin{cases} s(1-t)-\sum_{i=1}^{m-2}k_i(\xi_i-t)s, & s\le t, s\le \xi_1,\\ t[(1-s)-\sum_{i=1}^{m-2}k_i(\xi_i-s)], & t\le s\le \xi_1,\\ s(1-t)+\sum_{i=1}^{j}k_i\xi_i(t-s)-\sum_{i=j+1}^{m-2}k_i(\xi_i-t)s, & \\ \quad \xi_j\le s\le \xi_{j+1}, s\le t, j=1,2,\ldots,m-3, & \\ t[(1-s)-\sum_{i=j+1}^{m-2}k_i(\xi_i-s)], & \\ \quad \xi_j\le s\le \xi_{j+1}, t\le s, j=1,2,\ldots,m-3, & \\ s(1-t)+\sum_{i=1}^{m-2}k_i\xi_i(t-s), & \xi_{m-2}\le s\le t,\\ t(1-s), & \xi_{m-2}\le s, t\le s. \end{cases}$$

Moreover, the Green function satisfies the following properties:

(i) $G(t,s)>0$ *for* $t,s\in(0,1)$, *and* $G(t,s)$ *is continuous on* $[0,1]\times[0,1]$;

(ii) $G(t,s)\le G(s,s)$ *for all* $t,s\in[0,1]$.

Lemma 2.2 *Assume that (H1)-(H5) hold. If* $0<\sum_{i=1}^{m-2}k_i\xi_i<1$, *then* $u\in C^2[0,1]$ *is a solution of (1) if and only if* $u\in C[0,1]$ *satisfies the following nonlinear integral equation:*

$$u(t)=\lambda\int_0^1 G(t,s)h(s)f\bigl(u(s)\bigr)\,ds+\frac{\mu\int_0^1 g(u(s))\,ds}{1-\sum_{i=1}^{m-2}k_i\xi_i}t.$$

Proof Integrating both sides of (1) from 0 to t twice and applying the boundary conditions, then we can obtain

$$\begin{aligned} u(t)&=\frac{\mu\int_0^1 g(u(s))\,ds}{1-\sum_{i=1}^{m-2}k_i\xi_i}t-\lambda\int_0^t(t-s)h(s)f\bigl(u(s)\bigr)\,ds\\ &\quad+\lambda\frac{t}{1-\sum_{i=1}^{m-2}k_i\xi_i}\left[\int_0^1(1-s)h(s)f\bigl(u(s)\bigr)\,ds-\sum_{i=1}^{m-2}k_i\int_0^{\xi_i}(\xi_i-s)h(s)f\bigl(u(s)\bigr)\,ds\right]. \end{aligned}$$

Furthermore, by Lemma 2.1, we can obtain

$$u(t)=\lambda\int_0^1 G(t,s)h(s)f\bigl(u(s)\bigr)\,ds+\frac{\mu\int_0^1 g(u(s))\,ds}{1-\sum_{i=1}^{m-2}k_i\xi_i}t.$$

□

Let E denote the Banach space $C[0,1]$ with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|$. A function $u(t)$ is said to be a solution of (1) if $u\in C[0,1]\cap C^2(0,1)$ satisfies (1). Moreover, from Lemma 2.2, it is clear to see that $u(t)$ is a solution of (1) is equivalent to the fixed point of the operator T defined as

$$Tu(t)=\lambda\int_0^1 G(t,s)h(s)f\bigl(u(s)\bigr)\,ds+\mu\frac{\int_0^1 g(u(s))\,ds}{1-\sum_{i=1}^{m-2}k_i\xi_i}t.$$

In addition, define a cone $K\subset E$ as

$$K=\Bigl\{u\in E:u(t)\ge 0, t\in[0,1], \inf_{t\in[\xi_1,\xi_{m-2}]}u(t)\ge\theta\|u\|\Bigr\},$$

where $\theta=k_1\xi_1\min[1-\xi_1,\xi_2]$. Then we have

Lemma 2.3 *If (H1)-(H3) hold, then $T: K \to K$ is completely continuous.*

The proof procedure of Lemma 2.3 is standard, so we omit it.

