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Existence of solutions and nonnegative solutions for a class of $p(t)$-Laplacian differential systems with multipoint and integral boundary value conditions

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Abstract
This paper explores the existence of solutions for a class of $p(t)$-Laplacian differential systems with multipoint and integral boundary value conditions via Leray-Schauder's degree. Moreover, the existence of nonnegative solutions is discussed.
MSC: 34B10

Keywords: $p(t)$-Laplacian; Leray-Schauder degree; fixed point

1 Introduction

In this paper, we consider the existence of solutions for the following system:

$$(P)\quad \begin{cases} -\triangle_{p_1(t)}u = \delta_1 f_1(t,u,u',v,v'), & t\in(0,1),\\ -\triangle_{p_2(t)}v = \delta_2 f_2(t,u,u',v,v'), & t\in(0,1),\\ u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i) + e_1, \\ \lim_{t\to 1^-}|u'|^{p_1(t)-2}u'(t) = \int_0^1 k(t)|u'|^{p_1(t)-2}u'(t)\,dt + e_2, \\ v(0) - k_1 v'(0) = \int_0^1 e(t)v(t)\,dt, \qquad v(1) + k_2 v'(1) = \sum_{i=1}^{m-2}\beta_i v(\eta_i), \end{cases}$$

where $p_l \in C([0,1],\mathbb{R})$, $p_l(t) > 1$ $(l = 1,2)$; $-\triangle_{p(t)}\gamma := -(|\gamma'|^{p(t)-2}\gamma')'$ is called $p(t)$-Laplacian; $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $0 < \eta_1 < \cdots < \eta_{m-2} < 1$; $\alpha_i \geq 0$, $\beta_i \geq 0$ $(i = 1,\ldots,m-2)$ and $0 < \sum_{i=1}^{m-2}\alpha_i < 1$, $0 < \sum_{i=1}^{m-2}\beta_i < 1$; $k(t), e(t) \in L^1(0,1)$, they are both nonnegative, $\sigma_1 = \int_0^1 k(t)\,dt \in (0,1)$, $\sigma_2 = \int_0^1 e(t)\,dt \in (0,1)$; $e_1, e_2 \in \mathbb{R}^N$; k_1 and k_2 are nonnegative constants; δ_1 and δ_2 are positive parameters.

The study of differential equations and variational problems with variable exponent growth conditions has attracted more and more attention in recent years. Many results have been obtained on these problems, for example, [1–16]. We refer to [3, 12, 16] for the applied background of these problems. If $p(t) \equiv p$ (a constant), $-\triangle_{p(t)}$ becomes the well-known p-Laplacian. If $p(t)$ is a general function, $-\triangle_{p(t)}$ represents a non-homogeneity and possesses more nonlinearity, thus $-\triangle_{p(t)}$ is more complicated than $-\triangle_p$ (see [7]).

In recent years, because of the wide mathematical and physical background (see [17–19]), the existence of positive solutions for the p-Laplacian equation group has received

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extensive attention. Especially, when $p = 2$, the existence of positive solutions for the equation group boundary value problems has been obtained (see [20–25]). On the integral boundary value problems, we refer to [26–30]. But as for the $p(t)$-Laplacian equation group, there are few papers dealing with the existence of solutions, especially the existence of solutions for the systems with multipoint and integral boundary value problems. Therefore, when $p(t)$ is a general function, this paper mainly investigates the existence of solutions for a class of $p(t)$-Laplacian differential systems with multipoint and integral boundary value conditions. Moreover, we discuss the existence of nonnegative solutions.

Let $N \geq 1$ and $J = [0,1]$, the function $f_l = (f_l^1, \dots, f_l^N) : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $(l = 1,2)$ is assumed to be Carathéodory, by which we mean:

(i) For almost every $t \in J$, the function $f_l(t,\cdot,\cdot,\cdot,\cdot)$ is continuous;

(ii) For each $(x,y,z,w) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, the function $f_l(\cdot,x,y,z,w)$ is measurable on J;

(iii) For each $R > 0$, there are $\beta_R, \rho_R \in L^1(J,\mathbb{R})$ such that, for almost every $t \in J$ and every $(x,y,z,w) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$, $|z| \leq R$, $|w| \leq R$, one has

$$\left|f_1(t,x,y,z,w)\right| \leq \beta_R(t), \qquad \left|f_2(t,x,y,z,w)\right| \leq \rho_R(t).$$

Throughout the paper, we denote

$$\left|\gamma'\right|^{p(0)-2}\gamma'(0) = \lim_{t\to 0^+}\left|\gamma'\right|^{p(t)-2}\gamma'(t),$$
$$\left|\gamma'\right|^{p(1)-2}\gamma'(1) = \lim_{t\to 1^-}\left|\gamma'\right|^{p(t)-2}\gamma'(t).$$

