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On decay properties of solutions for degenerate Kirchhoff equations with strong damping and source terms

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Abstract

We investigate the degenerate Kirchhoff equations with strong damping and source terms of the form

$$\varepsilon u_{tt} - \|\nabla u\|_2^{2\gamma} \Delta u - \Delta u_t = f(u),$$

in a bounded domain. We obtain the optimal decay rate for $\|\nabla u_t\|_2^2$ by deriving its decay estimate from below, provided that either ε is suitably small or the initial data satisfy the proper smallness assumption. The key ingredient in the proof is based on the work of Ono (J. Math. Anal. Appl. 381(1):229-239, 2011), with necessary modification imposed by our problem.

MSC: 35L70; 35L80

Keywords: strong damping; degenerate; Kirchhoff equation; decay estimates

1 Introduction

In this paper, we consider the initial boundary value problem for the following degenerate Kirchhoff equations with strong damping and source terms:

$$\varepsilon u_{tt} - \|\nabla u\|_2^{2\gamma} \Delta u - \Delta u_t = f(u), \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, t \geq 0, \tag{1.3}$$

where Ω is a bounded domain in R^N $(N \geq 1)$ with a smooth boundary $\partial\Omega$. Here, $\varepsilon > 0$ and $\gamma > 0$ are positive constants and $f(u)$ is a nonlinear term like $|u|^{p-1}u$, $p > 1$.

In the case $N = 1$, the nonlinear vibrations of the elastic string are written in the form:

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, \tag{1.4}$$

for $0 < x < L$, $t \geq 0$; where u is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ρ the mass density, h the cross section area, L the length, p_0 the initial axial tension and f the external force. The equation was first introduced by Kirchhoff [1] in the study of stretched strings and plates, and is called the wave equation of Kirchhoff type after his name. Moreover, it is said that (1.4) is degenerate if $p_0 = 0$ and nondegenerate if $p_0 > 0$.

A number of results on the solutions to problem (1.1)-(1.3) have been established by many authors. For example, Hosoya and Yamada [2] studied the following equation:

$$u_{tt} - M\big(\|\nabla u\|_2^2\big)\Delta u + \delta u_t = 0, \tag{1.5}$$

with $M(r) \geq M_0 > 0$ (the nondegenerate case). They proved the global existence of a unique solution under the small data condition in $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Similarly, in the degenerate case ($M(r) = r^\gamma$), Nishihara and Yamada [3] obtained the global existence of solutions for small initial data in $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

Concerning decay properties of solutions, Nakao [4] has derived decay estimates from above of solutions when $f(u) = 0$ in (1.1). Later, Nishihara [5] established a decay estimate from below of the potential of solutions to problem (1.1) without imposing $f(u)$. Nishihara and Ono [6–8] studied detailed cases of nondegenerate type and degenerate type to problem (1.1)-(1.3). They proved the existence and uniqueness of a global solution for initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$ and the polynomial decay of the solution. Recently, in the absence of $f(u)$ in (1.1), Ono [9] proved the optimal decay estimate for $\|\nabla u\|_2^2$. And the decay property of the norm $\|\Delta u\|_2^2$ for $t \geq 0$ was also given in that paper.

Motivated by these works, in this paper, we intend to give the optimal decay estimate for $\|\nabla u_t\|_2$ to problem (1.1)-(1.3). In this way, we can extend the result in [8] where the author considered (1.1) with $\gamma \geq 1$; and we also improve the result of [9] in the presence of a nonlinear source term. We followed the technique skill introduced in [9] with the concept of a stable set in $H^2(\Omega) \cap H_0^1(\Omega)$ to derive the optimal decay rate of $\|\nabla u_t\|_2$. The content of this paper is organized as follows. In Section 2, we give some lemmas which will be used later. In Section 3, we derive the global solution and its decay properties with $\gamma > 0$.

2 Preliminary results

In this section, we shall give some lemmas which will be used throughout this work. We denote by $\|\cdot\|_p$ the L^p norm over Ω.

Lemma 2.1 (The Sobolev-Poincaré inequality) *If $2 \leq p \leq \frac{2N}{N-2}$ ($2 \leq p < \infty$ if $N = 1, 2$), then*

$$\|u\|_p \leq B_1 \|\nabla u\|_2, \quad \textit{for } u \in H_0^1(\Omega)$$

holds with some positive constant B_1.

Lemma 2.2 ([4, 9]) *Let $\phi(t)$ be a nonincreasing and nonnegative function on $[0, \infty)$ such that*

$$\phi(t)^{1+r} \leq \omega_0\big(\phi(t) - \phi(t+1)\big)$$

with certain constants $\omega_0 \geq 0$ and $r > 0$. Then, the function $\phi(t)$ satisfies

$$\phi(t) \leq \big(\phi(0)^{-r} + \omega_0^{-1} r[t-1]^+\big)^{-\frac{1}{r}}$$

for $t \geq 0$, where we put $[a]^+ = \max\{0, a\}$ and $\frac{1}{[a]^+} = \infty$ if $[a]^+ = 0$.

Now, we state the local existence for problem (1.1)-(1.3) which can be established by the arguments of [8].

