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Linear vibrations of continuum with fractional derivatives

Duygu Dönmez Demir[1*], Necdet Bildik[1] and Berra Gültekin Sinir[2]

*Correspondence: duygu.donmez@cbu.edu.tr
[1]Department of Mathematics, Faculty of Art & Sciences, Celal Bayar University, Manisa, 45047, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, linear vibrations of axially moving systems which are modelled by a fractional derivative are considered. The approximate analytical solution is obtained by applying the method of multiple scales. Including stability analysis, the effects of variation in different parameters belonging to the application problems on the system are calculated numerically and depicted by graphs. It is determined that the external excitation force acting on the system has an effect on the stiffness of the system. Moreover, the general algorithm developed can be applied to many problems for linear vibrations of continuum.

Keywords: linear vibrations; dynamic analysis of continuum; fractional derivative; perturbation method

1 Introduction

Fractional derivatives are useful for describing the occurrence of vibrations in engineering practice. The studies involving fractional calculus and its applications to mechanical problems appear widely in different studies [1]. The advances in fractional calculus focus on modern examples in differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology and electrochemistry [2].

The general solution procedure including all the problems instead of separately solving each problem is quite advantageous. Many different linear or nonlinear models addressing vibrations of continuum appear in the literature. Some of these works are as follows: Pakdemirli [3] developed a general operator technique to analyse the vibrations of a continuous system with an arbitrary number of coupled differential equations. Özhan and Pakdemirli [4–6] suggested the general solution procedure to investigate a more general class of continuous systems such as gyroscopic and viscoelastic systems. Ghayesh *et al.* [7] considered a general solution procedure for the vibrations of systems with cubic nonlinearities subjected to nonlinear and time-dependent boundary conditions. Hence, a general solution is adapted to solve the dynamic problems constituting continuum.

In recent years, there has been a growing interest in the area of fractional variational calculus and its applications [8, 9]. Fractional calculus, which is used successfully in various fields such as mathematics, science and engineering, is one of the generalisations of classical calculus. The merits of using a fractional differential operator lie in the fact that few parameters are needed to accurately describe the constitutive law of damping materials [10]. Bagley and Calico [11] modelled the mechanical properties of damping materials by fractional order time derivatives. The mechanical scientific community recognised the significance of fractional calculus for modelling viscoelastic material behaviour thanks to Bagley *et al.* [12]. Also, they studied longitudinal vibrations of rods and flexural vibrations of beams based on viscoelastic fractional derivative models [13]. For solving dynamic problems with a fractional derivative, the analysis of free damped vibrations of various mechanical systems, whose behaviour is described by linear viscoelastic models with fractional derivatives, were studied by Rossikhin and Shitikova [14]. Mainardi [15] considered the problems in continuum mechanics related to mathematical modelling of viscoelastic bodies. Cooke *et al.* [16] investigated the response of a viscoelastic beam with a fractional derivative. Skaar *et al.* [17] used a fractional standard linear solid model. French and Rogers [18] presented a small group of structural dynamics problems for which fractional calculus was adopted.

The general solution allows one to investigate the effects on a dynamic analysis of continuum whose damping term is modelled by a fractional derivative. An engineering problem which is a special application of the general model developed in this study was formerly considered in [19]. In our previous study [20], the analysis of primary and parametric resonance for the external excitation term having ε-order was performed. As the forced term is obtained in one-order, sum or difference type of resonance also appears in the present model. The method of multiple scales is used in the analysis. Thus, the amplitude and phase modulation equations are produced in terms of operators. In addition, the variations of the curves with respect to the dimensionless parameters are presented. Finally, the effects of fractional damping on the linear vibrations of continuum are investigated in detail.

2 Equation of motion

Let us consider a non-homogeneous and dimensionless model as follows:

$$\ddot{w} + L_0[w] + \varepsilon\left\{L_1[w]\cos\Omega_1 t + L_2\left[D^\alpha w\right]\right\} = F(x)\cos\Omega_2 t, \tag{1}$$

$$B_1(w) = 0 \quad \text{at } x = 0, \qquad B_2(w) = 0 \quad \text{at } x = 1, \tag{2}$$

where $w(x,t)$ represents the displacement, x and t are the spatial and time variables. ε is a small dimensionless parameter, F is the external excitation force, Ω_1 and Ω_2 are the internal and the external excitation frequencies, respectively. D^α defines the fractional derivative of order α. The dot denotes differentiation with respect to time t; L_0, L_1 and L_2 are self-adjoint operators involving only the spatial variable x, B_1 and B_2 are linear operators of boundary conditions. Here, the associated boundary conditions are linear, homogeneous and free from the time.

3 Method of multiple scales

The method is directly applied to the partial differential equation (1). Thus, we can write

$$w(x,t;\varepsilon) = w_0(x,T_0,T_1) + \varepsilon w_1(x,T_0,T_1) + \cdots, \tag{3}$$

where $t = T_0$ is the usual fast-time scales, $\varepsilon t = T_1$ is the slow-time scales. Time derivatives are expressed in terms of fast and slow time scales as follows [21]:

$$d/dt = D_0 + \varepsilon D_1 + \cdots, \qquad d^2/dt^2 = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{4}$$

where $D_n = \partial/\partial T_n$ and $T_n = \varepsilon^n t$. The perturbative expansion of the Riemann-Liouville fractional derivative is given by

$$\left(\frac{d}{dt}\right)^{\alpha} = D_+^{\alpha} + \varepsilon\alpha D_+^{\alpha-1}D_1 + \frac{1}{2}\varepsilon^2\alpha\left[(\alpha-1)D_+^{\alpha-2}D_1^2 + 2D_+^{\alpha-1}D_2\right] + \cdots. \tag{5}$$