The inner product in $\mathbb{R}^N$ will be denoted by $\langle\cdot,\cdot\rangle$, $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. For $N \geq 1$, we set $C = C(J,\mathbb{R}^N)$, $C^1 = \{\gamma \in C \mid \gamma' \in C\}$; $W = \{(u,v) \mid u,v \in C^1\}$. For any $\gamma(t) = (\gamma^1(t),\dots,\gamma^N(t)) \in C$, we denote $|\gamma^i|_0 = \max_{t\in[0,1]}|\gamma^i(t)|$, $\|\gamma\|_0 = (\sum_{i=1}^N |\gamma^i|_0^2)^{\frac{1}{2}}$ and $\|\gamma\|_1 = \|\gamma\|_0 + \|\gamma'\|_0$. For any $(u,v) \in W$, we denote $\|(u,v)\| = \|u\|_1 + \|v\|_1$. Spaces C, C^1 and W will be equipped with the norm $\|\cdot\|_0$, $\|\cdot\|_1$ and $\|\cdot\|$, respectively. Then $(C,\|\cdot\|_0)$, $(C^1,\|\cdot\|_1)$ and $(W,\|\cdot\|)$ are Banach spaces. Denote $L^1 = L^1(J,\mathbb{R}^N)$ with the norm $\|\gamma\|_{L^1} = [\sum_{i=1}^N(\int_0^1|\gamma^i|\,dt)^2]^{\frac{1}{2}}$.

We say a function $(u,v) : J \to \mathbb{R}^N$ is a solution of (P) if $(u,v) \in W$ satisfies the differential equation in (P) a.e. on J and the boundary value conditions.

In this paper, we always use C_i to denote positive constants if this does not lead to confusion. Denote

$$b^- = \inf_{t\in J} b(t), \qquad b^+ = \sup_{t\in J} b(t) \quad \text{for any } b \in C(J,\mathbb{R}).$$

We say f_l $(l = 1,2)$ satisfies a sub-$(p_l^- - 1)$ growth condition if f_l satisfies

$$\lim_{|x|+|y|+|z|+|w|\to+\infty} \frac{f_l(t,x,y,z,w)}{(|x|+|y|+|z|+|w|)^{q_l(t)-1}} = 0 \quad \text{for } t \in J \text{ uniformly},$$

where $q_l(t) \in C(J,\mathbb{R})$, and $1 < q_l^- \leq q_l^+ < p_l^-$. We say f_l satisfies a general growth condition if f_l does not satisfy a sub-$(p_l^- - 1)$ growth condition.

We will discuss the existence of solutions for (P) in the following two cases:

(i) f_l satisfies a sub-$(p_l^- - 1)$ growth condition for $l = 1, 2$;

(ii) f_l satisfies a general growth condition for $l = 1, 2$.

This paper is organized as follows. In Section 2, we do some preparation. In Section 3, we discuss the existence of solutions of (P). Finally, in Section 4, we discuss the existence of nonnegative solutions for (P).

2 Preliminary

For any $(t, x) \in J \times \mathbb{R}^N$, denote $\varphi_{p_l}(t, x) = |x|^{p_l(t)-2}x$ ($l = 1, 2$). Obviously, φ_{p_l} has the following properties.

Lemma 2.1 (see [5]) *φ_{p_l} is a continuous function and satisfies the following*:

(i) *For any $t \in [0, 1]$, $\varphi_{p_l}(t, \cdot)$ is strictly monotone, that is,*

$$\langle \varphi_{p_l}(t, x_1) - \varphi_{p_l}(t, x_2), x_1 - x_2 \rangle > 0 \quad \text{for any } x_1, x_2 \in \mathbb{R}^N, x_1 \neq x_2.$$

(ii) *There exists a function $\beta_l : [0, +\infty) \to [0, +\infty)$, $\beta_l(s) \to +\infty$ as $s \to +\infty$, such that*

$$\langle \varphi_{p_l}(t, x), x \rangle \geq \beta_l(|x|)|x| \quad \text{for all } x \in \mathbb{R}^N.$$

It is well known that $\varphi_{p_l}(t, \cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $t \in [0, 1]$. For any $t \in J$, denote by $\varphi_{p_l}^{-1}(t, \cdot)$ the inverse operator of $\varphi_{p_l}(t, \cdot)$, then

$$\varphi_{p_l}^{-1}(t, x) = |x|^{\frac{2-p_l(t)}{p_l(t)-1}} x, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \qquad \varphi_{p_l}^{-1}(t, 0) = 0.$$

It is clear that $\varphi_{p_l}^{-1}(t, \cdot)$ is continuous and sends bounded sets into bounded sets.

Let us now consider the following problem:

$$-\bigl(\varphi_{p_1}\bigl(t, u'(t)\bigr)\bigr)' = g_1(t), \quad t \in (0, 1), \tag{1}$$

with the boundary value condition

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1, \qquad \lim_{t \to 1^-} |u'|^{p_1(t)-2} u'(t) = \int_0^1 k(t) |u'|^{p_1(t)-2} u'(t)\, dt + e_2, \tag{2}$$

where $g_1 \in L^1$. If u is a solution of (1) with (2), by integrating (1) from 0 to t, we find that

$$\varphi_{p_1}\bigl(t, u'(t)\bigr) = \varphi_{p_1}\bigl(0, u'(0)\bigr) - \int_0^t g_1(s)\, ds. \tag{3}$$

Denote $a_1 = \varphi_{p_1}(0, u'(0))$. It is easy to see that a_1 is dependent on $g_1(\cdot)$. Define operator $F : L^1 \longrightarrow C$ as

$$F(g_1)(t) = \int_0^t g_1(s)\, ds, \quad \forall t \in J, \forall g_1 \in L^1.$$