Theorem 2.3 *Suppose that* $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$, *then there exists a unique solution* u *of problem (1.1)-(1.3) satisfying*

$$u(t) \in C\big([0,T); H^2(\Omega) \cap H_0^1(\Omega)\big),$$
$$u_t(t) \in C\big([0,T); L^2(\Omega)\big) \cap L^2\big(0,T; H_0^1(\Omega)\big).$$

Moreover, at least one of the following statements holds true:

(i) $T = \infty$,

(ii) $\|u_t\|_2^2 + \|\Delta u\|_2^2 \to \infty$, *as* $t \to T^-$.

3 Decay properties

In this section, we shall show the decay properties of solutions u to problem (1.1)-(1.3). For this purpose, we define

$$I(t) \equiv I\big(u(t)\big) = \big\|\nabla u(t)\big\|_2^{2(\gamma+1)} - \big\|u(t)\big\|_{p+1}^{p+1}, \tag{3.1}$$

$$J(t) \equiv J\big(u(t)\big) = \frac{1}{2(\gamma+1)}\big\|\nabla u(t)\big\|_2^{2(\gamma+1)} - \frac{1}{p+1}\big\|u(t)\big\|_{p+1}^{p+1}, \tag{3.2}$$

and the energy function

$$E(t) = \frac{\varepsilon}{2}\|u_t\|_2^2 + J(t), \tag{3.3}$$

for $u(t) \in H_0^1(\Omega)$, $t \geq 0$. Then, multiplying (1.1) by u_t and integrating it over Ω, we see that

$$E'(t) = -\|\nabla u_t\|_2^2. \tag{3.4}$$

Lemma 3.1 *Let* $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$. *Suppose that* p *and* γ *satisfy*

$$p > 2\gamma + 1 \quad \textit{and} \quad 1 < p \leq \frac{N+2}{N-2} \quad (p < \infty,\ \textit{if}\ N = 1, 2). \tag{3.5}$$

If $I(0) > 0$ *and*

$$l = B_1^{p+1}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(0)\right)^{\frac{p-2\gamma-1}{2(\gamma+1)}} < 1, \tag{3.6}$$

then $I(t) > 0$, *for* $t \in [0, T]$.

Proof Since $I(0) > 0$, then there exists $t_{\max} < T$ such that

$$I(t) \geq 0, \quad t \in [0, t_{\max}],$$

which gives

$$J(t) = \frac{p-2\gamma-1}{2(p+1)(\gamma+1)}\big\|\nabla u(t)\big\|_2^{2(\gamma+1)} + \frac{1}{p+1}I(t) \geq \frac{p-2\gamma-1}{2(p+1)(\gamma+1)}\big\|\nabla u(t)\big\|_2^{2(\gamma+1)}. \tag{3.7}$$

Thus, from (3.3) and $E(t)$ being nonincreasing by (3.4), we have that

$$\begin{aligned}\|\nabla u(t)\|_2^{2(\gamma+1)} &\le \frac{2(p+1)(\gamma+1)}{p-2\gamma-1}J(t) \le \frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\\ &\le \frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(0) \quad \text{on } [0,t_{\max}]. \end{aligned} \tag{3.8}$$

And so, exploiting Lemma 2.1, (3.8) and (3.6), we obtain

$$\begin{aligned}\|u\|_{p+1}^{p+1} &\le B_1^{p+1}\|\nabla u\|_2^{p+1} \le B_1^{p+1}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(0)\right)^{\frac{p-2\gamma-1}{2(\gamma+1)}}\|\nabla u(t)\|_2^{2(\gamma+1)}\\ &= l\|\nabla u(t)\|_2^{2(\gamma+1)} < \|\nabla u(t)\|_2^{2(\gamma+1)} \quad \text{on } [0,t_{\max}]. \end{aligned} \tag{3.9}$$

Therefore,

$$I(t) = \|\nabla u(t)\|_2^{2(\gamma+1)} - \|u(t)\|_{p+1}^{p+1} > 0 \quad \text{on } [0,t_{\max}].$$

By repeating this procedure and using the fact that

$$\lim_{t\to t_{\max}} B_1^{p+1}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\right)^{\frac{p-2\gamma-1}{2(\gamma+1)}} \le l < 1,$$

we can take $t_{\max} = T$. □

Lemma 3.2 *Let u satisfy the assumptions of Lemma 3.1. Then there exists $0 < \eta < 1$ such that*

$$\|u(t)\|_{p+1}^{p+1} \le (1-\eta)\|\nabla u(t)\|_2^{2(\gamma+1)} \quad \textit{for } t \in [0,T], \tag{3.10}$$

with $\eta = 1 - l$.

Proof From (3.9), we get

$$\|u(t)\|_{p+1}^{p+1} \le l\|\nabla u(t)\|_2^{2(\gamma+1)}, \quad \text{for } t \in [0,T].$$

Let $\eta = 1 - l$, then we have the inequality (3.10). □

Theorem 3.3 *Suppose that $(u_0,u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$, $I(0) > 0$, (3.5) and (3.6) hold. Then there exists a unique global solution u of problem (1.1)-(1.3) satisfying*

$$\begin{aligned}&u(t) \in C\big([0,\infty); H^2(\Omega)\cap H_0^1(\Omega)\big) \cap L^\infty\big(0,\infty; H^2(\Omega)\cap H_0^1(\Omega)\big),\\ &u_t(t) \in C\big([0,\infty); L^2(\Omega)\big) \cap L^2\big(0,\infty; H_0^1(\Omega)\big) \cap L^\infty\big(0,\infty; L^2(\Omega)\big).\end{aligned}$$

Furthermore, we have the following decay estimates:

$$E(t) \le \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma}{\gamma+1}\big(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\big)^{-2}[t-1]^+\right)^{-\frac{\gamma+1}{\gamma}} \quad \textit{for } t \ge 0, \tag{3.11}$$

where α and β are some positive constants given by (3.15).

