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Existence of positive solutions of elliptic mixed boundary value problem

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Abstract

In this paper, we use variational methods to prove two existence of positive solutions of the following mixed boundary value problem:

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \sigma, \\ \frac{\partial u}{\partial \nu} = g(x,u), & x \in \Gamma. \end{cases}$$

One deals with the asymptotic behaviors of $f(x,u)$ near zero and infinity and the other deals with superlinear of $f(x,u)$ at infinity.

MSC: 35M12; 35D30

Keywords: elliptic mixed boundary value problem; positive solutions; mountain pass theorem; Sobolev embedding theorem

1 Introduction and preliminaries

This paper is concerned with the existence of positive solutions of the following elliptic mixed boundary value problem:

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \sigma, \\ \frac{\partial u}{\partial \nu} = g(x,u), & x \in \Gamma, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, $\sigma \cup \Gamma = \partial\Omega$, $\sigma \cap \Gamma = \emptyset$, Γ is a sufficiently smooth $(n-1)$-dimensional manifold, and ν is the outward normal vector on $\partial\Omega$. We assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy

(S1) $f(x,t) \geq 0$, $\forall t \geq 0$, $x \in \Omega$, $f(x,0) = 0$. $f(x,t) \equiv 0$, $\forall t < 0$, $x \in \Omega$.

(S2) For almost every $x \in \Omega$, $\frac{f(x,t)}{t}$ is nondecreasing with respect to $t > 0$.

(S3) $\lim_{t\to 0} \frac{f(x,t)}{t} = p(x)$, $\lim_{t\to +\infty} \frac{f(x,t)}{t} = q(x) \not\equiv 0$ uniformly in a.e. $x \in \Omega$, where $\|p(x)\|_\infty < \lambda_1$, λ_1 is the first eigenvalue of (2), $0 \leq p(x), q(x) \in L^\infty(\Omega)$.

(S4) There exists $c_1, c_2 > 0$ such that $|f(x,t)| \leq c_1 + c_2|t|^{p-1}$ for some $p \in (2, \frac{2n}{n-2})$ as $n \geq 3$ and $p \in (2, +\infty)$ as $n = 1, 2$.

The eigenvalue problem of (1) is studied by Liu and Su in [1]

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \sigma, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma. \end{cases} \tag{2}$$

There exists a set of eigenvalues $\{\lambda_k\}$ and corresponding eigenfunctions $\{u_k\}$ which solve problem (2), where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$, $\lambda_k \to \infty$ as $k \to \infty$, $\lambda_1 = \inf_{0 \neq u \in V} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx + \int_\Gamma |u|^2\,ds}$.

There have been many papers concerned with similar problems at resonance under the boundary condition; see [2–10]. Moreover, some multiplicity theorems are obtained by the topological degree technique and variational methods; interested readers can see [11–17]. Problem (1) is different from the classical ones, such as those with Dirichlet, Neuman, Robin, No-flux, or Steklov boundary conditions.

In this paper, we assume $V := \{v \in H^1(\Omega) : v|_\sigma = 0\}$ is a closed subspace of $H^1(\Omega)$. We define the norm in V as $\|u\|^2 = \int_\Omega |\nabla u|^2\,dx + \int_\Gamma |\gamma u|^2\,ds$, $\|\cdot\|_{L^p(\Omega)}$ is the $L^p(\Omega)$ norm, $\|\cdot\|_{L^p(\Gamma)}$ is the $L^p(\Gamma)$ norm, $\gamma : V \to L^2(\Gamma)$ is the trace operator with $\gamma u = u_\Gamma$ for all $u \in H^1(\Omega)$, that is continuous and compact (see [18]). Furthermore, we define $g = \gamma f$, $0 \leq g(x,t) \leq |\gamma f(x,t)|$ for $t > 0$ (see [1]). Then, by (S3), we obtain

$$\lim_{t\to+\infty} \frac{g(x,t)}{t} \leq \lim_{t\to+\infty} \frac{|\gamma f(x,t)|}{t} = q(x) \not\equiv 0, \quad \text{a.e. } x \in \overline{\Omega}. \tag{3}$$

