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Existence results of fractional integro-differential equations with m -point multi-term fractional order integral boundary conditions

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Abstract

In this article, we present some new existence and uniqueness results for nonlinear fractional integro-differential equations with m -point multi-term fractional order integral boundary conditions. Our results are based on the Banach contraction principle and Krasnoselskii's fixed point theorem.

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1 Introduction

Differential equations of fractional order arise in several research areas of science and engineering, such as physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, electrical networks, control of dynamical systems, *etc.* Fractional differential equations provide an excellent tool for description of memory and hereditary properties of various materials and processes. Some recent contributions to the theory of fractional differential equations and its applications can be seen in [1–5].

Recently, many researchers have given attention to the existence of solutions of the initial and boundary value problems for fractional differential equations. There are some papers that have studied the existence of solutions to boundary value problems with two-point, three-point, multi-point or integral boundary conditions. See for examples [6–30]. However, to the best of the authors' knowledge, there are only a few papers that consider nonlinear fractional differential equations with nonlocal fractional order integral boundary conditions, see [31–33].

In this article, we study the existence and uniqueness of solutions for the following nonlinear fractional integro-differential equation with m -point multi-term fractional order integral boundary condition:

$${}^c D^q x(t) = f(t, x(t), (\psi^v x)(t)), \quad t \in [0, T], q \in (1, 2], \quad (1.1)$$

$$x(0) = 0, \quad x(T) = \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} [I^{p_{j,i}} x]_{\xi_{i-1}}^{\xi_i}, \quad (1.2)$$

where ${}^c D^q$ denotes the Caputo fractional derivative of order q , $[I^{p_{j,i}}x]_{\xi_{i-1}}^{\xi_i} = [I^{p_{j,i}}x](\xi_i) - [I^{p_{j,i}}x](\xi_{i-1})$, $I^{p_{j,i}}$ is the Riemann-Liouville fractional integral of order $p_{j,i} > 0$, $j = 1, 2, \dots, n_i$, $n_i \in \mathbb{N} = \{1, 2, \dots\}$, $i = 1, 2, \dots, m-1$, $0 = \xi_0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < \xi_{m-1} = T$, $f : [0, T] \times X \times X \rightarrow X$, for $\phi : [0, T] \times [0, T] \rightarrow [0, \infty)$,

$$(\psi^\nu x)(t) = [I^\nu \phi x](t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \phi(t,s)x(s) ds, \quad \nu > 0,$$

and $\alpha_{j,i} \in \mathbb{R}$ is such that $\sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \left(\frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i}+2)} \right) \neq T$. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{C} = C([0, T], X)$ denote the Banach space of all continuous functions from $[0, T] \rightarrow X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|_{\mathcal{C}}$.

We note that if $m = 2$, then condition (1.2) reduces to the two-point multi-term fractional integral boundary condition

$$x(0) = 0, \quad x(T) = \sum_{j=1}^{n_1} \alpha_{j,1} [I^{p_{j,1}}x](T), \quad n_1 \in \mathbb{N} = \{1, 2, \dots\}.$$

If $n_i = 1$, $i = 1, 2, \dots, m-1$, then condition (1.2) reduces to the m -point fractional integral boundary condition

$$x(0) = 0, \quad x(T) = \sum_{i=1}^{m-1} \alpha_{1,i} [I^{p_{1,i}}x]_{\xi_{i-1}}^{\xi_i}. \tag{1.3}$$

Moreover, if $p_{1,i} = 1$ and $\alpha_{1,i} = \alpha_i$ for $i = 1, 2, \dots, m-1$, then condition (1.3) reduces to the usual m -point integral boundary condition

$$x(0) = 0, \quad x(T) = \alpha_1 \int_0^{\xi_1} x(s) ds + \alpha_2 \int_{\xi_1}^{\xi_2} x(s) ds + \dots + \alpha_{m-1} \int_{\xi_{m-2}}^T x(s) ds.$$

In this case, the boundary condition corresponds to $m-1$ intervals of area under the curve of solution $x(t)$ from $t = \xi_{i-1}$ to $t = \xi_i$ for $i = 1, 2, \dots, m-1$.