Proof First, for $T = \infty$, we can obtain the result by following the arguments as in [8], so we omit the proof. Next, we will show the decay estimate. For $t \geq 0$, integrating (3.4) over $[t, t+1]$, we have

$$\int_t^{t+1} \|\nabla u_t(s)\|_2^2 \, ds = E(t) - E(t+1) \equiv D(t)^2. \tag{3.12}$$

Then there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|\nabla u_t(t_i)\|_2^2 \leq 4D(t)^2, \quad i = 1, 2. \tag{3.13}$$

Multiplying (1.1) by u, integrating it over $\Omega \times [t_1, t_2]$, using integration by parts and Lemma 2.1, we get

$$\begin{aligned}
&\int_{t_1}^{t_2} \left(\|\nabla u(s)\|_2^{2(\gamma+1)} - \|u(s)\|_{p+1}^{p+1}\right) ds \\
&\quad \leq \varepsilon B_1^2 \int_t^{t+1} \|\nabla u_t\|_2^2 \, ds + \varepsilon B_1^2 \sum_{i=1}^{2} \|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 \\
&\qquad + \int_t^{t+1} \|\nabla u_t\|_2 \|\nabla u\|_2 \, ds.
\end{aligned}$$

Hence, from the definition of $I(t)$ by (3.1), using (3.12), (3.13) and (3.8), we obtain

$$\int_{t_1}^{t_2} I(s)\, ds \leq \varepsilon B_1^2 D(t)^2 + \left(4\varepsilon B_1^2 + 1\right) D(t) \left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1} E(t)\right)^{\frac{1}{2(\gamma+1)}}. \tag{3.14}$$

On the other hand, integrating (3.4) over $[t, t_2]$, noting that $E(t_2) \leq 2\int_{t_1}^{t_2} E(t)\, dt$ due to $t_2 - t_1 \geq \frac{1}{2}$, and using (3.3) and (3.7), we have that

$$\begin{aligned}
E(t) &= E(t_2) + \int_t^{t_2} \|\nabla u_t(s)\|_2^2 \, ds \\
&\leq 2\int_{t_1}^{t_2} E(s)\, ds + \int_t^{t_2} \|\nabla u_t(s)\|_2^2 \, ds \\
&\leq \left(1 + \varepsilon B_1^2\right) \int_t^{t_2} \|\nabla u_t(s)\|_2^2 \, ds + \frac{p-2\gamma-1}{(p+1)(\gamma+1)} \int_t^{t_2} \|\nabla u(s)\|_2^{2(\gamma+1)} \, ds \\
&\quad + \frac{2}{(p+1)} \int_t^{t_2} I(s)\, ds.
\end{aligned}$$

We then use the fact that $\|\nabla u(t)\|_2^{2(\gamma+1)} \leq \frac{1}{1-l} I(t)$ by (3.10), (3.12) and (3.14) to obtain

$$\begin{aligned}
E(t) &\leq \left(1 + \varepsilon B_1^2\right) D(t)^2 + c_1 \int_t^{t_2} I(t)\, dt \\
&\leq \alpha D(t)^2 + \beta D(t) E(t)^{\frac{1}{2(\gamma+1)}},
\end{aligned} \tag{3.15}$$

where

$$\alpha = \left(1 + \varepsilon B_1^2 + c_1 \varepsilon B_1^2\right) \quad \text{and} \quad \beta = c_1 \left(4\varepsilon B_1^2 + 1\right) \left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}\right)^{\frac{1}{2(\gamma+1)}} \tag{3.16}$$

with

$$c_1 = \frac{p-2\gamma-1}{(p+1)(\gamma+1)(1-l)} + \frac{2}{(p+1)}.$$

Moreover, from (3.12) and $E(t)$ being nonincreasing by (3.4), we note that

$$D(t) \le E^{\frac{1}{2}}(t) \le E(0)^{\frac{\gamma}{2(\gamma+1)}} E(t)^{\frac{1}{2(\gamma+1)}}.$$

Thus, by Young's inequality, (3.15) becomes

$$\begin{aligned} E(t) &\le \bigl(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\bigr) D(t) E(t)^{\frac{1}{2(\gamma+1)}} \\ &\le \frac{2\gamma+1}{2(\gamma+1)} \bigl[\bigl(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\bigr) D(t)\bigr]^{\frac{2(\gamma+1)}{2\gamma+1}} + \frac{1}{2(\gamma+1)} E(t). \end{aligned}$$

This implies that

$$\begin{aligned} E(t)^{1+\frac{\gamma}{\gamma+1}} &\le \bigl(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\bigr)^2 D^2(t) \\ &= \bigl(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\bigr)^2 \bigl(E(t) - E(t+1)\bigr). \end{aligned}$$

Therefore, applying Lemma 2.2, we conclude that

$$E(t) \le \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma+1}{\gamma} \bigl(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\bigr)^{-2} [t-1]^+ \right)^{-\frac{\gamma+1}{\gamma}}, \quad \text{for } t \ge 0. \qquad \square$$

Remark (i) Based on the estimate (3.11), we have further the following estimates:

$$\|\nabla u\|_2^2 \le c_2 (1+t)^{-\frac{1}{\gamma}}, \qquad \|u_t\|_2^2 \le c_3 (1+t)^{-1-\frac{1}{\gamma}}, \quad \text{for } t \ge 0, \tag{3.17}$$

where c_2 and c_3 are some positive constants.