Let Ω be a bounded domain with a Lipschitz boundary; there is a continuous embedding $V \hookrightarrow L^y(\Omega)$ for $y \in [2, \frac{2n}{n-2}]$ when $n \geq 3$, and $y \in [2, +\infty)$ when $n = 1, 2$. Then there exists $\gamma_y > 0$, such that

$$\|u\|_{L^y(\Omega)} \leq \gamma_y \|u\|, \quad \forall u \in V. \tag{4}$$

Moreover, there is a continuous boundary trace embedding $V \hookrightarrow L^z(\Gamma)$ for $z \in [2, \frac{2(n-1)}{n-2}]$ when $n \geq 3$, and $z \in [2, +\infty)$ when $n = 1, 2$. Then there exists $k_z > 0$, such that

$$\|u\|_{L^z(\Gamma)} \leq k_z \|u\|, \quad \forall u \in V. \tag{5}$$

It is well known that to seek a nontrivial weak solution of problem (1) is equivalent to finding a nonzero critical value of the C^1 functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx - \int_\Gamma G(s,u)\,ds, \tag{6}$$

where $u \in V$, $F(x,u) = \int_0^u f(x,t)\,dt$, $G(x,u) = \int_0^u g(x,t)\,dt$. Moreover, by (S1) and the Strong maximum principle, a nonzero critical point of J is in fact a positive solution of (1). In order to find critical points of the functional (6), one often requires the technique condition, that is, for some $\mu > 2$, $\forall |u| \geq M > 0$, $x \in \Omega$,

$$0 < \mu F(x,u) \leq u f(x,u), \quad F(x,u) = \int_0^u f(x,t)\,dt. \tag{AR}$$

It is easy to see that the condition (AR) implies that $\lim_{u\to+\infty}\frac{F(x,u)}{u^2}=+\infty$, that is, $f(x,u)$ must be superlinear with respect to u at infinity. In the present paper, motivated by [19] and [20], we study the existence and nonexistence of positive solutions for problem (1) with the asymptotic behavior assumptions (S3) of f at zero and infinity. Moreover, we also study superlinear of f at infinity with $q(x)\equiv+\infty$ in (S3), which is weaker than the (AR) condition, that is the (AR) condition does not hold.

In order to get our conclusion, we define the minimization problem

$$\Lambda=\inf\left\{\int_\Omega|\nabla u|^2\,dx:u\in V,\int_\Omega q(x)u^2\,dx+\int_\Gamma q(s)u^2\,ds=1\right\},\tag{7}$$

then $\Lambda>0$, which is achieved by some $\varphi_\Lambda\in V$ with $\varphi_\Lambda(x)>0$ a.e. in Ω; see Lemma 1.

We denote by c, c_1, c_2 universal constants unless specified otherwise. Our main results are as follows.

Theorem 1 *Let conditions (S1) to (S3) hold, then:*

(i) If $\Lambda>1$, then the problem (1) has no any positive solution in V.

(ii) If $\Lambda<1$, then the problem (1) has at least one positive solution in V.

(iii) If $\Lambda=1$, then the problem (1) has one positive solution $u(x)\in V$ if and only if there exists a constant $c>0$ such that $u(x)=c\varphi_\Lambda(x)$ and $f(x,u)=q(x)u(x)$, $g(x,u)=q(x)u(x)$ a.e. $x\in\Omega$, where $\varphi_\Lambda(x)>0$ is the function which achieves Λ.

Corollary 2 *Let conditions (S1) to (S3) with $q(x)\equiv l>0$ hold, then:*

(i) If $l<\lambda_1$, then the problem (1) has no any positive solution in V.

(ii) If $\lambda_1<l<+\infty$, then the problem (1) has at least one positive solution in V.

(iii) If $l=\lambda_1$, then the problem (1) has one positive solution $u(x)\in V$ if and only if there exists a constant $c>0$ such that $u(x)=c\varphi_1(x)$ and $f(x,u)=\lambda_1u(x)$, $g(x,u)=\lambda_1u(x)$ a.e. $x\in\Omega$, where $\varphi_1(x)>0$ is the eigenfunction of the λ_1.

Theorem 3 *Let conditions (S1) to (S4) with $q(x)\equiv+\infty$ hold, then the problem (1) has at least one positive solution in V.*

2 Some lemmas

We need the following lemmas.

Lemma 1 *If $q(x)\in L^\infty(\Omega)$, $q(x)\geq 0$, $q(x)\not\equiv 0$, then $\Lambda>0$ and there exists $\varphi_\Lambda(x)\in V$ such that $\Lambda=\int_\Omega|\nabla\varphi_\Lambda|^2\,dx$ and $\int_\Omega q(x)\varphi_\Lambda^2\,dx+\int_\Gamma q(s)\varphi_\Lambda^2\,ds=1$. Moreover, $\varphi_\Lambda(x)>0$ a.e. in V.*

Proof By the Sobolev embedding function $V\hookrightarrow L^2(\Omega)$ and Fatou's lemma, it is easy to know that $\Lambda>0$ and there exists $\varphi_\Lambda(x)\in V$, which satisfies Λ, that is, $\int_\Omega q(x)\varphi_\Lambda^2\,dx+\int_\Gamma q(s)\varphi_\Lambda^2\,ds=1$. Furthermore, we assume $\varphi_\Lambda(x)\geq 0$, then $\varphi_\Lambda(x)$ could replace by $|\varphi_\Lambda(x)|$. By the Strong maximum principle, we know $\varphi_\Lambda(x)>0$ a.e. in V. □