Now, we will establish the classical lower and upper solutions method for our problem. As usual, we say that $x(t)$ is a lower solution for (1) if

$$\begin{cases} x''(t) + \lambda h(t) f(x(t)) \geq 0, \\ x(0) \leq 0, \qquad x(1) - \sum_{i=1}^{m-2} k_i x(\xi_i) \leq \int_0^1 g(x(s))\, ds. \end{cases}$$

Similarly, we define the upper solution $y(t)$ of the problem (1):

$$\begin{cases} y''(t) + \lambda h(t) f(y(t)) \leq 0, \\ y(0) \geq 0, \qquad y(1) - \sum_{i=1}^{m-2} k_i y(\xi_i) \geq \int_0^1 g(y(s))\, ds. \end{cases}$$

Lemma 2.4 *Let $x(t)$, $y(t)$ be lower and upper solutions, respectively, of (1) such that $0 \leq x(t) \leq y(t)$. Then (1) has a nonnegative solution $u(t)$ satisfying $x(t) \leq u(t) \leq y(t)$ for $t \in [0,1]$.*

Proof Define

$$D_x^y = \{ u \in R : x(t) \leq u(t) \leq y(t), \forall t \in [0,1] \}.$$

It is clear to see that D_x^y is a bounded, convex and closed subset in Banach space E. Now we can prove that $T: D_x^y \to D_x^y$.

For any $u(t) \in D_x^y$, from (H3), we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s) h(s) f\big(u(s)\big)\, ds + \mu \frac{\int_0^1 g(u(s))\, ds}{1 - \sum_{i=1}^{m-2} k_i \xi_i} t \\ &\leq \lambda \int_0^1 G(t,s) h(s) f\big(y(s)\big)\, ds + \mu \frac{\int_0^1 g(y(s))\, ds}{1 - \sum_{i=1}^{m-2} k_i \xi_i} t \\ &= y(t). \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s) h(s) f\big(u(s)\big)\, ds + \mu \frac{\int_0^1 g(u(s))\, ds}{1 - \sum_{i=1}^{m-2} k_i \xi_i} t \\ &\geq \lambda \int_0^1 G(t,s) h(s) f\big(x(s)\big)\, ds + \mu \frac{\int_0^1 g(x(s))\, ds}{1 - \sum_{i=1}^{m-2} k_i \xi_i} t \\ &= x(t). \end{aligned}$$

From above inequalities, we obtain that $T: D_x^y \to D_x^y$.

Therefore, by Schauder's fixed theorem, the operator T has a fixed point $u(t) \in D_x^y$, which is the solution of (1). $\square$

3 Proof of Theorem 1.1

Lemma 3.1 *Assume (H1)-(H5) hold and Σ be a compact subset of $R_+^2 \backslash \{(0,0)\}$. Then there exists a constant $C_\Sigma > 0$ such that for all $(\lambda, \mu) \in \Sigma$ and all possible positive solutions $u(t)$ of (1) at (λ, μ), one has $\|u\| \le C_\Sigma$.*

Proof Suppose on the contrary that there exists a sequence $\{u_n\}$ of positive solutions of Eq. (1) at (λ_n, μ_n) such that $(\lambda_n, \mu_n) \in \Sigma$ for all $n \in N$ and

$$\|u_n\| \to \infty.$$

Then $u_n(t) \in K$, and thus

$$\inf_{t\in[\xi_1,\xi_{m-2}]} u_n(t) \ge \theta \|u_n\|. \tag{2}$$

Since Σ is compact, the sequence $\{(\lambda_n, \mu_n)\}_{n=1}^\infty$ has a convergent subsequence which we denote without loss of generality still by $\{(\lambda_n, \mu_n)\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \lambda_n = \lambda^*, \qquad \lim_{n\to\infty} \mu_n = \mu^*$$

and at least $\lambda^* > 0$ or $\mu^* > 0$.

Case (I). If $\lambda^* > 0$, we have $\lambda_n \ge \lambda^*/2 > 0$ for n sufficient large. Then by (H5), there exists a $R > 0$ such that

$$f(u) \ge Lu, \quad \forall u \ge R,$$

where L satisfies

$$\frac{\lambda^*}{2} L\theta \min_{t\in[0,1]} \int_{\xi_1}^{\xi_{m-2}} G(t,s)h(s)\,ds > 1.$$

Since $\|u_n\| \to \infty$, for n sufficient large, we

$$\begin{aligned}
u_n(t) &= \lambda_n \int_0^1 G(t,s)h(s)f\big(u_n(s)\big)\,ds + \mu_n \frac{\int_0^1 g(u_n(s))\,ds}{1-\sum_{i=1}^{m-2} k_i\xi_i} t \\
&\ge \lambda_n \int_0^1 G(t,s)h(s)f\big(u_n(s)\big)\,ds \\
&\ge \lambda_n \int_0^1 G(t,s)h(s)Lu_n(s)\,ds \\
&\ge \frac{\lambda^*}{2} \int_{\xi_1}^{\xi_{m-2}} G(t,s)h(s)Lu_n(s)\,ds \\
&\ge \frac{\lambda^*}{2} L\theta \|u_n\| \min_{t\in[0,1]} \int_{\xi_1}^{\xi_{m-2}} G(t,s)h(s)\,ds \\
&> \|u_n\|.
\end{aligned}$$

This is a contradiction.