(ii) If $\omega > \frac{\gamma}{\gamma+1}$, then

$$\int_0^t E(s)^\omega \, ds \le k_1 E(0)^{\omega - \frac{\gamma}{\gamma+1}}, \tag{3.18}$$

with $k_1 = \frac{(\gamma+1)^2\omega - \gamma}{((\gamma+1)\omega-\gamma)(\gamma+1)} (\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta)^2$. Indeed, by (3.11), we see that

$$\begin{aligned} \int_0^t E(s)^\omega \, ds &\le \int_0^t \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma+1}{\gamma} \bigl(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\bigr)^{-2} [s-1]^+ \right)^{-\frac{(\gamma+1)\omega}{\gamma}} ds \\ &= E(0)^\omega + \int_1^t \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma+1}{\gamma} \bigl(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\bigr)^{-2} (s-1) \right)^{-\frac{(\gamma+1)\omega}{\gamma}} ds. \end{aligned}$$

A direct computation yields the result.

Next, we will improve the decay rate for $\|u_t\|_2^2$ given by (3.17).

Proposition 3.4 *If* $(u_0,u_1)\in H^2(\Omega)\cap H_0^1(\Omega)\times L^2(\Omega)$, *then the solution* u *of problem (1.1)-(1.3) satisfies*

$$\|u_t\|_2^2\le c_4(1+t)^{-2-\frac{1}{\gamma}}\quad\textit{and}\quad\int_0^t\|\nabla u_t\|_2^2\,ds\le c_5,\quad\textit{for } t\ge 0,\tag{3.19}$$

where c_4 *and* c_5 *are some positive constants.*

Proof Multiplying (1.1) by u_t and integrating it over Ω, we have

$$\frac{\varepsilon}{2}\frac{d}{dt}\|u_t\|_2^2+\|\nabla u_t\|_2^2=-\|\nabla u\|_2^{2\gamma}\int_\Omega\nabla u\nabla u_t\,dx+\int_\Omega|u|^{p-1}uu_t\,dx.\tag{3.20}$$

We now estimate the right-hand side of (3.20). Employing Hölder's inequality and Young's inequality, the first term gives

$$\begin{aligned}\|\nabla u\|_2^{2\gamma}\int_\Omega\nabla u\nabla u_t\,dx&\le\|\nabla u\|_2^{2\gamma+1}\|\nabla u_t\|_2\\&\le\|\nabla u\|_2^{2(2\gamma+1)}+\frac{1}{4}\|\nabla u_t\|_2^2.\end{aligned}$$

As for the second term, using Hölder's inequality, Lemma 2.1, Young's inequality and (3.8) yields

$$\begin{aligned}\left|\int_\Omega|u|^{p-1}uu_t\,dx\right|&\le\|u\|_{p+1}^p\|u_t\|_{p+1}\le B_1^{p+1}\|\nabla u\|_2^p\|\nabla u_t\|_2\\&\le B_1^{2(p+1)}\|\nabla u\|_2^{2p}+\frac{1}{4}\|\nabla u_t\|_2^2\\&\le B_1^{2(p+1)}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(0)\right)^{\frac{p-2\gamma-1}{\gamma+1}}\|\nabla u\|_2^{2(2\gamma+1)}+\frac{1}{4}\|\nabla u_t\|_2^2.\end{aligned}$$

Combining these estimates and using (3.17), we arrive at

$$\varepsilon\frac{d}{dt}\|u_t\|_2^2+\|\nabla u_t\|_2^2\le c_6\|\nabla u\|_2^{2(2\gamma+1)}\le c_7(1+t)^{-2-\frac{1}{\gamma}},$$

where $c_6=2(1+B_1^{2(p+1)}(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(0))^{\frac{p-2\gamma-1}{\gamma+1}})$ and $c_7=c_6c_2^{2\gamma+1}$. Then, from Lemma 2.1, we have

$$\varepsilon\frac{d}{dt}\|u_t\|_2^2+B_1^{-1}\|u_t\|_2^2\le c_7(1+t)^{-2-\frac{1}{\gamma}}.$$

Therefore, the desired estimate (3.19) is obtained. □

In order to obtain the decay estimates for problem (1.1)-(1.3), we need the function $H(t)$ and equality (3.22) as in [9]. Define

$$H(t)=\begin{cases}\varepsilon\frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2\gamma}}+\|\nabla u\|_2^2, & \text{if } \gamma\ge 1;\\ \varepsilon\frac{\|u_t\|_2^2}{\|\nabla u\|_2^2}+\|\nabla u\|_2^{2\gamma}, & \text{if } 0<\gamma<1.\end{cases}\tag{3.21}$$