Lemma 2 *If conditions (S1) to (S3) hold, then there exists $\beta,\rho>0$ such that $J|_{\partial B_\rho(0)}\geq\beta$, $\forall u\in V$, $\|u\|=\rho$.*

Proof By condition (S3), there exists $\delta>0$, $\varepsilon>0$ such that $\frac{f(x,u)}{u}\leq\lambda_1-\varepsilon$, $\frac{g(x,u)}{u}\leq\frac{\gamma f(x,u)}{u}\leq\lambda_1-\varepsilon$ as $0<|u|\leq\delta$. Which implies that $F(x,u)\leq\frac{1}{2}(\lambda_1-\varepsilon)u^2+c|u|^y$, $G(x,u)\leq\frac{1}{2}(\lambda_1-\varepsilon)u^2+c|u|^z$.

By (4) and (5), we obtain

$$
\begin{aligned}
J(u) &= \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 - \int_\Omega F(x,u)\,dx - \int_\Gamma G(s,u)\,ds \\
&\geq \frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\gamma u\|_{L^2(\Gamma)}^2 - \frac{1}{2}\|\gamma u\|_{L^2(\Gamma)}^2 - \frac{1}{2}(\lambda_1-\varepsilon)\|u\|_{L^2(\Omega)}^2 \\
&\quad - c\|u\|_{L^y(\Omega)}^y - \frac{1}{2}(\lambda_1-\varepsilon)\|u\|_{L^2(\Gamma)}^2 - c\|u\|_{L^z(\Gamma)}^z \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}(\lambda_1-\varepsilon)\frac{1}{\lambda_1}\|u\|^2 - c\gamma_y^y\|u\|^y - \frac{1}{2}(\lambda_1-\varepsilon+1)\frac{1}{\lambda_1+1}\|u\|^2 - ck_z^z\|u\|^z \\
&= \left[\frac{\varepsilon(2\lambda_1+1)}{2\lambda_1(\lambda_1+1)} - \frac{1}{2}\right]\|u\|^2 - c\gamma_y^y\|u\|^y - ck_z^z\|u\|^z.
\end{aligned}
$$

Hence, $y,z > 2$; we take ε which satisfies $\frac{\varepsilon(2\lambda_1+1)}{2\lambda_1(\lambda_1+1)} - \frac{1}{2} > 0$, that is, $\varepsilon > \frac{\lambda_1(\lambda_1+1)}{2\lambda_1+1}$. Then we take a positive constant β such that $J|_{\partial B_\rho(0)} \geq \beta$ as $\|u\| = \rho$, and is small enough. □

Lemma 3 *If conditions (S1) to (S3) hold,* $\Lambda < 1$, $\varphi_\Lambda(x) > 0$ *is defined by Lemma 1, then* $J(t\varphi_\Lambda(x)) \to -\infty$ *as* $t \to +\infty$.

Proof If $\Lambda < 1$, $\varphi_\Lambda(x) > 0$ is defined by Lemma 1, by Fatou's lemma, and (S3), we have

$$
\begin{aligned}
&\lim_{t\to+\infty} \frac{J(t\varphi_\Lambda(x))}{t^2} \\
&\quad= \frac{1}{2}\int_\Omega \left|\nabla\varphi_\Lambda(x)\right|^2 dx - \lim_{t\to+\infty}\frac{\int_\Omega F(x,t\varphi_\Lambda(x))\,dx}{t^2} - \lim_{t\to+\infty}\frac{\int_\Gamma G(s,t\varphi_\Lambda(s))\,ds}{t^2} \\
&\quad\leq \frac{1}{2}\int_\Omega \left|\nabla\varphi_\Lambda(x)\right|^2 dx - \int_\Omega \lim_{t\to+\infty}\frac{F(x,t\varphi_\Lambda(x))}{t^2\varphi_\Lambda^2(x)}\varphi_\Lambda^2(x)\,dx \\
&\qquad - \int_\Gamma \lim_{t\to+\infty}\frac{G(s,t\varphi_\Lambda(s))}{t^2\varphi_\Lambda^2(s)}\varphi_\Lambda^2(s)\,ds \\
&\quad= \frac{1}{2}\int_\Omega \left|\nabla\varphi_\Lambda(x)\right|^2 dx - \frac{1}{2}\int_\Omega \frac{f(x,t\varphi_\Lambda(x))}{t\varphi_\Lambda(x)}\varphi_\Lambda^2(x)\,dx - \frac{1}{2}\int_\Gamma \frac{g(s,t\varphi_\Lambda(s))}{t\varphi_\Lambda(s)}\varphi_\Lambda^2(s)\,ds \\
&\quad= \frac{1}{2}\int_\Omega \left|\nabla\varphi_\Lambda(x)\right|^2 dx - \frac{1}{2}\left[\int_\Omega q(x)\varphi_\Lambda^2(x)\,dx + \int_\Gamma q(s)\varphi_\Lambda^2(s)\,ds\right] \\
&\quad= \frac{1}{2\Lambda}(\Lambda-1)\int_\Omega \left|\nabla\varphi_\Lambda(x)\right|^2 dx \\
&\quad< 0.
\end{aligned}
$$