Case (II). If $\mu^* > 0$, then we have $\mu_n \geq \mu^*/2 > 0$ for n sufficient large. Since $\lim_{|u|\to+\infty} \frac{g(u)}{u} = +\infty$, there exists a $R > 0$ such that

$$g(u) \geq Mu, \quad \forall u \geq R,$$

where M satisfies

$$\frac{\mu^*}{2}\frac{M\theta\xi_1(\xi_{m-2}-\xi_1)}{1-\sum_{i=1}^{m-2}k_i\xi_i} > 1.$$

Since $\|u_n\| \to \infty$, then for n sufficient large, we have

$$\begin{aligned} u_n(\xi_1) &= \lambda_n \int_0^1 G(\xi_1,s)h(s)f\big(u_n(s)\big)\,ds + \mu_n \frac{\int_0^1 g(u_n(s))\,ds}{1-\sum_{i=1}^{m-2}k_i\xi_i}\xi_1 \\ &\geq \mu_n \frac{\int_0^1 g(u_n(s))\,ds}{1-\sum_{i=1}^{m-2}k_i\xi_i}\xi_1 \\ &\geq \mu_n \frac{\int_0^1 Mu_n(s)\,ds}{1-\sum_{i=1}^{m-2}k_i\xi_i}\xi_1 \\ &\geq \frac{\mu^*}{2}\frac{M\theta\|u\|\xi_1(\xi_{m-2}-\xi_1)}{1-\sum_{i=1}^{m-2}k_i\xi_i} \\ &> \|u_n\|. \end{aligned}$$

This is a contradiction. □

Lemma 3.2 *Assume (H1)-(H4) hold. If (1) has a positive solution at* $(\overline{\lambda},\overline{\mu})$, *then Eq. (1) has a positive solution at* $(\lambda,\mu) \in R_+^2\backslash\{(0,0)\}$ *for all* $(\lambda,\mu) \leq (\overline{\lambda},\overline{\mu})$.

Proof Let $\overline{u}(t)$ be the solution of Eq. (1) at $(\overline{\lambda},\overline{\mu})$, then $\overline{u}(t)$ be the upper solution of (1) at $(\lambda,\mu) \in R_+^2\backslash\{(0,0)\}$ with $(\lambda,\mu) \leq (\overline{\lambda},\overline{\mu})$. Since $f(0) > 0$ or $g(0) > 0$, $u = 0$ is not a solution of (1), but it is the lower solution of (1) at (λ,μ). Therefore, by Lemma 2.4, we obtain the result. □

Lemma 3.3 *Assume (H1)-(H5) hold. Then there exists* $(\lambda^*,\mu^*) > (0,0)$ *such that Eq. (1) has a positive solution for all* $(\lambda,\mu) \leq (\lambda^*,\mu^*)$.

Proof Let $\beta(t)$ be the unique solution of

$$\begin{cases} -u''(t) = h(t), \\ u(0) = 0, \qquad u(1) - \sum_{i=1}^{m-2}k_iu(\xi_i) = 1. \end{cases} \tag{3}$$

It is clear to see that $\beta(t)$ is a positive solution of (3). Let $M_f = \max_{t\in[0,1]} f(\beta(t))$, $M_g = \max_{t\in[0,1]} g(\beta(t))$, then by (H4), we know that $M_f > 0$ and $M_g > 0$. Set $(\lambda^*,\mu^*) = (1/M_f, 1/M_g)$, we have

$$\begin{cases} \beta''(t) + \lambda^*h(t)f(\beta(t)) = -h(t) + \lambda^*h(t)f(\beta(t)) = h(t)(\lambda^*f(\beta(t)) - 1) \leq 0, \\ \beta(0) = 0, \qquad \beta(1) - \sum_{i=1}^{m-2}k_i\beta(\xi_i) - \mu^*\int_0^1 g(\beta(s))\,ds = \int_0^1 1 - \mu^*g(\beta(s))\,ds \geq 0, \end{cases}$$

which implies that $\beta(t)$ is an upper solution of (3) at (λ^*, μ^*). On the other hand, 0 is a lower solution of (1) and $0 \leq \beta(t)$. By (H3), 0 is not a solution of (1). Hence, (1) has a positive solution at (λ^*, μ^*), Lemma 3.2 now implies the conclusion of Lemma 3.3. □