In order to calculate the fractional derivative of the exponential function, we may use Riemann-Liouville derivatives [22]

$$D_+^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-\infty}^{t}\frac{u(\tau)\,d\tau}{(t-\tau)^{\alpha}}. \tag{6}$$

If we take $u(t) = e^{i\omega t}$, then

$$D_+^{\alpha}e^{i\omega t} = (i\omega)^{\alpha}e^{i\omega t} \tag{7}$$

is obtained such that $D_+^{\alpha}, D_+^{\alpha-1}, D_+^{\alpha-2}, \ldots$ are the Riemann-Liouville fractional derivatives. Substituting Eqs. (3)-(5) into Eqs. (1) and (2), one obtains

$$O(1): D_0^2 w_0 + L_0[w_0] = F(x)\cos\Omega_2 T_0, \tag{8}$$

$$B_1(w_0) = 0 \quad \text{at } x = 0, \qquad B_2(w_0) = 0 \quad \text{at } x = 1, \tag{9}$$

$$O(\varepsilon): D_0^2 w_1 + L_0[w_1] = -2D_0 D_1 w_0 - L_1[w_0]\cos\Omega_1 T_0 - L_2[D_0 w_0], \tag{10}$$

$$B_1(w_1) = 0 \quad \text{at } x = 0, \qquad B_2(w_1) = 0 \quad \text{at } x = 1. \tag{11}$$

The solution at ε^0-order is

$$w_0(x, T_0, T_1) = \left[A_n(T_1)e^{i\omega_n T_0} + cc\right]X_n(x) + \left(e^{i\Omega_2 T_0} + cc\right)Y(x), \tag{12}$$

where cc denotes complex conjugates. On the other hand, the functions X_n and Y satisfy the following equations:

$$L_0[X_n] - \omega_n^2 X_n = 0; \quad n = 1, 2, \ldots, \tag{13}$$

$$B_1(X_n) = 0 \quad \text{at } x = 0 \quad \text{and} \quad B_2(X_n) = 0 \quad \text{at } x = 1, \tag{14}$$

$$L_0[Y] - \Omega_2^2 Y = \frac{1}{2}F, \tag{15}$$

$$B_1(Y) = 0 \quad \text{at } x = 0, \qquad B_2(Y) = 0 \quad \text{at } x = 1. \tag{16}$$

At ε-order, the solution is

$$w_1(x, T_0, T_1) = \phi_n(x, T_1)e^{i\omega_n T_0} + cc + W(x, T_0, T_1), \tag{17}$$

where $W(x, T_0, T_1)$ is related to the non-secular terms and other parts of the solution are associated with the secular terms. For the approximate solution at ε-order, one substitutes Eq. (12) into Eq. (10). Thus, the resulting equation is obtained as

$$\begin{aligned} D_0^2 w_1 + L_0[w_1] &= \big[-2i\omega_n D_1 A_n e^{i\omega_n T_0} + 2i\omega_n D_1 \bar{A}_n e^{-i\omega_n T_0}\big] X_n(x) \\ &\quad - \frac{1}{2} L_1[X_n]\big[A_n\big(e^{i(\omega_n+\Omega_1)T_0} + e^{i(\omega_n-\Omega_1)T_0}\big) + cc\big] \\ &\quad - \frac{1}{2} L_1[Y]\big[e^{i(\Omega_1+\Omega_2)T_0} + e^{i(\Omega_1-\Omega_2)T_0} + cc\big] \\ &\quad - L_2[X_n]\big[(i\omega_n)^\alpha A_n e^{i\omega_n T_0} + (-i\omega_n)^\alpha \bar{A}_n e^{-i\omega_n T_0}\big] \\ &\quad - L_2[Y]\big[(i\Omega_2)^\alpha e^{i\Omega_2 T_0} + (-i\Omega_2)^\alpha e^{-i\Omega_2 T_0}\big]. \end{aligned} \tag{18}$$

Then, five cases occur as follows.

4 Case studies

In this section, we assume that one dominant mode of vibrations exists. Depending on the numerical values of natural frequency, five different cases occur.

4.1 Ω_1 away from $2\omega_n$ and 0, Ω_2 away from ω_n

This case corresponds to absence of any resonances. Then Eq. (18) turns into

$$D_0^2 w_1 + L_0[w_1] = \big[-2i\omega_n D_1 A_n X_n - (i\omega_n)^\alpha A_n L_2[X_n]\big] e^{i\omega_n T_0} + cc + NST, \tag{19}$$

where NST denotes non-secular terms. If Eq. (17) is substituted into Eq. (19), then ϕ_n satisfies

$$L_0[\phi_n] - \omega_n^2 \phi_n = -2i\omega_n D_1 A_n X_n - (i\omega_n)^\alpha A_n L_2[X_n], \tag{20}$$

$$B_1(\phi_n) = 0 \quad \text{at } x = 0, \qquad B_2(\phi_n) = 0 \quad \text{at } x = 1. \tag{21}$$

Thus, the solvability condition [23] requires

$$2i\omega_n D_1 A_n + (i\omega_n)^\alpha d_1 A_n = 0, \tag{22}$$

where

$$d_1 = \int_0^1 X_n(x) L_2[X_n]\, dx. \tag{23}$$

Finally, the amplitude is obtained as

$$A_n(T_1) = A_0 \exp\left(\frac{1}{2}\omega_n^{\alpha-1} d_1 \left[i\cos\left(\frac{\pi}{2}\alpha\right) - \sin\left(\frac{\pi}{2}\alpha\right)\right] T_1\right), \tag{24}$$

and in the same sense, the displacement is calculated as

$$\begin{aligned} w(x,t) \cong A_0 \biggl\{ & \exp\left(-\frac{d_1}{2}\omega_n^{\alpha-1}\sin\left(\frac{\pi}{2}\alpha\right)\varepsilon t\right) \\ & \times \left[\exp\left[i\left(\frac{d_1}{2}\varepsilon\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right)+\omega_n\right)t\right]+cc\right]\biggr\} X_n(x) \\ & + \left(e^{i\Omega_2 t}+e^{-i\Omega_2 t}\right)Y(x), \end{aligned} \tag{25}$$

where A_0 is constant.