Multiplying (1.1) by $\frac{2u_t}{\|\nabla u\|_2^{2(\gamma+k)}}$ with $k \geq 0$ and integrating it over Ω, we have the following equality:

$$\frac{d}{dt}\left[\varepsilon\frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+k)}}+\frac{1}{\|\nabla u\|_2^{2(k-1)}}\right]+2\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+k)}}$$
$$=-2\varepsilon(\gamma+k)\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^{2(\gamma+k+1)}}\|u_t\|_2^2-2k\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^{2k}}+2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^{2(\gamma+k)}}. \tag{3.22}$$

Proposition 3.5 *Let* $(u_0,u_1)\in H^2(\Omega)\cap H_0^1(\Omega)\times L^2(\Omega)$, $I(0)>0$, *(3.5)-(3.6) hold and* $\|\nabla u_0\|_2>0$. *Suppose that* $\|\nabla u(t)\|_2>0$ *for* $0\leq t<T_1$ *and*

$$1-2\gamma B_1\big(\varepsilon H^\gamma(0)\big)^{\frac{1}{2}}>0,\quad \text{if } \gamma\geq 1; \tag{3.23}$$

$$1-2B_1\big(\varepsilon H(0)\big)^{\frac{1}{2}}>0,\quad \text{if } 0<\gamma<1. \tag{3.24}$$

Then it holds that, for $0\leq t<T_1$,

$$H(t)\leq H(0)+\alpha_1 E(0)^{\frac{p-2\gamma}{\gamma+1}},\quad \text{if } \gamma\geq 1; \tag{3.25}$$

$$H(t)\leq H(0)+\alpha_2 E(0)^{\frac{\gamma}{\gamma+1}},\quad \text{if } 0<\gamma<1, \tag{3.26}$$

where constants α_1 *and* α_2 *are given by (3.31) and (3.36).*

Proof (i) In the case $\gamma\geq 1$, we observe from (3.21) that

$$\varepsilon\frac{\|u_t\|_2}{\|\nabla u\|_2}=\varepsilon^{\frac{1}{2}}\left(\varepsilon\frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2\gamma}}\|\nabla u\|_2^{2\gamma-2}\right)^{\frac{1}{2}}\leq\big(\varepsilon H^\gamma(t)\big)^{\frac{1}{2}}. \tag{3.27}$$

Using (3.22) with $k=0$ yields

$$\frac{d}{dt}H(t)+2\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2\gamma}}=-2\gamma\varepsilon\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^{2(\gamma+1)}}\|u_t\|_2^2+2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^{2\gamma}}. \tag{3.28}$$

Now, we estimate the right-hand side of (3.28). So, thanks to Hölder's inequality, Lemma 2.1, (3.8) and Young's inequality, we see that

$$2\gamma\varepsilon\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^{2(\gamma+1)}}\|u_t\|_2^2\leq 2\gamma B_1\varepsilon\frac{\|u_t\|_2}{\|\nabla u\|_2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2\gamma}},$$

and

$$\begin{aligned}2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^{2\gamma}}&\leq 2B_1^{p+1}\|\nabla u\|_2^{p-\gamma}\frac{\|\nabla u_t\|_2}{\|\nabla u\|_2^{\gamma}}\\&\leq 2B_1^{p+1}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\right)^{\frac{p-\gamma}{2(\gamma+1)}}\frac{\|\nabla u_t\|_2}{\|\nabla u\|_2^{\gamma}}\\&\leq B_1^{2(p+1)}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\right)^{\frac{p-\gamma}{\gamma+1}}+\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2\gamma}}.\end{aligned}$$

Taking into account these two estimates in (3.28) and using $\varepsilon \frac{\|u_t\|_2}{\|\nabla u\|_2} \leq (\varepsilon H^\gamma(t))^{\frac{1}{2}}$ by (3.27), we obtain

$$\frac{d}{dt}H(t) + \left(1 - 2\gamma B_1\left(\varepsilon H^\gamma(t)\right)^{\frac{1}{2}}\right)\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2\gamma}} \leq B_1^{2(p+1)}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\right)^{\frac{p-\gamma}{\gamma+1}}.$$

If $1 - 2\gamma B_1(\varepsilon H^\gamma(0))^{\frac{1}{2}} > 0$, then there exists T_2 such that $0 < T_2 \leq T_1$ and

$$1 - 2\gamma B_1\left(\varepsilon H^\gamma(t)\right)^{\frac{1}{2}} \geq 0, \quad \text{for } 0 \leq t \leq T_2. \tag{3.29}$$

Hence

$$\frac{d}{dt}H(t) \leq B_1^{2(p+1)}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\right)^{\frac{p-\gamma}{\gamma+1}}, \tag{3.30}$$

for $0 \leq t \leq T_2$. Applying (3.18) with $\omega = \frac{p-\gamma}{\gamma+1}$, we derive that

$$H(t) \leq H(0) + \alpha_1 E(0)^{\frac{p-2\gamma}{\gamma+1}}, \tag{3.31}$$

with

$$\alpha_1 = B_1^{2(p+1)}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}\right)^{\frac{p-\gamma}{\gamma+1}}\frac{(\gamma+1)(p-\gamma)-\gamma}{(p-2\gamma)(\gamma+1)}\left(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\right)^2.$$

Therefore, we see that (3.29)-(3.31) hold true for $0 \leq t < T_1$, which gives the estimate (3.25) for $\gamma \geq 1$.