From (3), we have

$$u'(t) = \varphi_{p_1}^{-1}\bigl[t, a_1 - F(g_1)\bigr]. \tag{4}$$

By integrating (4) from 0 to t, we find that

$$u(t) = u(0) + F\{\varphi_{p_1}^{-1}[t, a_1 - F(g_1)]\}(t), \quad t \in J.$$

From (2), we have

$$a_1 = \frac{\int_0^1 g_1(t)\,dt - \int_0^1 k(t)\int_0^t g_1(s)\,ds\,dt + e_2}{1 - \sigma_1},$$

and

$$u(0) = \frac{\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}\varphi_{p_1}^{-1}[t, a_1 - F(g_1)(t)]\,dt - \int_0^1 \varphi_{p_1}^{-1}[t, a_1 - F(g_1)(t)]\,dt + e_1}{1 - \sum_{i=1}^{m-2}\alpha_i}.$$

For fixed $h_1 \in L^1$, we define $a_1 : L^1 \to \mathbb{R}^N$ as

$$a_1(h_1) = \frac{\int_0^1 h_1(t)\,dt - \int_0^1 k(t)\int_0^t h_1(s)\,ds\,dt + e_2}{1 - \sigma_1}. \tag{5}$$

It is easy to obtain the following lemma.

Lemma 2.2 *$a_1 : L^1 \to \mathbb{R}^N$ is continuous and sends bounded sets of L^1 to bounded sets of $\mathbb{R}^N$. Moreover,*

$$\left|a_1(h_1)\right| \le \frac{2N}{1 - \sigma_1}\cdot\left(\|h_1\|_{L^1} + |e_2|\right). \tag{6}$$

It is clear that $a_1(\cdot)$ is a compact continuous mapping.
Let us now consider another problem

$$-\left(\varphi_{p_2}\left(t, v'(t)\right)\right)' = g_2(t), \quad t \in (0,1), \tag{7}$$

with the boundary value condition

$$v(0) - k_1 v'(0) = \int_0^1 e(t)v(t)\,dt, \qquad v(1) + k_2 v'(1) = \sum_{i=1}^{m-2}\beta_i v(\eta_i), \tag{8}$$

where $g_2 \in L^1$. Similar to the discussion of the solutions of (1) with (2), we have

$$v'(t) = \varphi_{p_2}^{-1}[t, a_2 - F(g_2)],$$

and

$$v(t) = v(0) + F\{\varphi_{p_2}^{-1}[t, a_2 - F(g_2)]\}(t), \quad t \in J,$$

where $a_2 := \varphi_{p_2}(0, v'(0))$, $F(g_2)(t) = \int_0^t g_2(s)\,ds$ for any $t \in J$.
From $v(0) - k_1 v'(0) = \int_0^1 e(t)v(t)\,dt$, we have

$$v(0) = \frac{k_1\varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t)\int_0^t \varphi_{p_2}^{-1}[s, a_2 - F(g_2)(s)]\,ds\,dt}{1 - \sigma_2}. \tag{9}$$

From $v(1) + k_2 v'(1) = \sum_{i=1}^{m-2} \beta_i v(\eta_i)$, we have

$$v(0) = \frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi_{p_2}^{-1}[t, a_2 - F(g_2)(t)]\,dt - \int_0^1 \varphi_{p_2}^{-1}[t, a_2 - F(g_2)(t)]\,dt}{1 - \sum_{i=1}^{m-2} \beta_i} - \frac{k_2 \varphi_{p_2}^{-1}[1, a_2 - F(g_2)(1)]}{1 - \sum_{i=1}^{m-2} \beta_i}. \tag{10}$$

From (9) and (10), we have

$$\frac{k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) \int_0^t \varphi_{p_2}^{-1}[s, a_2 - F(g_2)(s)]\,ds\,dt}{1 - \sigma_2} = \frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi_{p_2}^{-1}[t, a_2 - F(g_2)(t)]\,dt - \int_0^1 \varphi_{p_2}^{-1}[t, a_2 - F(g_2)(t)]\,dt}{1 - \sum_{i=1}^{m-2} \beta_i} - \frac{k_2 \varphi_{p_2}^{-1}[1, a_2 - F(g_2)(1)]}{1 - \sum_{i=1}^{m-2} \beta_i}.$$

For fixed $h_2 \in C$, we denote

$$\Lambda_{h_2}(a_2) = \frac{k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) \int_0^t \varphi_{p_2}^{-1}[s, a_2 - h_2(s)]\,ds\,dt}{1 - \sigma_2} - \frac{\sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} \varphi_{p_2}^{-1}[t, a_2 - h_2(t)]\,dt - \int_0^1 \varphi_{p_2}^{-1}[t, a_2 - h_2(t)]\,dt}{1 - \sum_{i=1}^{m-2} \beta_i} + \frac{k_2 \varphi_{p_2}^{-1}[1, a_2 - h_2(1)]}{1 - \sum_{i=1}^{m-2} \beta_i}.$$

Lemma 2.3 *The function $\Lambda_{h_2}(\cdot)$ has the following properties*:

(i) *For any fixed $h_2 \in C$, the equation*

$$\Lambda_{h_2}(a_2) = 0 \tag{11}$$

has a unique solution $\widetilde{a_2}(h_2) \in \mathbb{R}^N$.