2 Preliminaries

Let us recall some basic definitions [2, 4].

Definition 2.1 For a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds, \quad n-1 < q < n, n = [q] + 1,$$

provided that $f^{(n)}(t)$ exists, where $[q]$ denotes the integer part of the real number q .

Definition 2.2 The Riemann-Liouville fractional integral of order q for a continuous function $f(t)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad q > 0,$$

provided that such integral exists.

Definition 2.3 The Riemann-Liouville fractional derivative of order q for a continuous function $f(t)$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-q-1} f(s) ds, \quad n = [q] + 1,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 (see [2]) *Let $q > 0$, the general solution of the fractional differential equation*

$${}^c D^q x(t) = 0$$

has the solution

$$x(t) = k_0 + k_1 t + k_2 t^2 + \dots + k_{n-1} t^{n-1},$$

where $k_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ ($n = [q] + 1$).

In view of Lemma 2.1, it follows that

$$I^q {}^c D^q x(t) = x(t) + k_0 + k_1 t + k_2 t^2 + \dots + k_{n-1} t^{n-1}, \tag{2.1}$$

for some $k_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ ($n = [q] + 1$).

We state a result due to Krasnoselskii [34] which is needed to prove the existence of at least one solution of the problem (1.1)-(1.2).

Theorem 2.2 *Let M be a closed convex and nonempty subset of a Banach space X . Let A, B be operators such that*

- (i) $Ax + By \in M$ whenever $x, y \in M$,
- (ii) A is compact and continuous,
- (iii) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.3 *Let $\sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \left(\frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i}+2)} \right) \neq T$, $1 < q \leq 2$. Then for $h \in C[0, T]$, the problem*

$${}^c D^q x(t) = h(t), \quad 0 < t < T, \tag{2.2}$$

$$x(0) = 0, \quad x(T) = \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} [I^{p_{j,i}} x] \Big|_{\xi_{i-1}}^{\xi_i}, \tag{2.3}$$

has a unique solution

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds - \frac{t}{\Gamma(q) \left(T - \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i}+2)} \right)} \int_0^T (T-s)^{q-1} h(s) ds$$

$$\begin{aligned}
 & + \frac{t}{\Gamma(q)(T - \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i}+2)})} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \\
 & \times \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s - r)^{q-1} h(r) dr ds \right. \\
 & \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s - r)^{q-1} h(r) dr ds \right]. \tag{2.4}
 \end{aligned}$$

Proof We may apply (2.1) to reduce (2.2) to an equivalent integral equation

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} h(s) ds - c_0 - c_1 t, \tag{2.5}$$

for some constants $c_0, c_1 \in \mathbb{R}$.

From $x(0) = 0$, it follows that $c_0 = 0$. Taking the Riemann-Liouville fractional integral of order $p_{j,i}$ for (2.5), we get

$$\begin{aligned}
 [I^{p_{j,i}} x](t) & = \frac{1}{\Gamma(p_{j,i})} \int_0^t (t - s)^{p_{j,i}-1} \left[\frac{1}{\Gamma(q)} \int_0^s (s - r)^{q-1} h(r) dr - c_1 s \right] ds \\
 & = \frac{1}{\Gamma(p_{j,i})\Gamma(q)} \int_0^t \int_0^s (t - s)^{p_{j,i}-1} (s - r)^{q-1} h(r) dr ds - c_1 \frac{t^{p_{j,i}+1}}{\Gamma(p_{j,i} + 2)}.
 \end{aligned}$$

On the other hand, for $0 \leq \xi_{i-1} \leq \xi_i \leq T$, we obtain

$$\begin{aligned}
 [I^{p_{j,i}} x] \Big|_{\xi_{i-1}}^{\xi_i} & = [I^{p_{j,i}} x](\xi_i) - [I^{p_{j,i}} x](\xi_{i-1}) \\
 & = \frac{1}{\Gamma(p_{j,i})\Gamma(q)} \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s - r)^{q-1} h(r) dr ds \right. \\
 & \quad \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s - r)^{q-1} h(r) dr ds \right] \\
 & \quad - c_1 \left(\frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i} + 2)} \right).
 \end{aligned}$$