(ii) In the case $0 < \gamma < 1$, we note from (3.21) that

$$\varepsilon\frac{\|u_t\|_2}{\|\nabla u\|_2} \leq \left(\varepsilon H(t)\right)^{\frac{1}{2}}. \tag{3.32}$$

From (3.22) with $k = 1 - \gamma$, we have

$$\begin{aligned}\frac{d}{dt}H(t) + 2\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^2} &= -2\varepsilon\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^4}\|u_t\|_2^2 - 2(1-\gamma)\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^{2(1-\gamma)}} \\ &\quad + 2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^2}.\end{aligned} \tag{3.33}$$

Similar to those estimates, as in the case for $\gamma \geq 1$, the right-hand side of (3.33) can be estimated as follows:

$$\begin{aligned}2\varepsilon\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^4}\|u_t\|_2^2 &\leq 2B_1\varepsilon\frac{\|u_t\|_2}{\|\nabla u\|_2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^2} \leq 2B_1\left(\varepsilon H(t)\right)^{\frac{1}{2}}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^2}, \\ 2(1-\gamma)\frac{\int_\Omega \nabla u\nabla u_t\,dx}{\|\nabla u\|_2^{2(1-\gamma)}} &\leq 2(1-\gamma)\|\nabla u\|_2^{2\gamma}\frac{\|\nabla u_t\|_2}{\|\nabla u\|_2} \leq 2(1-\gamma)^2\left(\|\nabla u\|_2^2\right)^{2\gamma} + \frac{1}{2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^2} \\ &\leq 2(1-\gamma)^2\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\right)^{\frac{2\gamma}{\gamma+1}} + \frac{1}{2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^2},\end{aligned}$$

and

$$2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^2} \le 2B_1^{2(p+1)}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}E(t)\right)^{\frac{p-1}{\gamma+1}} + \frac{1}{2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^2}.$$

Combining these estimates, (3.33) becomes

$$\begin{aligned}
&\frac{d}{dt}H(t) + \left(1 - 2B_1\left(\varepsilon H(t)\right)^{\frac{1}{2}}\right)\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^2} \\
&\quad\le 2(1-\gamma)^2\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}\right)^{\frac{2\gamma}{\gamma+1}} E(t)^{\frac{2\gamma}{\gamma+1}} \\
&\qquad + 2B_1^{2(p+1)}\left(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1}\right)^{\frac{p-1}{\gamma+1}} E(t)^{\frac{p-1}{\gamma+1}} \\
&\quad= c_8 E(0)^{\frac{2\gamma}{\gamma+1}}\left(\frac{E(t)}{E(0)}\right)^{\frac{2\gamma}{\gamma+1}} + c_9 E(0)^{\frac{p-1}{\gamma+1}}\left(\frac{E(t)}{E(0)}\right)^{\frac{p-1}{\gamma+1}} \\
&\quad\le c_{10}\left(E(t)\right)^{\frac{2\gamma}{\gamma+1}},
\end{aligned}$$

where we have used the fact that $0 < E(t) \le E(0)$ and $p > 2\gamma + 1$ on the last inequality with $c_8 = 2(1-\gamma)^2(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1})^{\frac{2\gamma}{\gamma+1}}$, $c_9 = 2B_1^{2(p+1)}(\frac{2(p+1)(\gamma+1)}{p-2\gamma-1})^{\frac{p-1}{\gamma+1}}$ and $c_{10} = c_8 + c_9 E(0)^{\frac{p-2\gamma-1}{\gamma+1}}$.

If $1 - 2B_1(\varepsilon H(0))^{\frac{1}{2}} > 0$, then there exists T_3 such that $0 < T_3 \le T_1$ and

$$1 - 2B_1\left(\varepsilon H(t)\right)^{\frac{1}{2}} \ge 0, \quad \text{for } 0 \le t \le T_3. \tag{3.34}$$

Thus

$$\frac{d}{dt}H(t) \le c_{10}\left(E(t)\right)^{\frac{2\gamma}{\gamma+1}}, \tag{3.35}$$

for $0 \le t \le T_3$. Employing (3.18) with $\omega = \frac{2\gamma}{\gamma+1}$ gives

$$H(t) \le H(0) + \alpha_2 E(0)^{\frac{\gamma}{\gamma+1}}, \quad \text{for } 0 \le t \le T_3, \tag{3.36}$$

with

$$\alpha_2 = \frac{(2\gamma+1)c_{10}}{\gamma+1}\left(\alpha E(0)^{\frac{\gamma}{2(\gamma+1)}} + \beta\right)^2.$$

Therefore, we see that (3.34)-(3.36) hold true for $0 \le t < T_1$, which gives the estimate (3.26) for $0 < \gamma < 1$. □

Proposition 3.6 *Under the same assumptions as in Proposition 3.5, assume further that the initial data satisfies*

$$\begin{cases} 1 - 2(\gamma+2)B_1(\varepsilon(H(0) + \alpha_1 E(0)^{\frac{p-2\gamma}{\gamma+1}})^\gamma)^{\frac{1}{2}} > 0, & \text{if } \gamma \ge 1; \\ 1 - 2(2\gamma+1)B_1(\varepsilon(H(0) + \alpha_2 E(0)^{\frac{\gamma}{\gamma+1}}))^{\frac{1}{2}} > 0, & \text{if } 0 < \gamma < 1. \end{cases} \tag{3.37}$$