Define a set S by

$$S = \{(\lambda, \mu) \in R_+^2 \backslash \{(0,0)\} : (1) \text{ has a positive solution at } (\lambda, \mu)\}.$$

Then it follows from Lemma 3.3 that $S \neq \emptyset$ and $(S, \leq)$ is a partially ordered set.

Lemma 3.4 *Assume (H1)-(H5) hold. Then $(S, \leq)$ is bounded above.*

Proof Let $(\lambda, \mu) \in S$ and $u(t)$ be a positive solution of (1) at (λ, μ), then we have

$$\begin{aligned}
\|u\| \geq u(\xi_1) &= \lambda \int_0^1 G(\xi_1, s)h(s)f\big(u(s)\big)\,ds + \mu \frac{\int_0^1 g(u(s))\,ds}{1 - \sum_{i=1}^{m-2} k_i\xi_i}\xi_1 \\
&\geq \lambda \int_0^1 G(\xi_1, s)h(s)m_1 u(s)\,ds + \mu \frac{\int_0^1 m_2 u(s)\,ds}{1 - \sum_{i=1}^{m-2} k_i\xi_i}\xi_1 \\
&\geq \lambda \int_{\xi_1}^{\xi_{m-2}} G(\xi_1, s)h(s)m_1 u(s)\,ds + \mu \frac{m_2\theta\|u\|\xi_1(\xi_{m-2} - \xi_1)}{1 - \sum_{i=1}^{m-2} k_i\xi_i} \\
&\geq \lambda m_1\theta\|u\| \int_{\xi_1}^{\xi_{m-2}} G(\xi_1, s)h(s)\,ds + \mu \frac{m_2\theta\|u\|\xi_1(\xi_{m-2} - \xi_1)}{1 - \sum_{i=1}^{m-2} k_i\xi_i}
\end{aligned}$$

by (H4). Furthermore, we can obtain that

$$\lambda m_1\theta \int_{\xi_1}^{\xi_{m-2}} G(\xi_1, s)h(s)\,ds + \mu \frac{m_2\theta\xi_1(\xi_{m-2} - \xi_1)}{1 - \sum_{i=1}^{m-2} k_i\xi_i} \leq 1.$$

□

Lemma 3.5 *Assume (H1)-(H5) hold. Then every chain in S has a unique supremum in S.*

Lemma 3.6 *Assume (H1)-(H5) hold. Then there exists a $\widetilde{\lambda} \in [\lambda^*, \overline{\lambda}]$ such (1) has a positive solution at $(\lambda, 0)$ for all $0 < \lambda \leq \widetilde{\lambda}$, no solution at $(\lambda, 0)$ for all $\lambda > \widetilde{\lambda}$. Similarly, there exists a $\widetilde{\mu} \in [\mu^*, \overline{\mu}]$ such that (1) has a positive solution at $(0, \mu)$ for all $0 < \mu \leq \widetilde{\mu}$, and no solution at $(0, \mu)$ for all $\mu > \widetilde{\mu}$.*

Lemma 3.7 *Assume (H1)-(H5) hold. Then there exists a continuous curve Γ separating $R_+^2 \backslash \{(0,0)\}$ into two disjoint subsets Ω_1 and Ω_2 such that Ω_1 is bounded and Ω_2 is unbounded, Eq. (1) has at least one solution for $(\lambda, \mu) \in \Omega_1 \cup \Gamma$, and no solution for $(\lambda, \mu) \in \Omega_2$. The function $\mu = \mu(\lambda)$ is nonincreasing, that is, if*

$$\lambda \leq \lambda' \leq \widetilde{\lambda},$$

then

$$\mu(\lambda) \geq \mu\big(\lambda'\big).$$

Lemma 3.8 *Let $(\lambda,\mu) \in \Omega_1$. Then there exists $\varepsilon_0 > 0$ such that $(u^* + \varepsilon, v^* + \varepsilon)$ is an upper solution of (1) at (λ,μ) for all $0 < \varepsilon \le \varepsilon_0$, where (u^*, v^*) is the positive solution of Eq. (1) corresponding to some $(\lambda^*, \mu^*) \in \Gamma$ satisfying*

$$(\lambda,\mu) \le (\lambda^*, \mu^*).$$

Proof From (H4), there exists constant $M > 0$ such that

$$f(u^*(t)) \ge M > 0, \qquad g(u^*(t)) \ge M > 0, \quad \text{for all } t \in [0,1].$$