The second condition of (2.3) implies that

$$\begin{aligned}
 & \frac{1}{\Gamma(q)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s - r)^{q-1} h(r) dr ds \right. \\
 & \quad \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s - r)^{q-1} h(r) dr ds \right] \\
 & \quad - c_1 \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \left(\frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i} + 2)} \right) \\
 & = \frac{1}{\Gamma(q)} \int_0^T (T - s)^{q-1} h(s) ds - c_1 T.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 c_1 = & \frac{1}{\Gamma(q)(T - \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i}+2)}}) \int_0^T (T-s)^{q-1} h(s) ds \\
 & - \frac{1}{\Gamma(q)(T - \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i}+2)}}) \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \\
 & \times \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s-r)^{q-1} h(r) dr ds \right. \\
 & \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s-r)^{q-1} h(r) dr ds \right].
 \end{aligned}$$

Substituting the values of c_0 and c_1 in (2.5), we obtain the solution (2.4). □

In the following, for the sake of convenience, set

$$\lambda = T - \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i} + 2)} \neq 0, \tag{2.6}$$

$$\Omega = \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+1}}{|\lambda|\Gamma(q+1)} + \frac{T}{|\lambda|} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{q+p_{j,i}} - \xi_{i-1}^{q+p_{j,i}}}{\Gamma(q+p_{j,i}+1)}. \tag{2.7}$$

We denote $B(\cdot, \cdot)$ as the standard beta function such that

$$B(b+1, a) = \int_0^1 (1-s)^{a-1} s^b ds = \frac{\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

3 Main results

Now we are in the position to establish the main result. Our first result is based on Banach's fixed point theorem.

Theorem 3.1 *Assume that $f : [0, T] \times X \times X \rightarrow X$ is jointly continuous and maps bounded subsets of $[0, T] \times X \times X$ into relatively compact subsets of X , and $\phi : [0, T] \times [0, T] \rightarrow [0, \infty)$ is continuous with $\phi_0 = \max\{\phi(t, s) : (t, s) \in [0, T] \times [0, T]\}$. In addition, suppose that there exist positive constants L_1, L_2 such that*

$$(H_1) \quad \|f(t, x(t), (\psi^\nu x)(t)) - f(t, y(t), (\psi^\nu y)(t))\| \leq L_1 \|x - y\| + L_2 \|(\psi^\nu x) - (\psi^\nu y)\|, \text{ for all } t \in [0, T], x, y \in X,$$

$$(H_2) \quad \Lambda = (L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)}) \Omega < 1, \text{ where } \Omega \text{ is defined by (2.7).}$$

Then the problem (1.1)-(1.2) has a unique solution.

Proof We transform the BVP (1.1)-(1.2) into a fixed point problem. In view of Lemma 2.3 and (2.6), we consider the operator $F : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\begin{aligned}
 (Fx)(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), (\psi^\nu x)(s)) ds \\
 & - \frac{t}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s), (\psi^\nu x)(s)) ds
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{t}{\lambda\Gamma(q)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \\
 & \times \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s - r)^{q-1} f(r, x(r), (\psi^v x)(r)) \, dr \, ds \right. \\
 & \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s - r)^{q-1} f(r, x(r), (\psi^v x)(r)) \, dr \, ds \right],
 \end{aligned}$$

for $t \in [0, T]$. By setting $\sup_{t \in [0, T]} \|f(t, 0, 0)\| = M$, and choosing

$$r \geq \frac{M\Omega}{1 - \Lambda}, \tag{3.1}$$

we will show that $FB_r \subset B_r$, where $B_r = \{x \in C : \|x\| \leq r\}$. Then, for $x \in B_r$, we have