Then it holds that

$$\|\nabla u\|_2^2 \geq \alpha_3(1+t)^{-\frac{1}{\gamma}}, \quad \text{for } 0 \leq t < T_1, \tag{3.38}$$

where α_3 *is a positive constant.*

Proof (i) In the case $\gamma \geq 1$, we obtain from (3.22) with $k = 2$ that

$$\begin{aligned} &\frac{d}{dt}\left[\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} + \frac{1}{\|\nabla u\|_2^2}\right] + 2\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} \\ &\quad = -2(\gamma+2)\varepsilon \frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^{2(\gamma+3)}}\|u_t\|_2^2 - 4\frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^4} + 2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^{2(\gamma+2)}}. \end{aligned} \tag{3.39}$$

As in deriving the estimates for case (i) of Proposition 3.5, we get the following estimates:

$$\begin{aligned} &2(\gamma+2)\varepsilon \frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^{2(\gamma+3)}}\|u_t\|_2^2 \leq 2(\gamma+2)B_1\varepsilon \frac{\|u_t\|_2}{\|\nabla u\|_2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}}, \\ &4\frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^4} \leq \frac{1}{2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} + 8\|\nabla u\|_2^{2(\gamma-1)}, \end{aligned}$$

and

$$2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^{2(\gamma+2)}} \leq \frac{1}{2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} + 2B_1^{2(p+1)}\|\nabla u\|_2^{2(p-\gamma-2)}.$$

Thus, using the above estimates in (3.39), together with the fact that $\varepsilon\frac{\|u_t\|_2}{\|\nabla u\|_2} \leq (\varepsilon H^\gamma(t))^{\frac{1}{2}}$ by (3.27) and the estimate (3.25), yield

$$\begin{aligned} &\frac{d}{dt}\left[\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} + \frac{1}{\|\nabla u\|_2^2}\right] + \frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} \\ &\quad \leq 2(\gamma+2)B_1\bigl(\varepsilon H^\gamma(t)\bigr)^{\frac{1}{2}}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} \\ &\qquad + 8\|\nabla u\|_2^{2(\gamma-1)} + 2B_1^{2(p+1)}\|\nabla u\|_2^{2(p-\gamma-2)} \\ &\quad \leq 2(\gamma+2)B_1\bigl(\varepsilon\bigl(H(0)+\alpha_1E(0)^{\frac{p-2\gamma}{\gamma+1}}\bigr)^\gamma\bigr)^{\frac{1}{2}}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} \\ &\qquad + 8\|\nabla u\|_2^{2(\gamma-1)} + 2B_1^{2(p+1)}\|\nabla u\|_2^{2(p-\gamma-2)}, \end{aligned}$$

for $0 \leq t < T_1$.

If $1 - 2(\gamma+2)B_1(\varepsilon(H(0)+\alpha_1E(0)^{\frac{p-2\gamma}{\gamma+1}})^\gamma)^{\frac{1}{2}} > 0$, then, using $\|\nabla u\|_2^2 \leq c_2(1+t)^{-\frac{1}{\gamma}}$ by (3.17), we see that

$$\begin{aligned} \frac{d}{dt}\left[\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} + \frac{1}{\|\nabla u\|_2^2}\right] &\leq c_{11}\bigl((1+t)^{-\frac{\gamma-1}{\gamma}} + (1+t)^{-\frac{p-\gamma-2}{\gamma}}\bigr) \\ &\leq 2c_{11}(1+t)^{-\lambda_1} = 2c_{11}(1+t)^{-\frac{\gamma-1}{\gamma}}, \end{aligned}$$

and

$$\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(\gamma+2)}} + \frac{1}{\|\nabla u\|_2^2} \le c_{12}(1+t)^{\frac{1}{\gamma}},$$

which implies that

$$\|\nabla u\|_2^2 \ge c_{13}(1+t)^{-\frac{1}{\gamma}}, \quad \text{for } 0 \le t < T_1,$$

with $\lambda_1 = \min\{\frac{\gamma-1}{\gamma}, \frac{p-\gamma-2}{\gamma}\} = \frac{\gamma-1}{\gamma}$ and some positive constants c_i, $i = 11\text{-}13$.

(ii) In the case $0 < \gamma < 1$, it follows from (3.22) with $k = 1+\gamma$ that

$$\begin{aligned} &\frac{d}{dt}\left[\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} + \frac{1}{\|\nabla u\|_2^{2\gamma}}\right] + 2\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} \\ &\quad = -2(2\gamma+1)\varepsilon \frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^{2(2\gamma+2)}}\|u_t\|_2^2 - 2(\gamma+1)\frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^{2(\gamma+1)}} \\ &\qquad + 2\frac{\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^{2(2\gamma+1)}}. \end{aligned} \tag{3.40}$$