So, $J(t\varphi_\Lambda(x)) \to -\infty$ as $t \to +\infty$. □

Lemma 4 *Let conditions (S1) and (S2) hold. If a sequence* $\{u_n\} \subset V$ *satisfies* $\langle J'(u_n), u_n\rangle \to 0$ *as* $n \to +\infty$, *then there exists a subsequence of* $\{u_n\}$, *still denoted by* $\{u_n\}$ *such that* $J(tu_n) \leq \frac{1+t^2}{2n} + J(u_n)$ *for all* $t > 0$, $n \geq 1$.

Proof Since $\langle J'(u_n), u_n\rangle \to 0$ as $n \to +\infty$, for a subsequence, we may assume that

$$
-\frac{1}{n} < \langle J'(u_n), u_n\rangle = \|\nabla u_n\|_{L^2(\Omega)}^2 - \int_\Omega f(x,u_n)u_n\,dx - \int_\Gamma g(s,u_n)u_n\,ds < \frac{1}{n}, \quad \forall n \geq 1. \tag{8}
$$

For any fixed $x \in \Omega$ and $n \geq 1$, set

$$\psi_1(t) = \frac{t^2}{2} f(x, u_n)u_n - F(x, tu_n), \qquad \psi_2(t) = \frac{t^2}{2} g(s, u_n)u_n - G(s, tu_n).$$

Then (S2) implies that

$$\begin{aligned}\psi_1'(t) &= tf(x, u_n)u_n - f(x, tu_n)u_n \\ &= tu_n \left[f(x, u_n) - \frac{f(x, tu_n)}{t} \right] \\ &= \begin{cases} \geq 0, & 0 < t \leq 1; \\ \leq 0, & t > 1. \end{cases}\end{aligned}$$

It implies that $\psi_1(t) \leq \psi_1(1)$, $\forall t > 0$. Following the same procedures, we obtain $\psi_2(t) \leq \psi_2(1)$, $\forall t > 0$.

For all $t > 0$ and positive integer n, by (8), we have

$$\begin{aligned} J(tu_n) &= \frac{t^2}{2} \|\nabla u_n\|^2_{L^2(\Omega)} - \int_\Omega F(x, tu_n)\, dx - \int_\Gamma G(s, tu_n)\, ds \\ &\leq \frac{t^2}{2} \left[\frac{1}{n} + \int_\Omega f(x, u_n)u_n\, dx + \int_\Gamma g(s, u_n)u_n\, ds \right] \\ &\quad - \int_\Omega F(x, tu_n)\, dx - \int_\Gamma G(s, tu_n)\, ds \\ &\leq \frac{t^2}{2n} + \int_\Omega \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n) \right] dx + \int_\Gamma \left[\frac{1}{2} g(s, u_n)u_n - G(s, u_n) \right] ds. \end{aligned} \tag{9}$$

On the other hand, by (8), one has

$$\begin{aligned} J(u_n) &= \frac{1}{2} \|\nabla u_n\|^2_{L^2(\Omega)} - \int_\Omega F(x, u_n)\, dx - \int_\Gamma G(s, u_n)\, ds \\ &\geq \frac{1}{2} \left[-\frac{1}{n} + \int_\Omega f(x, u_n)u_n\, dx + \int_\Gamma g(s, u_n)u_n\, ds \right] - \int_\Omega F(x, u_n)\, dx - \int_\Gamma G(s, u_n)\, ds \\ &= -\frac{1}{2n} + \int_\Omega \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n) \right] dx + \int_\Gamma \left[\frac{1}{2} g(s, u_n)u_n - G(s, u_n) \right] ds. \end{aligned}$$

One has

$$\int_\Omega \left[\frac{1}{2} f(x, u_n)u_n - F(x, u_n) \right] dx + \int_\Gamma \left[\frac{1}{2} g(s, u_n)u_n - G(s, u_n) \right] ds \leq J(u_n) + \frac{1}{2n}. \tag{10}$$

Combining (9) and (10), we have $J(tu_n) \leq \frac{1+t^2}{2n} + J(u_n)$. □

Lemma 5 (see [21]) *Suppose E is a real Banach space, $J \in C^1(E, \mathbb{R})$ satisfies the following geometrical conditions:*