Then by the uniform continuity of f and g on a compact set, there exist $\epsilon_0 > 0$ such that

$$\begin{aligned} \left|f(u^*(t)+\epsilon) - f(u^*(t))\right| &< \frac{M(\lambda^* - \lambda)}{\lambda}, \\ \left|g(u^*(t)+\epsilon) - g(u^*(t))\right| &< \frac{M(\mu^* - \mu)}{\mu}, \end{aligned}$$

for all $t \in [0,1]$ and $0 < \epsilon \le \epsilon_0$.

Let $u^*_\epsilon = u^* + \epsilon$, then we have

$$\begin{aligned} &u^{*\prime\prime}_\epsilon(t) + \lambda h(t) f(u^*_\epsilon(t)) = -\lambda^* h(t) f(u^*_\epsilon(t)) + \lambda h(t) f(u^*_\epsilon(t)) \le h(t)(\lambda^* - \lambda)(M - f(u^*)) \le 0, \\ &u^*_\epsilon(0) = \epsilon > 0, \end{aligned}$$

and

$$\begin{aligned} &u^*_\epsilon(1) - \sum_{i=1}^{m-2} k_i u^*_\epsilon(\xi_i) - \mu \int_0^1 g(u^*_\epsilon)\,ds \\ &= u^*(1) + \epsilon - \sum_{i=1}^{m-2} k_i(u^*(\xi_i) + \epsilon) - \mu \int_0^1 g(u^* + \epsilon)\,ds \\ &= u^*(1) - \sum_{i=1}^{m-2} k_i(u^*(\xi_i)) + \epsilon - \sum_{i=1}^{m-2} k_i \epsilon - \mu \int_0^1 g(u^* + \epsilon)\,ds \\ &= \mu^* \int_0^1 g(u^*)\,ds - \mu \int_0^1 g(u^* + \epsilon)\,ds + \left(1 - \sum_{i=1}^{m-2} k_i\right)\epsilon \\ &= (\mu^* - \mu) \int_0^1 g(u^*)\,ds + \mu \left(\int_0^1 g(u^*) - g(u^* + \epsilon)\right) ds \\ &\quad + \left(1 - \sum_{i=1}^{m-2} k_i\right)\epsilon \\ &\ge \left(1 - \sum_{i=1}^{m-2} k_i\right)\epsilon > 0. \end{aligned}$$

From above inequalities, it is clear to see that u^*_ϵ, is an upper solution of (1) at (λ,μ) for all $0 < \epsilon \le \epsilon_0$. □

Proof of Theorem 1.1 From above lemmas, we need only to show the existence of the second positive solution of (1) for $(\lambda,\mu) \in \Omega_1$. Let $(\lambda,\mu) \in \Omega_1$, then there exists $(\lambda^*,\mu^*) \in \Gamma$ such that

$$(\lambda,\mu) \le \left(\lambda^*,\mu^*\right).$$

Let (u^*,v^*) be the positive solution of (1) at (λ^*,μ^*). Then for $\varepsilon_0 > 0$ given by Lemma 3.8 and for all $\varepsilon : 0 < \varepsilon \le \varepsilon_0$, denote

$$\widetilde{u}^* = u^* + \varepsilon, \qquad \widetilde{v}^* = v^* + \varepsilon.$$

Define the set

$$D = \left\{u \in E : -\varepsilon < u < \widetilde{u}^*\right\}.$$

Then D is bounded open set in E and $0 \in D$. The map T satisfies $K \cap \overline{D} \to K$ and is condensing, since it is completely continuous. Now let $(u,v) \in K \cap \partial D$, then there exists $\xi \in [0,1]$ such that either $u(\xi) = \widetilde{u}^*(\xi)$. Then by (H) and Lemma 3.8, we obtain

$$\begin{aligned} Tu(\xi) &= \lambda \int_0^1 G(\xi,s)h(s)f\big(u(s)\big)\,ds + \mu \frac{\int_0^1 g(u(s))\,ds}{1 - \sum_{i=1}^{m-2} k_i\xi_i}\xi \\ &\le \lambda \int_0^1 G(\xi,s)h(s)f\big(\widetilde{u}^*(s)\big)\,ds + \mu \frac{\int_0^1 g(\widetilde{u}^*(s))\,ds}{1 - \sum_{i=1}^{m-2} k_i\xi_i}\xi \\ &< \widetilde{u}^*(\xi) = u(\xi) \le \vartheta u(\xi) \end{aligned}$$