Estimate the right-hand side of (3.40) as in the case (ii) of Proposition 3.5 to obtain

$$\begin{aligned} &2(2\gamma+1)\varepsilon \frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^{2(2\gamma+2)}}\|u_t\|_2^2 \le 2(2\gamma+1)B_1\big(\varepsilon H(t)\big)^{\frac{1}{2}}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}}, \\ &2(\gamma+1)\frac{\int_\Omega \nabla u \nabla u_t\,dx}{\|\nabla u\|_2^{2(\gamma+1)}} \le 2(\gamma+1)^2 + \frac{1}{2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}}, \end{aligned}$$

and

$$\frac{2\int_\Omega |u|^{p-1}uu_t\,dx}{\|\nabla u\|_2^{2(2\gamma+1)}} \le 2B_1^{2(p+1)}\|\nabla u\|_2^{2(p-2\gamma-1)} + \frac{1}{2}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}}.$$

Back to (3.40), employing these estimates and using (3.26), we deduce that

$$\begin{aligned} &\frac{d}{dt}\left[\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} + \frac{1}{\|\nabla u\|_2^{2\gamma}}\right] + \frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} \\ &\quad \le 2(2\gamma+1)B_1\big(\varepsilon H(t)\big)^{\frac{1}{2}}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} \\ &\qquad + 2(\gamma+1)^2 + 2B_1^{2(p+1)}\big(\|\nabla u\|_2^2\big)^{(p-2\gamma-1)} \\ &\quad \le 2(2\gamma+1)B_1\Big(\varepsilon\big(H(0)+\alpha_2 E(0)^{\frac{\gamma}{\gamma+1}}\big)\Big)^{\frac{1}{2}}\frac{\|\nabla u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} \\ &\qquad + 2(\gamma+1)^2 + 2B_1^{2(p+1)}\big(\|\nabla u\|_2^2\big)^{(p-2\gamma-1)} \end{aligned}$$

for $0 \le t < T_1$.

If $1 - 2(2\gamma+1)B_1(\varepsilon(H(0)+\alpha_2 E(0)^{\frac{\gamma}{\gamma+1}}))^{\frac{1}{2}} > 0$, then, by $\|\nabla u\|_2^2 \le c_2(1+t)^{-\frac{1}{\gamma}}$, we have

$$\frac{d}{dt}\left[\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} + \frac{1}{\|\nabla u\|_2^{2\gamma}}\right] \le 2(\gamma+1)^2 + c_{14}(1+t)^{-\frac{p-2\gamma-1}{\gamma}} \le c_{15},$$

and

$$\varepsilon \frac{\|u_t\|_2^2}{\|\nabla u\|_2^{2(2\gamma+1)}} + \frac{1}{\|\nabla u\|_2^{2\gamma}} \le c_{16}(1+t),$$

for $0 \le t < T_1$ with some positive constants c_i, $i = 14$-16. This implies the desired estimate (3.38) for $0 < \gamma < 1$. □

Now, we are ready to state and prove our main result.

Theorem 3.7 *Let* $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$, $I(0) > 0$ *and (3.5)-(3.6) hold. Then the solution u of problem (1.1)-(1.3) satisfies*

$$\|\nabla u\|_2^2 \le \alpha_4(1+t)^{-\frac{1}{\gamma}} \quad \textit{and} \quad \|u_t\|_2^2 \le \alpha_5(1+t)^{-2-\frac{1}{\gamma}} \quad \textit{for } t \ge 0. \tag{3.41}$$

Moreover, suppose that the initial data satisfies $u_0 \neq 0$ *and*

$$\begin{cases} 1 - 2(\gamma+2)B_1(\varepsilon(H(0) + \alpha_1 E(0)^{\frac{p-2\gamma}{\gamma+1}})^{\gamma})^{\frac{1}{2}} > 0, & \textit{if } \gamma \ge 1; \\ 1 - 6B_1(\varepsilon(H(0) + \alpha_2 E(0)^{\frac{\gamma}{\gamma+1}}))^{\frac{1}{2}} > 0, & \textit{if } 0 < \gamma < 1. \end{cases} \tag{3.42}$$

Then

$$\|\nabla u\|_2^2 \ge \alpha_6(1+t)^{-\frac{1}{\gamma}}, \quad \textit{for } t \ge 0, \tag{3.43}$$

with some positive constants α_i, $i = 4, 5, 6$.

Proof Thanks to the decay estimates (3.11) and (3.17), we obtain (3.41). Setting

$$T_1 = \sup\{t \in [0,\infty) \mid \|\nabla u(s)\|_2^2 > 0 \text{ for } 0 \le s < t\},$$

then, we see that $T_1 > 0$ and $\|\nabla u(t)\|_2^2 > 0$ for $0 \le t < T_1$, because of $\|\nabla u_0\|_2 > 0$. If $T_1 < \infty$, then it holds that

$$\|\nabla u(T_1)\|_2^2 = 0. \tag{3.44}$$

However, from $\|\nabla u\|_2^2 \ge \alpha_3(1+t)^{-\frac{1}{\gamma}}$ by (3.38), we observe that

$$\lim_{t \to T_1} \|\nabla u(t)\|_2^2 \ge C(1+T_1)^{-\frac{1}{\gamma}} > 0,$$

which contradicts (3.44). Hence, we have $T_1 = \infty$ and $\|\nabla u(t)\|_2^2 > 0$ for all $t \ge 0$. Therefore, from (3.38), we have

$$\|\nabla u(t)\|_2^2 \ge \alpha_6(1+t)^{-\frac{1}{\gamma}}, \quad \text{for } t \ge 0,$$

which gives the decay estimate (3.44). □

Competing interests
The author declares that they have no competing interests.

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