(ii) *The function $\widetilde{a_2} : C \to \mathbb{R}^N$, defined in* (i), *is continuous and sends bounded sets to bounded sets. Moreover,*

$$\left|\widetilde{a_2}(h_2)\right| \le 3N \|h_2\|_0.$$

Proof (i) It is easy to see that

$$\Lambda_{h_2}(a_2) = \frac{k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) \int_0^t \varphi_{p_2}^{-1}[s, a_2 - h_2(s)]\,ds\,dt}{1 - \sigma_2} + \frac{\sum_{i=1}^{m-2} \beta_i \int_{\eta_i}^1 \varphi_{p_2}^{-1}[t, a_2 - h_2(t)]\,dt + k_2 \varphi_{p_2}^{-1}[1, a_2 - h_2(1)]}{1 - \sum_{i=1}^{m-2} \beta_i} + \int_0^1 \varphi_{p_2}^{-1}\big[t, a_2 - h_2(t)\big]\,dt.$$

From Lemma 2.1, it is immediate that

$$\langle \Lambda_{h_2}(x) - \Lambda_{h_2}(y), x - y \rangle > 0 \quad \text{for } x, y \in \mathbb{R}^N \text{ with } x \neq y,$$

and hence, if (11) has a solution, then it is unique.

Let $t_0 = 3N\|h_2\|_0$. Suppose $|a_2| > t_0$. Since $h_2 \in C$, it is easy to see that there exists an $i \in \{1, \dots, N\}$ such that the ith component a_2^i of a_2 satisfies

$$|a_2^i| \geq \frac{|a_2|}{N} > 3\|h_2\|_0.$$

Thus $(a_2^i - h_2^i(t))$ keeps sign on J and

$$|a_2^i - h_2^i(t)| \geq |a_2^i| - \|h_2\|_0 \geq \frac{2|a_2|}{3N} > 2\|h_2\|_0, \quad \forall t \in J.$$

Obviously, $|a_2 - h_2(t)| \leq \frac{4|a_2|}{3} \leq 2N|a_2^i - h_2^i(t)|$, then

$$|a_2 - h_2(t)|^{\frac{2-p_2(t)}{p_2(t)-1}} |a_2^i - h_2^i(t)| > \frac{1}{2N} |a_2^i - h_2^i(t)|^{\frac{1}{p_2(t)-1}} > \frac{1}{2N} [2\|h_2\|_0]^{\frac{1}{p_2(\zeta)-1}}, \quad \zeta \in J, t \in J.$$

Thus the ith component $\Lambda_{h_2}^i(a_2)$ of $\Lambda_{h_2}(a_2)$ is nonzero and keeps sign, and then we have

$$\Lambda_{h_2}(a_2) \neq 0.$$

Let us consider the equation

$$\lambda \Lambda_{h_2}(a_2) + (1 - \lambda) a_2 = 0, \quad \lambda \in [0, 1]. \tag{12}$$

It is easy to see that all the solutions of (12) belong to $b(t_0 + 1) = \{x \in \mathbb{R}^N \mid |x| < t_0 + 1\}$. So, we have

$$d_B[\Lambda_{h_2}(a_2), b(t_0 + 1), 0] = d_B[I, b(t_0 + 1), 0] \neq 0,$$

which implies the existence of solutions of $\Lambda_{h_2}(a_2) = 0$.

In this way, we define a function $\widetilde{a_2}(h_2) : C[0,1] \to \mathbb{R}^N$, which satisfies

$$\Lambda_{h_2}(\widetilde{a_2}(h_2)) = 0.$$

(ii) By the proof of (i), we also obtain that $\widetilde{a_2}$ sends bounded sets to bounded sets, and

$$|\widetilde{a_2}(h_2)| \leq 3N\|h_2\|_0.$$

It only remains to prove the continuity of $\widetilde{a_2}$. Let $\{v_n\}$ be a convergent sequence in C and $v_n \to v$ as $n \to +\infty$. Since $\{\widetilde{a_2}(v_n)\}$ is a bounded sequence, then it contains a convergent subsequence $\{\widetilde{a_2}(v_{n_j})\}$. Let $\widetilde{a_2}(v_{n_j}) \to a_0$ as $j \to +\infty$. Since $\Lambda_{v_{n_j}}(\widetilde{a_2}(v_{n_j})) = 0$, letting $j \to +\infty$, we have $\Lambda_v(a_0) = 0$. From (i), we get $a_0 = \widetilde{a_2}(v)$, it means that $\widetilde{a_2}$ is continuous. The proof is completed. □

Now, we define the operator $a_2 : L^1 \to \mathbb{R}^N$ as

$$a_2(v) = \widetilde{a_2}\big(F(v)\big). \tag{13}$$

It is clear that $a_2(\cdot)$ is continuous and sends bounded sets of L^1 into bounded sets of $\mathbb{R}^N$, and hence it is a compact continuous mapping.