4.2 Ω_1 close to $2\omega_n$ and Ω_2 away from ω_n

Principal parametric resonance occurs in this case. Thus, the internal excitation frequency is considered as

$$\Omega_1 = 2\omega_n + \varepsilon\sigma_n, \tag{26}$$

where σ_n is a detuning parameter. Then Eq. (18) becomes

$$2i\omega_n D_1 A_n + \frac{1}{2}d_2\bar{A}_n e^{i\sigma_n T_1} + (i\omega_n)^\alpha d_1 A_n = 0, \tag{27}$$

where

$$d_2 = \int_0^1 X_n(x) L_1[X_n]\,dx. \tag{28}$$

For the stability analysis, one introduces the transformation

$$A_n(T_1) = B_n(T_1)e^{i\sigma_n T_1/2}, \tag{29}$$

where

$$B_n(T_1) = \left(b_n^R + ib_n^I\right)e^{\lambda T_1}. \tag{30}$$

Substituting (29) and (30) into Eq. (27), and separating real and imaginary parts, the representation of the system of equations with the coefficient matrix is given as

$$\begin{bmatrix} \lambda + \frac{d_1}{2}\omega_n^{\alpha-1}\sin(\frac{\pi}{2}\alpha) & -\frac{d_2}{4\omega_n} + (\frac{d_1}{2}\omega_n^{\alpha-1}\cos(\frac{\pi}{2}\alpha) - \frac{\sigma_n}{2}) \\ -\frac{d_2}{4\omega_n} - (\frac{d_1}{2}\omega_n^{\alpha-1}\cos(\frac{\pi}{2}\alpha) - \frac{\sigma_n}{2}) & \lambda + \frac{d_1}{2}\omega_n^{\alpha-1}\sin(\frac{\pi}{2}\alpha) \end{bmatrix} \begin{bmatrix} b_n^R \\ b_n^I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{31}$$

For a non-trivial solution ($b_n^R \neq 0$, $b_n^I \neq 0$), the determinant of the coefficient matrix must be

$$\left[\lambda + \frac{d_1}{2}\omega_n^{\alpha-1}\sin\left(\frac{\pi}{2}\alpha\right)\right]^2 - \left[\left(\frac{d_2}{4\omega_n}\right)^2 - \left(\frac{d_1}{2}\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) - \frac{\sigma_n}{2}\right)^2\right] = 0. \tag{32}$$

For the steady state condition, λ must be zero. Therefore, the stability boundaries are

$$\sigma_n = d_1\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) \pm \sqrt{\frac{d_2^2}{4\omega_n^2} - d_1^2\omega_n^{2\alpha-2}\sin^2\left(\frac{\pi}{2}\alpha\right)}. \tag{33}$$

Inserting σ_n into Eq. (26), the internal excitation frequency is obtained as

$$\Omega_1 \cong 2\omega_n + \varepsilon\left[d_1\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) \pm \sqrt{\left(\frac{d_2}{2\omega_n}\right)^2 - \left(d_1\omega_n^{\alpha-1}\sin\left(\frac{\pi}{2}\alpha\right)\right)^2}\right]. \tag{34}$$

4.3 Ω_1 close to 0 and Ω_2 away from ω_n

The nearness of Ω_1 to zero is expressed as

$$\Omega_1 = \varepsilon\sigma_n. \tag{35}$$

Arranging Eq. (18), one obtains

$$2i\omega_n D_1 A_n + \left[\cos(\sigma_n T_1)d_2 + (i\omega_n)^\alpha d_1\right]A_n = 0. \tag{36}$$

Solving Eq. (36),

$$\begin{aligned} A_n(T_1) = A_0 \exp\bigg[&-\frac{d_1}{2}\omega_n^{\alpha-1}\sin\left(\frac{\pi}{2}\alpha\right)T_1 \\ &+ \frac{i}{2}\left(\frac{d_2}{\omega_n\sigma_n}\sin(\sigma_n T_1) + d_1\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right)T_1\right)\bigg] \end{aligned} \tag{37}$$

and

$$\begin{aligned} w(x,t) \cong A_0\bigg\{&\exp\left(-\frac{d_1}{2}\omega_n^{\alpha-1}\sin\left(\frac{\pi}{2}\alpha\right)\varepsilon t\right) \\ &\times\left[\exp\left(\frac{i}{2}\left[\frac{d_2}{\omega_n\sigma_n}\sin(\sigma_n\varepsilon t) + \varepsilon d_1 t\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) + \omega_n t\right]\right) + cc\right]\bigg\}X_n(x) \\ &+ \left(e^{i\Omega_2 T_0} + e^{-i\Omega_2 T_0}\right)Y(x) \end{aligned} \tag{38}$$

is calculated.