(i) $J(0) = 0$; there exists $\rho > 0$ such that $J|_{\partial B_\rho(0)} \geq r > 0$;

(ii) *There exists* $e \in E \backslash \overline{B_\rho(0)}$ *such that* $J(e) \le 0$. *Let* Γ_1 *be the set of all continuous paths joining* 0 *and e:*

$$\Gamma_1 = \{h \in C([0,1], E) | h(0) = 0, h(1) = e\},$$

and

$$c = \inf_{h \in \Gamma_1} \max_{t \in [0,1]} J(h(t)).$$

Then there exists a sequence $\{u_n\} \subset E$ *such that* $J(u_n) \to c \ge \beta$ *and* $(1 + \|u_n\|) \times \|J'(u_n)\|_{E^*} \to 0$.

3 Proofs of main results

Proof of Theorem 1 (i) If $u \in V$ is one positive solution of problem (1), by (3), one has

$$0 = \langle J'(u), u \rangle = \int_\Omega |\nabla u|^2 \,dx - \int_\Omega f(x,u) u \,dx - \int_\Gamma g(s,u) u \,ds.$$

That is,

$$\begin{aligned} \int_\Omega |\nabla u|^2 \,dx &= \int_\Omega f(x,u) u \,dx + \int_\Gamma g(s,u) u \,ds \\ &\le \int_\Omega q(x) u^2 \,dx + \int_\Gamma q(s) u^2 \,ds = 1. \end{aligned}$$

It implies that $\Lambda \le 1$. This completes the proof of Theorem 1(i).

(ii) By Lemma 2, there exists $\beta, \rho > 0$ such that $J|_{\partial B_\rho(0)} \ge \beta$ with $\|u\| = \rho$. By Lemma 3, we obtain $J(t_0 \varphi_\Lambda(x)) < 0$ as $t_0 \to +\infty$. Define

$$\Gamma_1 = \{h \in C([0,1], V) | h(0) = 0, h(1) = t_0 \varphi_\Lambda(x)\}, \tag{11}$$

$$c = \inf_{h \in \Gamma_1} \max_{t \in [0,1]} J(h(t)), \tag{12}$$

where $\varphi_\Lambda(x) > 0$ is given by Lemma 1. Then $c \ge \beta > 0$ and by Lemma 3, there exists $\{u_n\} \subset V$ such that

$$J(u_n) = \frac{1}{2} \|\nabla u_n\|^2_{L^2(\Omega)} - \int_\Omega F(x, u_n) \,dx - \int_\Gamma G(s, u_n) \,ds = c + o(1), \tag{13}$$

$$(1 + \|u_n\|) \|J'(u_n)\|_{V^*} \to 0. \tag{14}$$

(14) implies that

$$\langle J'(u_n), u_n \rangle = \|\nabla u_n\|^2_{L^2(\Omega)} - \int_\Omega f(x, u_n) u_n \,dx - \int_\Gamma g(s, u_n) u_n \,ds = o(1). \tag{15}$$

Here, in what follows, we use $o(1)$ to denote any quantity which tends to zero as $n \to +\infty$.

If $\{u_n\}$ is bounded in V, when Ω is bounded and $f(x,u)$, $g(x,u)$ are subcritical, we can get $\{u_n\}$ has a subsequence strong convergence to a critical value of J, and our proof is complete. So, to prove the theorem, we only need show that $\{u_n\}$ is bounded in V. Supposing that $\{u_n\}$ is unbounded, that is, $\|u_n\| \to +\infty$ as $n \to +\infty$. We order

$$t_n = \frac{2\sqrt{c}}{\|u_n\|}, \qquad w_n = t_n u_n = \frac{2\sqrt{c}u_n}{\|u_n\|}. \tag{16}$$

Then $\{w_n\}$ is bounded in V. By extracting a subsequence, we suppose $w_n \to w$ is a strong convergence in $L^2(\Omega)$, $w_n \to w$ is a convergence a.e. $x \in \Omega$, $w_n \rightharpoonup w$ is a weak convergence in V.