$$\begin{aligned}
 \|(Fx)(t)\| & \leq \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \|f(s, x(s), (\psi^v x)(s))\| \, ds \\
 & + \frac{T}{|\lambda|\Gamma(q)} \int_0^T (T - s)^{q-1} \|f(s, x(s), (\psi^v x)(s))\| \, ds \\
 & + \frac{T}{|\lambda|\Gamma(q)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \\
 & \times \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s - r)^{q-1} \|f(r, x(r), (\psi^v x)(r))\| \, dr \, ds \right. \\
 & \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s - r)^{q-1} \|f(r, x(r), (\psi^v x)(r))\| \, dr \, ds \right] \\
 & \leq \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} (\|f(s, x(s), (\psi^v x)(s)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|) \, ds \\
 & + \frac{T}{|\lambda|\Gamma(q)} \int_0^T (T - s)^{q-1} (\|f(s, x(s), (\psi^v x)(s)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|) \, ds \\
 & + \frac{T}{|\lambda|\Gamma(q)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s - r)^{q-1} \right. \\
 & \times (\|f(r, x(r), (\psi^v x)(r)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|) \, dr \, ds \\
 & \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s - r)^{q-1} \right. \\
 & \times (\|f(r, x(r), (\psi^v x)(r)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|) \, dr \, ds \left. \right] \\
 & \leq \frac{1}{\Gamma(q)} \left[\left(L_1 + \frac{L_2 \phi_0 T^v}{\Gamma(v+1)} \right) r + M \right] \int_0^t (t - s)^{q-1} \, ds \\
 & + \frac{T}{|\lambda|\Gamma(q)} \left[\left(L_1 + \frac{L_2 \phi_0 T^v}{\Gamma(v+1)} \right) r + M \right] \int_0^T (T - s)^{q-1} \, ds \\
 & + \frac{T}{|\lambda|\Gamma(q)} \left[\left(L_1 + \frac{L_2 \phi_0 T^v}{\Gamma(v+1)} \right) r + M \right] \\
 & \times \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s - r)^{q-1} \, dr \, ds \right.
 \end{aligned}$$

$$\begin{aligned}
 & - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s-r)^{q-1} dr ds \Big] \\
 & \leq \left(\left(L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)} \right) r + M \right) \left[\frac{T^q}{\Gamma(q+1)} + \frac{T^{q+1}}{|\lambda| \Gamma(q+1)} \right. \\
 & \quad \left. + \frac{T}{|\lambda| \Gamma(q+1)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} (\xi_i^{q+p_{j,i}} B(q+1, p_{j,i}) - \xi_{i-1}^{q+p_{j,i}} B(q+1, p_{j,i})) \right] \\
 & = \left(\left(L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)} \right) r + M \right) \left[\frac{T^q}{\Gamma(q+1)} + \frac{T^{q+1}}{|\lambda| \Gamma(q+1)} \right. \\
 & \quad \left. + \frac{T}{|\lambda|} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{q+p_{j,i}} - \xi_{i-1}^{q+p_{j,i}}}{\Gamma(q+p_{j,i}+1)} \right] = \Omega \left[\left(L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)} \right) r + M \right] \leq r.
 \end{aligned}$$

In the following, from (H_1) for $x, y \in C$ and for each $t \in [0, T]$, we have

$$\begin{aligned}
 & \| (Fx)(t) - (Fy)(t) \| \\
 & \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \| f(s, x(s), (\psi^\nu x)(s)) - f(s, y(s), (\psi^\nu y)(s)) \| ds \\
 & \quad + \frac{T}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} \| f(s, x(s), (\psi^\nu x)(s)) - f(s, y(s), (\psi^\nu y)(s)) \| ds \\
 & \quad + \frac{T}{|\lambda| \Gamma(q)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \\
 & \quad \times \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s-r)^{q-1} \| f(r, x(r), (\psi^\nu x)(r)) - f(r, y(r), (\psi^\nu y)(r)) \| dr ds \right. \\
 & \quad \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s-r)^{q-1} \| f(r, x(r), (\psi^\nu x)(r)) - f(r, y(r), (\psi^\nu y)(r)) \| dr ds \right] \\
 & \leq \frac{1}{\Gamma(q)} \left(L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)} \right) \|x - y\|_C \int_0^t (t-s)^{q-1} ds \\
 & \quad + \frac{T}{|\lambda| \Gamma(q)} \left(L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)} \right) \|x - y\|_C \int_0^T (T-s)^{q-1} ds \\
 & \quad + \frac{T}{|\lambda| \Gamma(q)} \left(L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)} \right) \|x - y\|_C \\
 & \quad \times \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s-r)^{q-1} dr ds \right. \\
 & \quad \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s-r)^{q-1} dr ds \right] \\
 & \leq \left(L_1 + \frac{L_2 \phi_0 T^\nu}{\Gamma(\nu+1)} \right) \Omega \|x - y\|_C = \Lambda \|x - y\|_C.
 \end{aligned}$$