4.4 Ω_1 away from $2\omega_n$ and Ω_2 close to ω_n

In this case, we consider the primary resonance $\Omega_2 \cong \omega_n$ when the frequency of the loading is approximately equal to the natural frequency. Thus, Eq. (18) turns into

$$2i\omega_n D_1 A_n + (i\omega_n)^\alpha d_1 A_n + (i\omega_n)^\alpha d_3 e^{i\sigma_n T_1} = 0, \tag{39}$$

where

$$d_3 = \int_0^1 X_n L_2[Y]\,dx. \tag{40}$$

Substituting the polar form

$$A_n(T_1) = \frac{1}{2}a_n(T_1)e^{i\beta_n(T_1)} \tag{41}$$

into Eq. (39) and separating the equation into real and imaginary parts, we obtain

$$\mathrm{Re}: a_n' = -\frac{d_1}{2}\omega_n^{\alpha-1} a_n \sin\left(\frac{\pi}{2}\alpha\right) - d_3\omega_n^{\alpha-1}\left[\cos\left(\frac{\pi}{2}\alpha\right)\sin\gamma_n + \sin\left(\frac{\pi}{2}\alpha\right)\cos\gamma_n\right], \tag{42}$$

$$\begin{aligned}\mathrm{Im}: \sigma_n - \gamma_n' = {}& \frac{d_1}{2}\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) \\ &+ \frac{d_3}{a_n}\omega_n^{\alpha-1}\left(\cos\left(\frac{\pi}{2}\alpha\right)\cos\gamma_n - \sin\left(\frac{\pi}{2}\alpha\right)\sin\gamma_n\right),\end{aligned} \tag{43}$$

where $\gamma_n = \sigma_n T_1 - \beta_n$. For steady-state solutions, we consider

$$a_n' = \gamma_n' = 0. \tag{44}$$

By the same mathematical manipulation, the stability boundaries are found as

$$\sigma_n = \frac{d_1}{2}\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) \pm \sqrt{\frac{d_3^2\omega_n^{2\alpha-2}}{a_n^2} - \frac{d_1^2}{4}\omega_n^{2\alpha-2}\sin^2\left(\frac{\pi}{2}\alpha\right)}. \tag{45}$$

4.5 Sum and difference type of resonance

Let us consider sum or difference of internal and external excitation frequency as $\Omega_1 \neq 0$, $\Omega_1 \neq 2\omega_n$ and $\Omega_2 \neq \omega_n$. Then $\Omega_1 + \Omega_2$ and $\Omega_1 - \Omega_2$, $\Omega_2 - \Omega_1$ are close to ω_n. If Eq. (18) is arranged, we get

$$2i\omega_n D_1 A_n + \frac{1}{2}d_4 e^{i\sigma_n T_1} + (i\omega_n)^\alpha d_1 A_n = 0, \tag{46}$$

where

$$d_4 = \int_0^1 X_n L_1[Y]\,dx. \tag{47}$$

Substituting Eq. (41) into Eq. (46) and separating into real and imaginary parts, we get

$$\mathrm{Re}: a_n' + \frac{d_1}{2}\omega_n^{\alpha-1} a_n \sin\left(\frac{\pi}{2}\alpha\right) = -\frac{d_4}{2\omega_n}\sin\gamma_n, \tag{48}$$

$$\mathrm{Im}: \sigma_n - \gamma_n' - \frac{d_1}{2}\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) = \frac{d_4}{2\omega_n a_n}\cos\gamma_n. \tag{49}$$

For steady-state solutions, the equations must be rearranged according to the condition (44). Then the stability boundaries are obtained as

$$\sigma_n = \frac{d_1}{2}\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) \pm \frac{1}{2}\sqrt{\frac{d_4^2}{\omega_n^2 a_n^2} - d_1^2\omega_n^{2\alpha-2}\sin^2\left(\frac{\pi}{2}\alpha\right)}. \tag{50}$$

5 Applications

5.1 The longitudinal vibrations of a tensioned rod

We will investigate longitudinal vibrations of an axial loaded rod with linear fractional damping for application. This problem is quite important in engineering applications. Also, the rods are used as a structural element in many civil and mechanical engineering problems. The governing equation motion of a tensioned rod with fractional damping is introduced as

$$m\frac{\partial^2 \hat{w}}{\partial \hat{t}^2} - P\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \hat{\eta}\frac{\partial^\alpha \hat{w}}{\partial \hat{t}^\alpha} = 0, \tag{51}$$

where $\hat{w}(\hat{x}, \hat{t})$ is the longitudinal displacement of the rod, ε is a small dimensionless parameter, m denotes the mass and $\hat{\eta}$ defines the damping coefficient. It is assumed that the tension P is characterised as a small periodic perturbation $\varepsilon P_1 \cos \hat{\Omega}\hat{t}$ on the steady-state tension P_0

$$P = P_0 + \varepsilon P_1 \cos \hat{\Omega}\hat{t}, \tag{52}$$

where $\hat{\Omega}$ is the frequency of the rod [24]. Introducing the dimensionless quantities as

$$w = \frac{\hat{w}}{L}, \qquad x = \frac{\hat{x}}{L}, \qquad t = \frac{\hat{t}}{L}\sqrt{\frac{P_0}{m}}, \tag{53}$$

the new dimensionless parameters are

$$N = \frac{P_1}{P_0}, \qquad \bar{\eta} = \frac{\hat{\eta}}{L^{\alpha-2}}\sqrt{\frac{P_0^{\alpha-2}}{m^\alpha}} \quad (\bar{\eta} = \varepsilon\eta), \qquad \Omega = \hat{\Omega}L\sqrt{\frac{m}{P_0}}, \tag{54}$$

where L is the length of the rod. Thus, the dimensionless equation is presented as

$$\frac{\partial^2 w}{\partial t^2} - (1 + \varepsilon N \cos \Omega t)\frac{\partial^2 w}{\partial x^2} + \varepsilon\eta\frac{\partial^\alpha w}{\partial t^\alpha} = 0. \tag{55}$$