We claim that $w \neq 0$. In fact, by (S1) and (S3), we know $\forall x \in \Omega$, $u_n \geq 0$, and there exists $M_1, M_2 > 0$ such that $|\frac{f(x,u_n)}{u_n}| \leq M_1$, $|\frac{g(x,u_n)}{u_n}| \leq M_2$. If $w = 0$, $w_n \to 0$ is a strong convergence in $L^2(\Omega)$, and by (15) and (16) we know

$$\begin{aligned} 4c &= t_n^2\|u_n\|^2 = t_n^2\left(\|\nabla u_n\|^2_{L^2(\Omega)} + \|\gamma u_n\|^2_{L^2(\Gamma)}\right) \\ &= t_n^2 \int_\Omega f(x,u_n)u_n\,dx + t_n^2\int_\Gamma g(s,u_n)u_n\,ds + t_n^2\|\gamma u_n\|^2_{L^2(\Gamma)} + o(1) \\ &= \int_\Omega \frac{f(x,u_n)}{u_n} w_n^2\,dx + \int_\Gamma \frac{g(s,u_n)}{u_n}w_n^2\,ds + t_n^2\|u_n\|^2_{L^2(\Gamma)} + o(1) \\ &\leq M_1\int_\Omega w_n^2\,dx + M_2\int_\Gamma w_n^2\,ds + \|w_n\|^2_{L^2(\Gamma)} + o(1) \\ &\to 0. \end{aligned}$$

It is contradiction with $c > 0$, so $w \neq 0$.

As follows, we prove $w \neq 0$ satisfies

$$\int_\Omega \nabla\varphi(x)\nabla w(x)\,dx - \int_\Omega q_1(x)\varphi(x)w(x)\,dx - \int_\Gamma q_2(s)\varphi(s)w(s)\,ds = 0.$$

We order

$$p_n(x) = \begin{cases} f(x,u_n)/u_n, & u_n \geq 0, x \in \Omega, \\ 0, & u_n < 0, x \in \Omega, \end{cases}$$

$$q_n(x) = \begin{cases} g(x,u_n)/u_n, & u_n \geq 0, x \in \Gamma, \\ 0, & u_n < 0, x \in \Gamma. \end{cases}$$

By (S1) and (S3), there exists $M_3 > 0$ such that $0 \leq p_n(x) \leq M_3$, $0 \leq q_n(x) \leq M_3$, $\forall x \in \overline{\Omega}$. We select a suitable subsequence and there exists $h_1(x) \in L^2(\Omega)$, $h_2(x) \in L^2(\Gamma)$ such that $p_n(x) \to h_1(x)$ is a strong convergence in $L^2(\Omega)$, $q_n(x) \to h_2(x)$ is a strong convergence in $L^2(\Gamma)$, and $0 \leq h_1(x) \leq M_3$, $0 \leq h_2(x) \leq M_3$, $\forall x \in \overline{\Omega}$.

It follows from $w_n \to w$ is a strong convergence in $L^2(\Omega)$ that

$$\begin{aligned} &\int_\Omega p_n(x)w_n(x)\varphi(x)\,dx = \int_\Omega p_n(x)w_n^+(x)\varphi(x)\,dx \to \int_\Omega h_1(x)w^+(x)\varphi(x)\,dx, \\ &\int_\Gamma q_n(s)w_n(s)\varphi(s)\,ds = \int_\Gamma q_n(s)w_n^+(s)\varphi(s)\,ds \to \int_\Gamma h_2(s)w^+(s)\varphi(s)\,ds. \end{aligned}$$

Hence, $\{p_n(x)w_n(x)\}$ is bounded in $L^2(\Omega)$, $p_n(x)w_n(x) \rightharpoonup h_1(x)w^+(x)$ in $L^2(\Omega)$; $\{q_n(x)w_n(x)\}$ is bounded in $L^2(\Gamma)$, $q_n(x)w_n(x) \rightharpoonup h_2(x)w^+(x)$ in $L^2(\Gamma)$.

By (16), we have

$$\begin{aligned}
&\left|\int_\Omega \nabla w_n(x)\nabla\varphi(x)\,dx - \int_\Omega p_n(x)w_n(x)\varphi(x)\,dx - \int_\Gamma q_n(s)w_n(s)\varphi(s)\,ds\right| \\
&\quad = \left|\int_\Omega \nabla\bigl(t_nu_n(x)\bigr)\nabla\varphi(x)\,dx - \int_\Omega p_n(x)t_nu_n(x)\varphi(x)\,dx - \int_\Gamma q_n(s)t_nu_n(s)\varphi(s)\,ds\right| \\
&\quad = \frac{2\sqrt{c}}{\|u_n\|}\left|\int_\Omega \nabla u_n(x)\nabla\varphi(x)\,dx - \int_\Omega p_n(x)u_n(x)\varphi(x)\,dx - \int_\Gamma q_n(s)u_n(s)\varphi(s)\,ds\right| \\
&\quad \to 0.
\end{aligned}$$

Since $w_n \rightharpoonup w$ is a weak convergence in V, we obtain

$$\int_\Omega \nabla\varphi(x)\nabla w(x)\,dx - \int_\Omega h_1(x)\varphi(x)w^+(x)\,dx - \int_\Gamma h_2(s)\varphi(s)w^+(s)\,ds = 0, \quad \varphi \in V.$$

We order $\varphi = w^-$; this yields $\|w^-\|^2 = 0$, so $w = w^+ \geq 0$. By the Strong maximum principle, we know $w > 0$ a.e. in Ω, so $u_n \to \infty$ a.e. in Ω. Combining (S3) and (3), we obtain

$$\int_\Omega \nabla\varphi(x)\nabla w(x)\,dx - \int_\Omega q(x)\varphi(x)w(x)\,dx - \int_\Gamma q(s)\varphi(s)w(s)\,ds = 0, \quad \forall\varphi \in V.$$

This is a contradiction with $\Lambda < 1$. This completes the proof of Theorem 1(ii).