If u is a solution of (1) with (2), we have

$$u(t) = u(0) + F\big\{\varphi_{p_1}^{-1}\big[t, a_1 - F(g_1)\big]\big\}(t), \quad t \in J,$$

and

$$u(0) = \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi_{p_1}^{-1}[t, a_1 - F(g_1)(t)]\,dt - \int_0^1 \varphi_{p_1}^{-1}[t, a_1 - F(g_1)(t)]\,dt + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i}.$$

If u is a solution of (7) with (8), we have

$$v(t) = v(0) + F\big\{\varphi_{p_2}^{-1}\big[t, a_2 - F(g_2)\big]\big\}(t), \quad t \in J,$$

and

$$v(0) = \frac{k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) \int_0^t \varphi_{p_2}^{-1}[s, a_2 - F(g_2)(s)]\,ds\,dt}{1 - \sigma_2}.$$

We denote

$$K_1(h_1)(t) := (K_1 \circ h_1)(t) = F\big\{\varphi_{p_1}^{-1}\big[t, a_1(h_1) - F(h_1)\big]\big\}(t), \quad \forall t \in [0,1],$$
$$K_2(h_2)(t) := (K_2 \circ h_2)(t) = F\big\{\varphi_{p_2}^{-1}\big[t, a_2(h_2) - F(h_2)\big]\big\}(t), \quad \forall t \in [0,1].$$

Lemma 2.4 *The operators K_l ($l = 1, 2$) are continuous and send equi-integrable sets in L^1 to relatively compact sets in C^1.*

Proof We only prove that the operator K_1 is continuous and sends equi-integrable sets in L^1 to relatively compact sets in C^1, the rest is similar.

It is easy to check that $K_1(h_1)(t) \in C^1$ for all $h_1 \in L^1$. Since

$$K_1(h_1)'(t) = \varphi_{p_1}^{-1}\big[t, a_1(h_1) - F(h_1)\big], \quad \forall t \in [0,1],$$

it is easy to check that K_1 is a continuous operator from L^1 to C^1.

Let now U be an equi-integrable set in L^1, then there exists $\rho_* \in L^1$ such that

$$\big|u(t)\big| \le \rho_*(t) \quad \text{a.e. in } J \text{ for any } u \in L^1.$$

We want to show that $\overline{K_1(U)} \subset C^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $K_1(U)$, then there exists a sequence $\{h_n\} \in U$ such that $u_n = K_1(h_n)$. For any $t_1, t_2 \in J$, we have

$$\big|F(h_n)(t_1) - F(h_n)(t_2)\big| = \left|\int_0^{t_1} h_n(t)\,dt - \int_0^{t_2} h_n(t)\,dt\right| = \left|\int_{t_1}^{t_2} h_n(t)\,dt\right| \le \left|\int_{t_1}^{t_2} \rho_*(t)\,dt\right|.$$

Hence the sequence $\{F(h_n)\}$ is uniformly bounded and equicontinuous. By the Ascoli-Arzela theorem, there exists a subsequence of $\{F(h_n)\}$ (which we still denote by $\{F(h_n)\}$) convergent in C. According to the bounded continuous of the operator a_1, we can choose a subsequence of $\{a_1(h_n) - F(h_n)\}$ (which we still denote by $\{a_1(h_n) - F(h_n)\}$) which is convergent in C, then $\varphi_{p_1}(t, K_1(h_n)'(t)) = a_1(h_n) - F(h_n)$ is convergent in C.

From the definition of $K_1(h_n)(t)$ and the continuity of $\varphi_{p_1}^{-1}$, we can see that $K_1(h_n)$ is convergent in C. Thus, $\{u_n\}$ is convergent in C^1. This completes the proof. □

Let us define $P_1, P_2 : C^1 \to C^1$ as

$$P_1(h_1) = \frac{\sum_{i=1}^{m-2} \alpha_i K_1(h_1)(\xi_i) - K_1(h_1)(1) + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i},$$

$$P_2(h_2) = \frac{k_1 \varphi_{p_2}^{-1}(0, a_2(h_2)) + \int_0^1 e(t) K_2(h_2)(t)\,dt}{1 - \sigma_2}.$$

It is easy to see that P_1 and P_2 are both compact continuous.

We denote $N_{f_l}(u,v) : [0,1] \times C^1 \to L^1$ ($l = 1,2$) the Nemytskii operator associated to f_l defined by

$$N_{f_l}(u,v)(t) = f_l\big(t, u(t), u'(t), v(t), v'(t)\big) \quad \text{a.e. on } J.$$

Lemma 2.5 *(u,v) is a solution of (P) if and only if (u,v) is a solution of the following abstract equation*:

$$(S) \qquad \begin{cases} u = P_1(\delta_1 N_{f_1}(u,v)) + K_1(\delta_1 N_{f_1}(u,v)), \\ v = P_2(\delta_2 N_{f_2}(u,v)) + K_2(\delta_2 N_{f_2}(u,v)). \end{cases}$$

Proof If (u,v) is a solution to (P), according to the proof before Lemma 2.5, it is easy to obtain that (u,v) is a solution to (S).