For the simply supported beam, the boundary conditions are

$$w(0, t) = w(1, t) = 0. \tag{56}$$

Considering Eq. (55), the operators corresponding to the general model are

$$L_0[w] = -w'', \tag{57}$$

$$L_1[w] = -Nw'', \tag{58}$$

$$L_2[w] = \eta D^\alpha w \tag{59}$$

and the other terms are

$$F(x) = 0, \tag{60}$$

$$\Omega_1 = \Omega. \tag{61}$$

Thus, the spatial function $X_n(x)$ satisfies

$$X_n'' + \omega_n^2 X_n = 0, \tag{62}$$

$$X_n(0) = X_n(1) = 0. \tag{63}$$

Finally, the solution of eigenvalue-eigenfunction problem (62) is

$$X_n(x) = c_{2n} \sin \omega_n x; \quad \omega_n = n\pi, n = 1, 2, \ldots. \tag{64}$$

In this problem, three different cases arise at ε-order as follows.

5.1.1 Ω_1 *away from* $2\omega_n$ *and* 0, Ω_2 *away from* ω_n

By the general solution (23), we may write

$$d_1 = \eta \int_0^1 X_n^2(x)\, dx. \tag{65}$$

Thus, the displacement is obtained as

$$\begin{aligned} w(x,t) \cong{}& A_0 \exp\left(-\frac{\eta}{2}\omega_n^{\alpha-1}\varepsilon t \sin\left(\frac{\pi}{2}\alpha\right)\right) \\ &\times \left[\exp\left(i\frac{\eta}{2}\varepsilon t \omega_n^{\alpha-1} \cos\left(\frac{\pi}{2}\alpha\right) + i\omega_n t\right) + cc\right] \sin \omega_n x. \end{aligned} \tag{66}$$

In Figure 1, it is observed that the damping and the natural frequency changed for different modes. The damping decreases and the natural frequency increases as the number of modes enlarges.

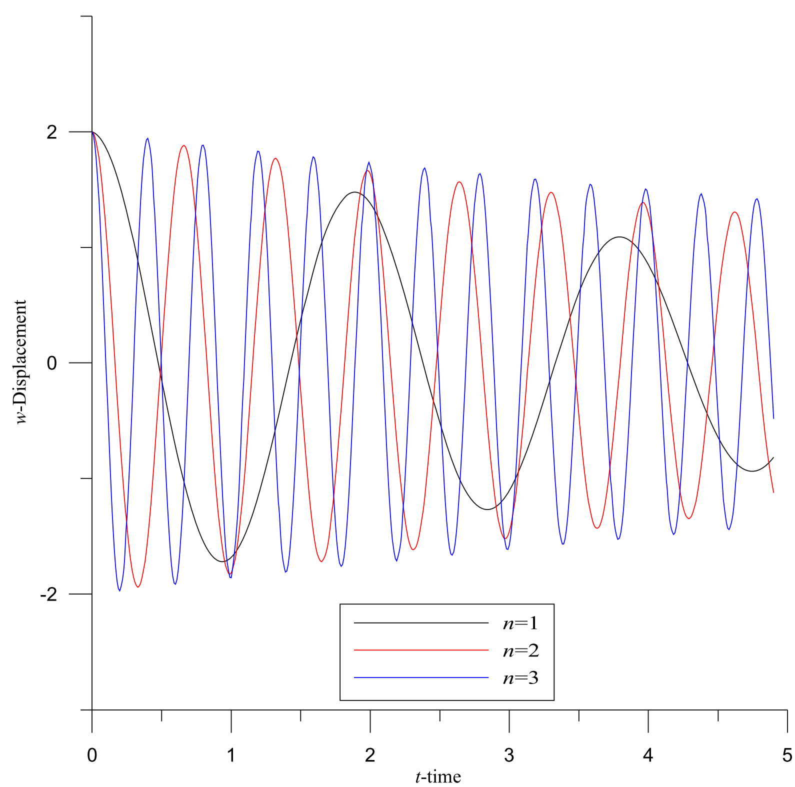

Figure 1 Displacement-time graph for different mode values ($\varepsilon = 0.1$, $\eta = 8$, $\alpha = 0.5$).

5.1.2 Ω_1 *close to* 0 *and* Ω_2 *away from* ω_n

This case corresponds to $\Omega_1 = \varepsilon\sigma_n$. Using Eq. (28), we get

$$d_2 = -N\int_0^1 X_n(x)X_n''(x)\,dx. \tag{67}$$

Thus, the amplitude and displacement are calculated respectively as follows:

$$A_n(T_1) = A_0\exp\left\{-\frac{\eta}{2}\omega_n^{\alpha-1}T_1\sin\left(\frac{\pi}{2}\alpha\right) + i\left[-\frac{N}{2\omega_n\sigma_n}\sin(\sigma_nT_1)\int_0^1 X_n(x)X_n''(x)\,dx + \frac{\eta}{2}\omega_n^{\alpha-1}T_1\cos\left(\frac{\pi}{2}\alpha\right)\right]\right\} \tag{68}$$

and

$$w(x,t) \cong A_0\left\{\exp\left(-\frac{\eta}{2}\omega_n^{\alpha-1}\varepsilon t\sin\left(\frac{\pi}{2}\alpha\right)\right)\left[\exp\left(-i\frac{N}{2\omega_n\sigma_n}\sin(\sigma_n\varepsilon t)\int_0^1 X_nX_n''\,dx + i\frac{\eta}{2}\varepsilon t\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) + i\omega_nt\right) + cc\right]\right\}\sin\omega_nx. \tag{69}$$

Furthermore, the supplementary term of the natural frequency from a fractional derivative is

$$\omega_{na} = -\frac{N}{2\omega_n\sigma_n}\sin(\sigma_n\varepsilon t)\int_0^1 X_nX_n''\,dx + \frac{\eta}{2}\varepsilon t\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right). \tag{70}$$

The variation of a supplementary term from a fractional derivative according to α-order is shown in Figure 2. The effects of the order of a fractional derivative on the displacement-time curves are seen readily in Figure 3. The damping accelerates acutely in the classic damping approach, namely $\alpha = 1$.