(iii) If $\Lambda = 1$, by Lemma 1, there exists some $\varphi_\Lambda(x) > 0$, such that

$$\int_\Omega \nabla v(x)\nabla\varphi_\Lambda(x)\,dx = \int_\Omega q(x)v(x)\varphi_\Lambda(x)\,dx + \int_\Gamma q(s)v(s)\varphi_\Lambda(s)\,ds. \tag{17}$$

If u is a positive solution of (1), for the above $\varphi_\Lambda(x)$, we have

$$\int_\Omega \nabla u(x)\nabla\varphi_\Lambda(x)\,dx = \int_\Omega f\bigl(x,u(x)\bigr)\varphi_\Lambda(x)\,dx + \int_\Gamma g\bigl(s,u(s)\bigr)\varphi_\Lambda(s)\,ds. \tag{18}$$

We order $v = u$ in (17), and it follows from (18) that

$$\begin{aligned}
\int_\Omega \nabla u(x)\nabla\varphi_\Lambda(x)\,dx &= \int_\Omega q(x)u(x)\varphi_\Lambda(x)\,dx + \int_\Gamma q(s)u(s)\varphi_\Lambda(s)\,ds \\
&= \int_\Omega f\bigl(x,u(x)\bigr)\varphi_\Lambda(x)\,dx + \int_\Gamma g\bigl(s,u(s)\bigr)\varphi_\Lambda(s)\,ds \\
&\leq \int_\Omega q(x)u(x)\varphi_\Lambda(x)\,dx + \int_\Gamma q(s)u(s)\varphi_\Lambda(s)\,ds,
\end{aligned}$$

which implies that $\int_\Omega (f(x,u) - q(x)u(x))\varphi_\Lambda(x)\,dx + \int_\Gamma (g(s,u) - q(s)u(s))\varphi_\Lambda(s)\,ds = 0$.

When $\varphi_\Lambda(x) > 0$ a.e. in Ω, combining (S2), (S3), and (3), we obtain

$$f(x,u) \leq q(x)u(x), \qquad g(x,u) \leq q(x)u(x).$$

Then we must have $f(x,u) = q(x)u(x)$, $g(x,u) = q(x)u(x)$ a.e. in Ω, $u(x) > 0$ also achieves Λ $(= 1)$. When $u = c\varphi_\Lambda$, $c > 0$, we have $\int_\Omega |\nabla\varphi_\Lambda|^2\,dx = \int_\Omega q(x)\varphi_\Lambda^2\,dx + \int_\Gamma q(s)\varphi_\Lambda^2\,ds$, which achieves Λ.

On the other hand, if for some $c > 0$, $u(x) = c\varphi_\Lambda(x)$ and $f(x, c\varphi_\Lambda(x)) = cq(x)\varphi_\Lambda(x)$, $g(x,u) = cq(x)\varphi_\Lambda(x)$ a.e. $x \in \Omega$, since $c\varphi_\Lambda(x)$ also achieves Λ. This means $u(x) = c\varphi_\Lambda(x)$ is a solution of problem (1) as $\Lambda = 1$. This completes the proof of Theorem 1(iii). □

Proof of Corollary 2 Note that when $q(x) \equiv l$, then $\Lambda = \frac{\lambda_1}{l}$. The conclusion follows from Theorem 1. □

Proof of Theorem 3 When $q(x) \equiv +\infty$, we can replace φ_Λ by φ_1 in (11) and define c as in (12), then following the same procedures as in the proof of Theorem 1(ii), we need to show only that $\{u_n\}$ is bounded in V. For this purpose, let $\{w_n\}$ be defined as in (16). If $\{w_n\}$ is bounded in V, we know $w_n \to w$ is a strong convergence in $L^2(\Omega)$, $w_n \to w$ is convergence a.e. $x \in \Omega$, $w_n \rightharpoonup w$ is a weak convergence in V, and $w \in V$.