By (H_2) , $\Lambda < 1$, therefore, F is a contraction. Hence, by the Banach fixed point theorem, we get that F has a fixed point which is a unique solution of the problem (1.1)-(1.2). \square

Our second result is based on Krasnoselskii's fixed point theorem.

Theorem 3.2 Assume that (H_1) and (H_2) hold with

$$\|f(t, x(t), (\psi^v x)(t))\| \leq \mu(t), \tag{3.2}$$

for all $(t, x, (\psi^v x)) \in [0, T] \times X \times X$, where $\mu \in L^1([0, T], \mathbb{R}^+)$.

Then the BVP (1.1)-(1.2) has at least one solution on $[0, T]$.

Proof First, we choose

$$r \geq \|\mu\|_{L^1 \Omega},$$

where Ω is defined by (2.7), and consider $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. On B_r we define the operators S and U as follows:

$$\begin{aligned} (Sx)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), (\psi^v x)(s)) ds \\ (Ux)(t) &= \frac{t}{\lambda \Gamma(q)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \\ &\quad \times \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s-r)^{q-1} f(r, x(r), (\psi^v x)(r)) dr ds \right. \\ &\quad \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s-r)^{q-1} f(r, x(r), (\psi^v x)(r)) dr ds \right] \\ &\quad - \frac{t}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s), (\psi^v x)(s)) ds. \end{aligned}$$

Now we shall show that $S + U$ has a fixed point in B_r . For $x, y \in B_r$, we have

$$\begin{aligned} \|Sx + Uy\| &\leq \|\mu\|_{L^1} \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{T}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} ds \right. \\ &\quad \left. + \frac{T}{|\lambda| \Gamma(q)} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \frac{\alpha_{j,i}}{\Gamma(p_{j,i})} \right. \\ &\quad \times \left[\int_0^{\xi_i} \int_0^s (\xi_i - s)^{p_{j,i}-1} (s-r)^{q-1} dr ds \right. \\ &\quad \left. - \int_0^{\xi_{i-1}} \int_0^s (\xi_{i-1} - s)^{p_{j,i}-1} (s-r)^{q-1} dr ds \right] \Big) \\ &\leq \|\mu\|_{L^1 \Omega} \leq r. \end{aligned}$$

Therefore, $Sx + Uy \in B_r$.

From assumptions (H_1) and (H_2) for $x, y \in \mathcal{C}$ and for each $t \in [0, T]$, we get that

$$\|(Ux)(t) - (Uy)(t)\| \leq \Lambda \|x - y\|_{\mathcal{C}}.$$

Hence, U is a contraction mapping.

Next, we show that S is compact and continuous.

The continuity of f implies that the operator S is continuous. By using condition (3.2), we have that S is uniformly bounded on B_r as

$$\|Sx\| \leq \frac{\|\mu\|_{L^1} T^q}{\Gamma(q+1)}.$$

Furthermore, in view of (H_1) , we define $\sup_{(t,x,(\psi^v x)) \in [0,T] \times B_r \times B_r} \|f(t,x,(\psi^v x))\| = \bar{f}$, and consequently we get