Conversely, if (u,v) is a solution to (S), then

$$\begin{aligned} u(1) &= P_1\big(\delta_1 N_{f_1}(u,v)\big) + K_1\big(\delta_1 N_{f_1}(u,v)\big)(1) \\ &= \frac{\sum_{i=1}^{m-2} \alpha_i K_1(\delta_1 N_{f_1}(u,v))(\xi_i) - K_1(\delta_1 N_{f_1}(u,v))(1) + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i} + K_1\big(\delta_1 N_{f_1}(u,v)\big)(1) \\ &= \frac{\sum_{i=1}^{m-2} \alpha_i K_1(\delta_1 N_{f_1}(u,v))(\xi_i) - \sum_{i=1}^{m-2} \alpha_i K_1(\delta_1 N_{f_1}(u,v))(1) + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i} \\ &= \frac{\sum_{i=1}^{m-2} \alpha_i [u(\xi_i) - u(0)] - \sum_{i=1}^{m-2} \alpha_i [u(1) - u(0)] + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i} \\ &= \frac{\sum_{i=1}^{m-2} \alpha_i u(\xi_i) - \sum_{i=1}^{m-2} \alpha_i u(1) + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i}, \end{aligned}$$

which implies

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1.$$

It follows from (S) that

$$\varphi_{p_1}(t, u'(t)) = a_1(\delta_1 N_{f_1}(u,v)) - F(\delta_1 N_{f_1}(u,v))(t),$$

then

$$\varphi_{p_1}(1, u'(1)) = a_1(\delta_1 N_{f_1}(u,v)) - F(\delta_1 N_{f_1}(u,v))(1).$$

By the condition of the mapping a_1, we have

$$\begin{aligned}
\varphi_{p_1}(1, u'(1)) &= \frac{\int_0^1 \delta_1 N_{f_1}(u,v)(t)\,dt - \int_0^1 k(t)\int_0^t \delta_1 N_{f_1}(u,v)(s)\,ds\,dt + e_2}{1-\sigma_1} \\
&\quad - \int_0^1 \delta_1 N_{f_1}(u,v)(t)\,dt \\
&= \frac{\sigma_1 \int_0^1 \delta_1 N_{f_1}(u,v)(t)\,dt - \int_0^1 k(t)\int_0^t \delta_1 N_{f_1}(u,v)(s)\,ds\,dt + e_2}{1-\sigma_1} \\
&= \frac{\sigma_1[a_1 - \varphi_{p_1}(1, u'(1))] - \int_0^1 k(t)[a_1 - \varphi_{p_1}(t, u'(t))]\,dt + e_2}{1-\sigma_1} \\
&= \frac{-\sigma_1 \varphi_{p_1}(1, u'(1)) + \int_0^1 k(t)\varphi_{p_1}(t, u'(t))\,dt + e_2}{1-\sigma_1},
\end{aligned}$$

and then

$$\varphi_{p_1}(1, u'(1)) = \int_0^1 k(t)\varphi_{p_1}(t, u'(t))\,dt + e_2.$$

From (S), we have

$$v'(t) = \varphi_{p_2}^{-1}[t, a_2 - F(\delta_2 N_{f_2}(u,v))],$$

and

$$\begin{aligned}
v(0) &= P_2(\delta_2 N_{f_2}(u,v)) \\
&= \frac{k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) K_2(\delta_2 N_{f_2}(u,v))(t)\,dt}{1-\sigma_2} \\
&= \frac{k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) v(t)\,dt - \sigma_2 v(0)}{1-\sigma_2},
\end{aligned}$$

then

$$v(0) = k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) v(t)\,dt.$$

Thus

$$v(0) - k_1 v'(0) = k_1 \varphi_{p_2}^{-1}(0, a_2) + \int_0^1 e(t) v(t)\,dt - k_1 \varphi_{p_2}^{-1}(0, a_2) = \int_0^1 e(t) v(t)\,dt.$$

From (S), we have

$$v(1) = P_2\big(\delta_2 N_{f_2}(u,v)\big) + K_2\big(\delta_2 N_{f_2}(u,v)\big)(1).$$

By the condition of the mapping a_2, we have

$$\begin{aligned} v(1) &= P_2\big(\delta_2 N_{f_2}(u,v)\big) + K_2\big(\delta_2 N_{f_2}(u,v)\big)(1) \\ &= -\frac{\sum_{i=1}^{m-2}\beta_i \int_{\eta_i}^1 \varphi_{p_2}^{-1}[t, a_2 - F(\delta_2 N_{f_2}(u,v))(t)]\,dt + k_2\varphi_{p_2}^{-1}[1, a_2 - F(\delta_2 N_{f_2}(u,v))(1)]}{1-\sum_{i=1}^{m-2}\beta_i} \\ &= -\frac{\sum_{i=1}^{m-2}\beta_i[v(1)-v(\eta_i)] + k_2\varphi_{p_2}^{-1}[1, a_2 - F(\delta_2 N_{f_2}(u,v))(1)]}{1-\sum_{i=1}^{m-2}\beta_i}, \end{aligned}$$

which implies that

$$v(1) = \sum_{i=1}^{m-2}\beta_i v(\eta_i) - k_2\varphi_{p_2}^{-1}\big[1, a_2 - F\big(\delta_2 N_{f_2}(u,v)\big)(1)\big].$$

Since $v'(1) = \varphi_{p_2}^{-1}[1, a_2 - F(\delta_2 N_{f_2}(u,v))(1)]$, then we have

$$v(1) + k_2 v'(1) = \sum_{i=1}^{m-2}\beta_i v(\eta_i).$$

Moreover, from (S), it is easy to obtain

$$-\big(\varphi_{p_1}\big(t, u'(t)\big)\big)' = \delta_1 N_{f_1}(u,v)$$

and

$$-\big(\varphi_{p_2}\big(t, v'(t)\big)\big)' = \delta_2 N_{f_2}(u,v).$$

Hence (u,v) is a solution of (P).