5.1.3 Ω_1 *close to* $2\omega_n$ *and* Ω_2 *away from* ω_n

In this part, we get the principal parametric resonance such that $\Omega_1 = 2\omega_n + \varepsilon\sigma_n$. Then the stability boundaries are

$$\sigma_n = \eta\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right) \pm \sqrt{\frac{N^2}{4\omega_n^2}\left(\int_0^1 X_nX_n''\,dx\right)^2 - \left(\eta\omega_n^{\alpha-1}\sin\left(\frac{\pi}{2}\alpha\right)\right)^2}. \tag{71}$$

Figure 4 shows the effects of α-order on the critical value N and the variation of an unstable region with some different values of α. It is observed that the critical value N becomes zero for $\alpha = 0$. The unstable region diminishes as α increases.

5.2 The dynamic analysis of an axially loaded viscoelastic beam resting on foundation

The fractional viscoelastic beam with axial load is resting on linear elastic foundation. This type of foundation is known as Winkler foundation. In the linear Winkler foundation model, $\hat{k}$ denotes the soil coefficient. The beam is modelled by fractional Kelvin-Voight

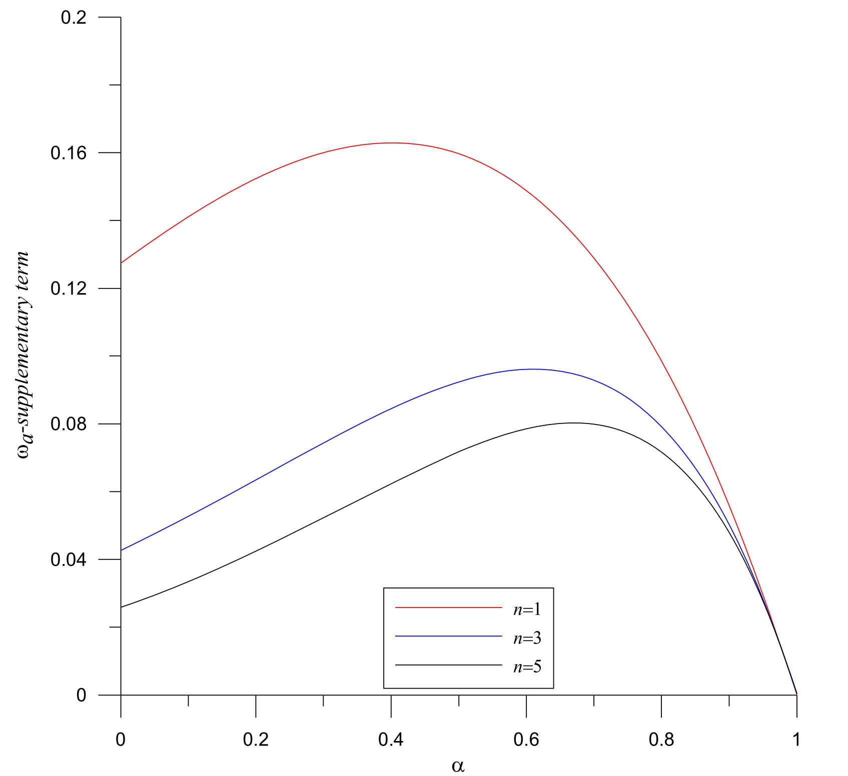

Figure 2 Supplementary term versus α for various mode values at $t = 1$ ($\varepsilon = 0.1$, $\eta = 8$, $N = 1$).

viscoelastic material. The governing equation is given by

$$EI\frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + \hat{\eta}\frac{\partial^\alpha}{\partial \hat{t}^\alpha}\left(\frac{\partial^4 \hat{w}}{\partial \hat{x}^4}\right) + P_0\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \hat{k}\hat{w} + m\frac{\partial^2 \hat{w}}{\partial \hat{t}^2} = \hat{f}(\hat{x})\cos\hat{\Omega}\hat{t}, \tag{72}$$

where E represents the modulus of elasticity, I is the moment of inertia and P_0 denotes axial force. Now, let us introduce the dimensionless quantities

$$w = \frac{\hat{w}}{L}, \qquad x = \frac{\hat{x}}{L}, \qquad t = \frac{\hat{t}}{L}\sqrt{\frac{P_0}{m}} \tag{73}$$

and the new dimensionless parameters are

$$\begin{aligned} &v_f^2 = \frac{EI}{P_0 L^2}, \qquad \bar{\eta} = \frac{\hat{\eta}}{L^{\alpha+2}}\sqrt{\frac{P_0^{\alpha-2}}{m^\alpha}} \quad (\bar{\eta} = \varepsilon\eta), \\ &k = \frac{\hat{k}L^2}{P_0}, \qquad f = \frac{\hat{f}L}{P_0}, \qquad \Omega = \hat{\Omega}L\sqrt{\frac{m}{P_0}}, \end{aligned} \tag{74}$$