$$\begin{aligned} & \| (Sx)(t_2) - (Sx)(t_1) \| \\ &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s), (\psi^v x)(s)) ds \right. \\ &\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s), (\psi^v x)(s)) ds \right\| \\ &\leq \frac{1}{\Gamma(q)} \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1}) \|f(s, x(s), (\psi^v x)(s))\| ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|f(s, x(s), (\psi^v x)(s))\| ds \\ &\leq \frac{\bar{f}}{\Gamma(q+1)} |2(t_2 - t_1)^q + t_2^q - t_1^q|, \end{aligned}$$

which is independent of x . As $t_2 \rightarrow t_1$, the right-hand side of the above inequality tends to zero. So, S is relatively compact on B_r . Hence, by Arzelá-Ascoli theorem, S is compact on B_r . Thus, all the assumptions of Theorem 2.2 are satisfied. As a consequence of Theorem 2.2, we have that the boundary value problem (1.1)-(1.2) has at least one solution on $[0, T]$. This completes the proof. \square

4 An example

Example 4.1 Consider the following four-point multi-term fractional integral boundary value problem:

$${}^c D^{\frac{3}{2}} x(t) = \frac{1}{(t+9)^2} \cdot \frac{|x|}{|x|+1} + \int_0^t \frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \cdot \frac{e^{-(s-t)}}{81} x(s) ds, \quad t \in [0, 3], \tag{4.1}$$

$$x(0) = 0,$$

$$x(3) = \frac{3}{8} [I^{\frac{1}{2}} x]_0^1 - \frac{2}{5} [I^{\frac{3}{2}} x]_0^1 + \frac{7}{2} [I^{\frac{5}{2}} x]_1^2 - 5 [I^{\frac{1}{2}} x]_2^3 + 2 [I^{\frac{3}{2}} x]_2^3 - 3 [I^{\frac{5}{2}} x]_2^3. \tag{4.2}$$

Set $q = 3/2$, $m = 4$, $\xi_0 = 0$, $\xi_1 = 1$, $\xi_2 = 2$, $\xi_3 = T = 3$, $n_1 = 2$, $p_{1,1} = 1/2$, $p_{2,1} = 3/2$, $\alpha_{1,1} = 3/8$, $\alpha_{2,1} = -2/5$, $n_2 = 1$, $p_{1,2} = 5/2$, $\alpha_{1,2} = 7/2$, $n_3 = 3$, $p_{1,3} = 1/2$, $p_{2,3} = 3/2$, $p_{3,3} = 5/2$, $\alpha_{1,3} = -5$, $\alpha_{2,3} = 2$, $\alpha_{3,3} = -3$, $v = 1/2$, $\phi(t, s) = e^{-(s-t)}/81$, $f(t, x) = (1/(t+9)^2)(|x|/(|x|+1)) + (\psi^{\frac{1}{2}} x)(t)$. Since $\|f(t, x, (\psi^{\frac{1}{2}} x)) - f(t, y, (\psi^{\frac{1}{2}} y))\| \leq (1/81)\|x - y\| + \|(\psi^{\frac{1}{2}} x) - (\psi^{\frac{1}{2}} y)\|$, then (H_1) and (H_2) are satisfied with

$$\lambda = T - \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{p_{j,i}+1} - \xi_{i-1}^{p_{j,i}+1}}{\Gamma(p_{j,i} + 2)} \approx 11.80716 \neq 0,$$

$$\Omega = \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+1}}{|\lambda|\Gamma(q+1)} + \frac{T}{|\lambda|} \sum_{i=1}^{m-1} \sum_{j=1}^{n_i} \alpha_{j,i} \frac{\xi_i^{q+p_{j,i}} - \xi_{i-1}^{q+p_{j,i}}}{\Gamma(q+p_{j,i}+1)} \approx 1.85722,$$

$L_1 = 1/81, L_2 = 1, \phi_0 = e^3/81, T^v = \sqrt{3}, \Gamma(v+1) = \sqrt{\pi}/2$. We can show that

$$\Lambda = \left(L_1 + \frac{L_2 \phi_0 T^v}{\Gamma(v+1)} \right) \Omega \approx 0.92300 < 1.$$

Hence, by Theorem 3.1, the boundary value problem (4.1)-(4.2) has a unique solution on $[0, 3]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this paper. They read and approved the final manuscript.

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