If $\|u_n\| \to +\infty$, then $t_n \to 0$ and $w(x) \equiv 0$. We set $\Omega_1 = \{x \in \Omega : w(x) = 0\}$, $\Omega_2 = \{x \in \Omega : w(x) \neq 0\}$. Obviously, by (16), $|u_n| \to +\infty$ a.e. in Ω_2. When $q(x) \equiv +\infty$ in (S3), there exists $K_1, K_2 > 0$ and n large enough we have $|\frac{f(x,u_n)}{u_n}| \geq K_1$, $|\frac{g(x,u_n)}{u_n}| \geq K_2$ uniformly in $x \in \Omega_2$. Hence, by (15) and (16), we obtain

$$\begin{aligned}
4c &= \lim_{n\to+\infty} t_n^2\|u_n\|^2 \\
&= \lim_{n\to+\infty} t_n^2\left(\|\nabla u_n\|_{L^2(\Omega)}^2 + \|\gamma u_n\|_{L^2(\Gamma)}^2\right) \\
&= \lim_{n\to+\infty} t_n^2\left(\int_\Omega f(x,u_n)u_n\,dx + \int_\Gamma g(s,u_n)u_n\,ds + \|\gamma u_n\|_{L^2(\Gamma)}^2\right) \\
&= \lim_{n\to+\infty}\left(\int_\Omega \frac{f(x,u_n)}{u_n}w_n^2\,dx + \int_\Gamma \frac{g(s,u_n)}{u_n}w_n^2\,ds + t_n^2\|\gamma u_n\|_{L^2(\Gamma)}^2\right) \\
&\geq K_1\int_\Omega w^2\,dx + K_2\int_\Gamma w^2\,ds + \|w\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Noticing that $w(x) \neq 0$ in Ω_2 and K_1, K_2 can be chosen large enough, so $m\Omega_2 \equiv 0$ and then $w(x) \equiv 0$ in Ω.

Then we know $\lim_{n\to+\infty}\int_\Omega F(x,w_n)\,dx + \lim_{n\to+\infty}\int_\Gamma G(s,w_n)\,ds = 0$, and consequently,

$$\begin{aligned}
J(w_n) &= \frac{1}{2}\|\nabla w_n\|_{L^2(\Omega)}^2 + o(1) \\
&= \frac{1}{2}\|w_n\|^2 - \frac{1}{2}\|w_n\|_{L^2(\Gamma)}^2 + o(1) \\
&\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1+1}\right)\|w_n\|^2 + o(1) \\
&= 2c\left(1 - \frac{1}{\lambda_1+1}\right) + o(1).
\end{aligned} \tag{19}$$

By $\|u_n\| \to +\infty$, $t_n \to 0$ as $n \to +\infty$, then it follows Lemma 4 and (13), we obtain

$$J(w_n) = J(t_n u_n) \leq \frac{1+t_n^2}{2n} \leq c. \tag{20}$$

Obviously, (19) and (20) are contradictory. So $\{u_n\}$ is bounded in V. This completes the proof of Theorem 3. □

4 Example

In this section, we give two examples on $f(x,u)$: One satisfies (S1) to (S3) with $q(x) \equiv +\infty$, but does not satisfy the (AR) condition; the other illustrates how the assumptions on the boundary are not trivial and compatible with the inner assumptions in Ω.

Example 1 Set:

$$f(x,t) = \begin{cases} 0, & t \leq 0; \\ tln(1+t), & t > 0. \end{cases}$$

Then it is easy to verify that $f(x,t)$ satisfies (S1) to (S3) with $p(x) = 0$ as $t \to 0$ and $q(x) = +\infty$ as $t \to +\infty$. In addition,

$$F(x,t) = \frac{1}{2}t^2 \ln(1+t) - \frac{1}{4}t^2 + \frac{1}{2}t - \frac{1}{2}\ln(1+t).$$

So, for some $\mu > 2$, $\mu F(x,t) = t^2 \ln(1+t)(\frac{\mu}{2} - \frac{\mu}{4\ln(1+t)} + \frac{\mu}{2tln(1+t)} - \frac{\mu}{2t^2}) > t^2 \ln(1+t)$, for all t large.

This means $f(x,t)$ does not satisfy the (AR) condition.

Example 2 Consider the following problem:

$$\begin{cases} -u''(x) = \alpha u(x), & 0 < x < l, \\ u(0) = 0, \\ u'(l) = \alpha u(l), \end{cases} \tag{21}$$

where $\alpha > 0$ is a constant. It is obvious that $g = \gamma f$ as $f(x,u) = \alpha u(x)$. Problem (21) is a case of (1); we can obtain the nontrivial solution: $u(x) = \widetilde{C}\sin\sqrt{\alpha}x$, $\widetilde{C} \neq 0$.

Competing interests
The author declares that he has no competing interests.

Author's contributions
Li G carried out all studies in this article.

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