where v_f is the flexural stiffness coefficient. Then the dimensionless equation and the boundary conditions are obtained as

$$v_f^2\frac{\partial^4 w}{\partial x^4} + \varepsilon\eta\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{\partial^4 w}{\partial x^4}\right) + \frac{\partial^2 w}{\partial x^2} + kw + \frac{\partial^2 w}{\partial t^2} = f(x)\cos\Omega t, \tag{75}$$

$$w(0,t) = w(1,t) = 0, \qquad w''(0,t) = w''(1,t) = 0. \tag{76}$$

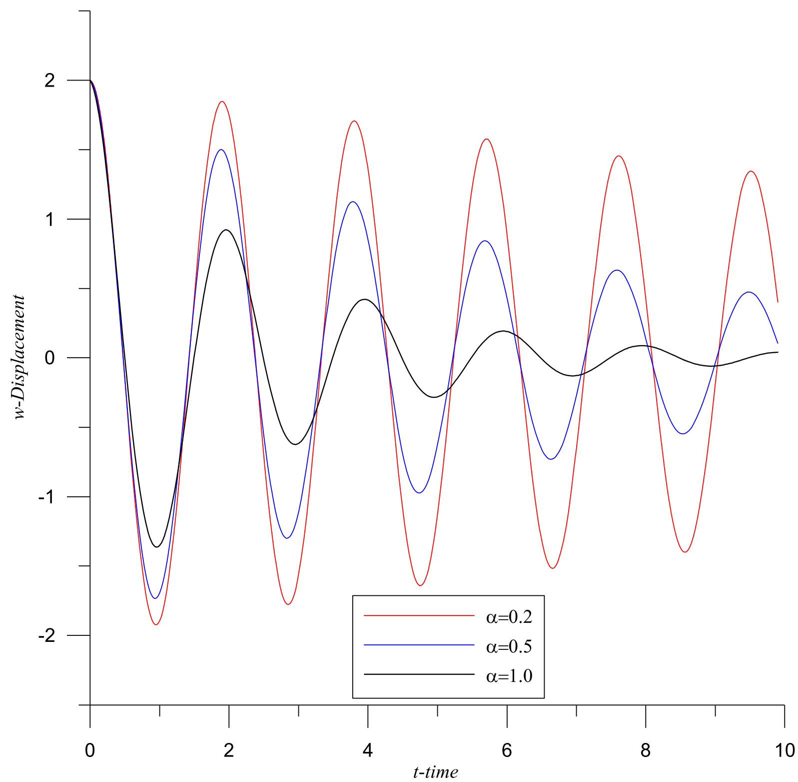

Figure 3 Displacement-time curves for the different values of α ($\varepsilon = 0.1$, $n = 1$, $\eta = 8.0$, $a = 0.2$, $\sigma = 0.1$).

Thus, the operators corresponding to a general model are

$$L_0[w] = v_f^2 w'''' + w'' + kw, \tag{77}$$

$$L_1[w] = 0, \tag{78}$$

$$L_2[w] = \eta D^\alpha w'''' \tag{79}$$

and the other terms are

$$F(x) = f(x), \tag{80}$$

$$\Omega_2 = \Omega. \tag{81}$$

Then Eqs. (13) and (15) reduce to

$$v_f^2 X_n'''' + X_n'' + \left(k - \omega_n^2\right) X_n = 0, \tag{82}$$

$$X_n(0) = X_n(1) = 0, \qquad X_n''(0) = X_n''(1) = 0 \tag{83}$$

and

$$v_f^2 Y'''' + Y'' + \left(k - \Omega^2\right) Y = \frac{1}{2} f(x), \tag{84}$$

$$Y(0) = Y(1) = 0, \qquad Y''(0) = Y''(1) = 0. \tag{85}$$

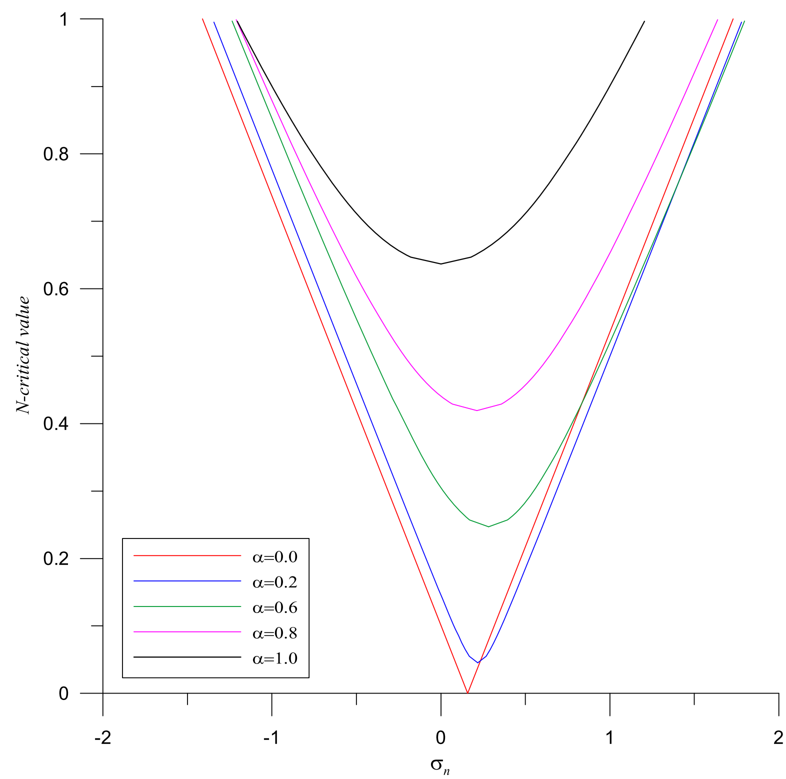

Figure 4 Stability boundaries for various fractional order for second mode (η = 1.0).