This completes the proof. □

3 Existence of solutions

In this section, we apply Leray-Schauder's degree to deal with the existence of solutions for (P), when f_l satisfies a sub-$(p_l^- - 1)$ growth condition or a general growth condition $(l = 1,2)$.

We denote (S) as

$$(u,v) = A(u,v) = \big(\Psi_{f_1}(u,v), \Phi_{f_2}(u,v)\big),$$

where

$$\begin{aligned} \Psi_{f_1}(u,v) &= P_1\big(\delta_1 N_{f_1}(u,v)\big) + K_1\big(\delta_1 N_{f_1}(u,v)\big), \\ \Phi_{f_2}(u,v) &= P_2\big(\delta_2 N_{f_2}(u,v)\big) + K_2\big(\delta_2 N_{f_2}(u,v)\big). \end{aligned}$$

Theorem 3.1 *If f_l satisfies a sub-$(p_l^- - 1)$ growth condition, then the problem (P) has at least one solution for any fixed parameter δ_l $(l = 1, 2)$.*

Proof Denote

$$A_\lambda(u,v) = \big(\Psi_{\lambda f_1}(u,v), \Phi_{\lambda f_2}(u,v)\big),$$

where

$$\Psi_{\lambda f_1}(u,v) = P_1\big(\lambda\delta_1 N_{f_1}(u,v)\big) + K_1\big(\lambda\delta_1 N_{f_1}(u,v)\big),$$
$$\Phi_{\lambda f_2}(u,v) = P_2\big(\lambda\delta_2 N_{f_2}(u,v)\big) + K_2\big(\lambda\delta_2 N_{f_2}(u,v)\big).$$

According to Lemma 2.5, we know that (P) has the same solution of

$$(u,v) = A_\lambda(u,v) \tag{14}$$

when $\lambda = 1$.

It is easy to see that the operators P_1 and P_2 are compact continuous. According to Lemma 2.2, Lemma 2.3 and Lemma 2.4, we can see that $\Psi_{\lambda f_1}(u,v)$ and $\Phi_{\lambda f_2}(u,v)$ are compact continuous from $C^1 \times [0,1]$ to C^1, thus $A_\lambda(u,v)$ is compact continuous from $W \times [0,1]$ to W.

We claim that all the solutions of (14) are uniformly bounded for $\lambda \in [0,1]$. In fact, if it is false, we can find a sequence of solutions $\{((u_n,v_n),\lambda_n)\}$ for (14) such that $\|(u_n,v_n)\| \to +\infty$ as $n \to +\infty$.

From Lemma 2.2, we have

$$\begin{aligned}\big|a_1\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big)\big| &\le C_1\big(\big\|N_{f_1}(u_n,v_n)\big\|_{L^1} + |e_2|\big)\\ &\le C_2\big(1 + \big\|(u_n,v_n)\big\|\big)^{q_1^+-1},\end{aligned}$$

which together with the sub-$(p_1^- - 1)$ growth condition of f_1 implies that

$$\begin{aligned}&\big|a_1\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big) - F\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big)\big|\\ &\quad\le \big|a_1\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big)\big| + \big|F\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big)\big|\\ &\quad\le C_3\big(1 + \big\|(u_n,v_n)\big\|\big)^{q_1^+-1}.\end{aligned} \tag{15}$$

From (14), we have

$$\big|u_n'(t)\big|^{p_1(t)-2}u_n'(t) = a_1\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big) - F\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big), \quad t \in J,$$

then

$$\big|u_n'(t)\big|^{p_1(t)-1} \le \big|a_1\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big)\big| + \big|F\big(\lambda_n\delta_1 N_{f_1}(u_n,v_n)\big)\big| \le C_4\big(1 + \big\|(u_n,v_n)\big\|\big)^{q_1^+-1}.$$

Denote $\alpha_1 = \frac{q_1^+-1}{p_1^--1}$. From the above inequality we have

$$\big\|u_n'(t)\big\|_0 \le C_5\big(1 + \big\|(u_n,v_n)\big\|\big)^{\alpha_1}. \tag{16}$$

It follows from (14) and (15) that

$$\left|u_n(0)\right| \le C_6\left(1 + \left\|(u_n, v_n)\right\|\right)^{\alpha_1}, \quad \text{where } \alpha_1 = \frac{q_1^+ - 1}{p_1^- - 1}.$$

For any $j = 1, \dots, N$, we have

$$\begin{aligned} \left|u_n^j(t)\right| &= \left|u_n^j(0) + \int_0^t \left(u_n^j\right)'(r)\,dr\right| \\ &\le \left|u_n^j(0)\right| + \left|\int_0^t \left(u_n^j\right)'(r)\,dr\right| \\ &\le [C_7 + C_5]\left(1 + \left\|(u_n, v_n)\right\|\right)^{\alpha_1} \le C_8\left(1 + \left\|(u_n, v_n)\right\|\right)^{\alpha_1}, \end{aligned}$$

which implies that

$$\left|u_n^j\right|_0 \le C_9\left(1 + \left\|(u_n, v_n)\right\|\right)^{\alpha_1}, \quad j = 1, \dots, N; n = 1, 2, \dots.$$

Thus

$$\|u_n\|_0 \le C_{10}\left(1 + \left\|(u_n, v_n)\right\|\right)^{\alpha_1}, \quad n = 1, 2, \dots. \tag{17}$$

It follows from (16) and (17) that $\|u_n\|_1 \le C_{11}(1 + \|(u_n, v_n)\|)^{\alpha_1}$.