for all $\vartheta \ge 1$. Thus, $T(u) \ne \vartheta u$ for all $u \in K \cap \partial D$ and $\vartheta \ge 1$, Lemma 1.2 now implies that

$$i(T, K \cap D, K) = 1.$$

Now for some fixed λ and μ, it follows from assumption (H4) that there exists a $R > 0$ such that

$$f(u) \ge Lu, \quad \text{and} \quad g(u) \ge Lu, \quad \forall u \ge R, \tag{4}$$

where L satisfies

$$L\theta\left(\lambda \int_{\xi_1}^{\xi_{m-2}} G(\xi_1,s)h(s)\,ds + \mu \frac{\xi_1(\xi_{m-2} - \xi_1)}{1 - \sum_{i=1}^{m-2} k_i\xi_i}\right) > 1.$$

Let $R^* = \max\{C_\Sigma, \theta^{-1}R, \|(\widetilde{u}^*,\widetilde{v}^*)\|\}$ where C_Σ is given by Lemma 3.1 with Σ a compact set in $R_+^2 \setminus \{(0,0)\}$ containing (λ,μ). Let

$$K_{R^*} = \left\{u \in K : \|u\| < R^*\right\}.$$

Then it follows from Lemma 3.1,

$$T(u) \ne u \quad \forall u \in \partial K_{R^*}.$$

Moreover, for $u \in \partial K_{R^*}$, we have

$$\inf_{t\in[\xi_1,\xi_{m-2}]} u(t) \geq \theta\|u\| \geq R.$$

Furthermore, we have

$$\begin{aligned}
Tu(\xi_1) &= \lambda \int_0^1 G(\xi_1,s)h(s)f\bigl(u(s)\bigr)\,ds + \mu \frac{\int_0^1 g(u(s))\,ds}{1-\sum_{i=1}^{m-2} k_i\xi_i}\xi_1 \\
&\geq \int_0^1 G(\xi_1,s)h(s)Lu(s)\,ds + \mu \frac{\int_0^1 Lu(s)\,ds}{1-\sum_{i=1}^{m-2} k_i\xi_i}\xi_1 \\
&\geq \lambda L\theta\|u\| \int_{\xi_1}^{\xi_{m-2}} G(\xi_1,s)h(s)\,ds + \mu \frac{L\theta\|u\|\xi_1(\xi_{m-2}-\xi_1)}{1-\sum_{i=1}^{m-2} k_i\xi_i} \\
&= L\theta\left(\lambda \int_{\xi_1}^{\xi_{m-2}} G(\xi_1,s)h(s)\,ds + \mu \frac{\xi_1(\xi_{m-2}-\xi_1)}{1-\sum_{i=1}^{m-2} k_i\xi_i}\right)\|u\| \\
&> \|u\|.
\end{aligned}$$

Thus, $\|Tu\| > \|u\|$ and it follows from Lemma 1.3 that

$$i(T, K_{R^*}, K) = 0.$$

By the additivity of the fixed-point index,

$$\begin{aligned}
0 &= i(T, K_{R^*}, K) = i(T, K\cap D, K) + i(T, K_{R^*}\setminus\overline{K\cap D}, K) \\
&= 1 + i(T, K_{R^*}\setminus\overline{K\cap D}, K),
\end{aligned}$$

which yields

$$i(T, K_{R^*}\setminus\overline{K\cap D}, K) = -1.$$

Hence, T has at least one fixed point in $K\cap D$ and another one in $K_{R^*}\setminus\overline{K\cap D}$; this shows that in Ω_1, (1) has at least two positive solution. □

Example Consider the following boundary value problem:

$$\begin{cases} -u''(t) = \lambda(u+1)^2, \\ u(0) = 0, \qquad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = \mu \int_0^1 (u(s)+2)^3\,ds, \end{cases} \tag{5}$$

where $f(u) = (u+1)^2$, $g(u) = (u+2)^3$, and $h(t) = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
In this paper, the author studies the existence, multiplicity, and nonexistence of positive solutions for nonhomogeneous m-point boundary value problems with two parameters. The proof is based on the upper-lower solutions method and fixed-point index. All authors typed, read, and approved the final manuscript.

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