Finally, the solutions of Eqs. (82) and (84) are

$$\begin{aligned}X_n(x) = c_{1n}\Bigg[&e^{r_{1n}x} + \frac{r_{4n}^2 - r_{1n}^2}{r_{2n}^2 - r_{4n}^2}\frac{e^{r_{3n}x} - e^{r_{1n}x}}{e^{r_{3n}x} - e^{r_{2n}x}}e^{r_{2n}x} + \frac{r_{4n}^2 - r_{1n}^2}{r_{3n}^2 - r_{4n}^2}\frac{e^{r_{1n}x} - e^{r_{2n}x}}{e^{r_{3n}x} - e^{r_{2n}x}}e^{r_{3n}x}\\ &- \left(1 + \frac{r_{4n}^2 - r_{1n}^2}{r_{2n}^2 - r_{4n}^2}\frac{e^{r_{3n}x} - e^{r_{1n}x}}{e^{r_{3n}x} - e^{r_{2n}x}} + \frac{r_{4n}^2 - r_{1n}^2}{r_{3n}^2 - r_{4n}^2}\frac{e^{r_{1n}x} - e^{r_{2n}x}}{e^{r_{3n}x} - e^{r_{2n}x}}\right)e^{r_{4n}x}\Bigg]\end{aligned} \tag{86}$$

and

$$Y(x) = \alpha_{1n}e^{\beta_{1n}x} + \alpha_{2n}e^{\beta_{2n}x} + \alpha_{3n}e^{\beta_{3n}x} + \alpha_{4n}e^{\beta_{4n}x} + \chi(x), \tag{87}$$

where $\chi(x)$ represents the particular solution from a non-homogeneous part. For the solution at ε-order, two different cases arise as follows.

5.2.1 Ω_1 away from $2\omega_n$ and 0, Ω_2 away from ω_n

Using the general solution (23), we get

$$d_1 = \eta\int_0^1 X_n(x)X_n''''(x)\,dx. \tag{88}$$

The amplitude and displacement are obtained

$$A_n(T_1) = A_0\exp\left\{\frac{\eta}{2}\omega_n^{\alpha-1}T_1\left[i\cos\left(\frac{\pi}{2}\alpha\right) - \sin\left(\frac{\pi}{2}\alpha\right)\right]\int_0^1 X_n(x)X_n''''(x)(x)\,dx\right\} \tag{89}$$

and

$$\begin{aligned} w(x,t) &\\ &\cong A_0 \exp\left(-\frac{\eta}{2}\omega_n^{\alpha-1}\varepsilon t \sin\left(\frac{\pi}{2}\alpha\right)\int_0^1 X_n(x)X_n''''(x)(x)\,dx\right) \\ &\quad\times \left\{\exp\left[i\left(\frac{\eta}{2}\varepsilon\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right)\int_0^1 X_n(x)X_n''''(x)(x)\,dx + \omega_n\right)t\right] + cc\right\}X_n(x). \end{aligned} \tag{90}$$

Thus, the supplementary term of the natural frequency due to a fractional derivative is

$$\omega_{na} = \frac{\eta}{2}\varepsilon\omega_n^{\alpha-1}\cos\left(\frac{\pi}{2}\alpha\right)\int_0^1 X_n(x)X_n''''(x)(x)\,dx. \tag{91}$$

5.2.2 Ω_1 away from 0 and $2\omega_n$, Ω_2 close to ω_n

In this case, using Eq. (45), the stability boundaries which correspond to the primary resonance $\Omega_2 = \omega_n + \varepsilon\sigma_n$ are given as

$$\begin{aligned} \sigma_n &= \frac{1}{2}\omega_n^{\alpha-1}\eta\cos\left(\frac{\pi}{2}\alpha\right)\int_0^1 X_n X_n''''(x)\,dx \\ &\quad \pm \omega_n^{\alpha-1}\eta\sqrt{\frac{1}{a_n^2}\left(\int_0^1 X_n Y''''(x)\,dx\right)^2 - \frac{1}{4}\sin^2\left(\frac{\pi}{2}\alpha\right)\left(\int_0^1 X_n X_n''''(x)\,dx\right)^2}, \end{aligned} \tag{92}$$

where the coefficient is

$$d_3 = \eta\int_0^1 X_n(x)Y''''(x)\,dx. \tag{93}$$

6 Conclusion and discussions

In this study, the general model subject to internal and external excitation is developed. The general model proposed for continuum is linear and one-dimensional. The effect of the damping term which is obtained from viscoelastic material properties is modelled with a fractional derivative. The dynamic analysis of the general model is examined by the method of multiple time scales. The approximate solutions are derived in terms of operators. The external force term is considered at order one. This consideration leads to sum and difference type of resonance in addition to primary and parametric resonance cases. The application of the general solution to two specific engineering problems is presented. The solvability boundaries are approximately obtained and numerically illustrated. It is shown that the order of the fractional derivative has an effect on natural frequencies and stability boundaries. It is shown that the stable region becomes smaller with increasing fractional order. And also, the coefficient of a fractional damping term has similar effects to fractional order.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Art & Sciences, Celal Bayar University, Manisa, 45047, Turkey. [2]Department of Civil Engineering, Faculty of Engineering, Celal Bayar University, Manisa, 45140, Turkey.

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