Similarly, we have $\|v_n\|_1 \le C_{12}(1 + \|(u_n, v_n)\|)^{\alpha_2}$, where $\alpha_2 = \frac{q_2^+ - 1}{p_2^- - 1}$.

Thus, $\{\|(u_n, v_n)\|\}$ is bounded.

Thus, we can choose a large enough $R_0 > 0$ such that all the solutions of (14) belong to $B(R_0) = \{(u, v) \in W \mid \|(u_n, v_n)\| < R_0\}$. Therefore, the Leray-Schauder degree $d_{LS}[I - A_\lambda(u, v), B(R_0), 0]$ is well defined for each $\lambda \in [0, 1]$, and

$$d_{LS}\left[I - A_1(u, v), B(R_0), 0\right] = d_{LS}\left[I - A_0(u, v), B(R_0), 0\right].$$

Denote

$$\left.\begin{aligned} u_0 &= \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi_{p_1}^{-1}[t, a_1(0)]\,dt - \int_0^1 \varphi_{p_1}^{-1}[t, a_1(0)]\,dt + e_1}{1 - \sum_{i=1}^{m-2} \alpha_i} + \int_0^r \varphi_{p_1}^{-1}[t, a_1(0)]\,dt, \\ v_0 &= \frac{k_1 \varphi_{p_2}^{-1}[0, a_2(0)] + \int_0^1 e(t) \int_0^t \varphi_{p_2}^{-1}[r, a_2(0)]\,dr\,dt}{1 - \sigma_2} + \int_0^r \varphi_{p_2}^{-1}[t, a_2(0)]\,dt, \end{aligned}\right\} \tag{18}$$

where $a_1(0)$ and $a_2(0)$ are defined in (5) and (13), then (u_0, v_0) is the unique solution of $(u, v) = A_0(u, v)$.

It is easy to see that (u, v) is a solution of $(u, v) = A_0(u, v)$ if and only if (u, v) is a solution of the following system:

$$\left.\begin{aligned} &-\triangle_{p_1(t)} u = 0, \quad t \in (0, 1), \\ &-\triangle_{p_2(t)} v = 0, \quad t \in (0, 1), \\ &u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1, \\ &\lim_{t \to 1^-} |u'|^{p_1(t)-2} u'(t) = \int_0^1 k(t) |u'|^{p_1(t)-2} u'(t)\,dt + e_2, \\ &v(0) - k_1 v'(0) = \int_0^1 e(t) v(t)\,dt, \qquad v(1) + k_2 v'(1) = \sum_{i=1}^{m-2} \beta_i v(\eta_i). \end{aligned}\right\} \tag{19}$$

Obviously, (19) possesses a unique solution (u_0, v_0). Note that $(u_0, v_0) \in B(R_0)$, we have

$$d_{LS}\big[I - A_1(u,v), B(R_0), 0\big] = d_{LS}\big[I - A_0(u,v), B(R_0), 0\big] \neq 0.$$

Therefore (P) has at least one solution. This completes the proof. □

In the following, we investigate the existence of solutions for (P) when f_l satisfies a general growth condition.

Denote

$$\Omega_\varepsilon = \Big\{(u,v) \in W \mid \max_{1\le i\le N}\big(|u^i|_0 + |(u^i)'|_0\big) < \varepsilon \text{ and } \max_{1\le i\le N}\big(|v^i|_0 + |(v^i)'|_0\big) < \varepsilon\Big\},$$
$$\theta = \frac{\varepsilon}{3}.$$

Assume the following.

(A_1) Let a positive constant ε be such that $(u_0, v_0) \in \Omega_\varepsilon$, $|P_1(0)| < \theta$, $|P_2(0)| < \theta$ and $|a_1(0)| < \min_{t\in J}(\frac{\theta}{3})^{p_1(t)-1}$, $|a_2(0)| < \min_{t\in J}(\frac{\theta}{2})^{p_2(t)-1}$, where (u_0, v_0) is defined in (18), $a_1(\cdot)$ and $a_2(\cdot)$ are defined in (5) and (13), respectively.

It is easy to see that Ω_ε is an open bounded domain in W. We have the following theorem.

Theorem 3.2 *Assume that* (A_1) *is satisfied. If positive parameters* δ_1 *and* δ_2 *are small enough, then the problem* (P) *has at least one solution on* $\overline{\Omega_\varepsilon}$.

Proof Similarly, we denote $A_\lambda(u,v) = (\Psi_{\lambda f_1}(u,v), \Phi_{\lambda f_2}(u,v))$. By Lemma 2.5, (u,v) is a solution of

$$\begin{cases} -\triangle_{p_1(t)}u = \lambda\delta_1 f_1(t,u,u',v,v'), & t \in (0,1), \\ -\triangle_{p_2(t)}v = \lambda\delta_2 f_2(t,u,u',v,v'), & t \in (0,1), \end{cases}$$

with (2) and (8) if and only if (u,v) is a solution of the following abstract equation:

$$(u,v) = A_\lambda(u,v). \tag{20}$$

From the proof of Theorem 3.1, we can see that $A_\lambda(u,v)$ is compact continuous from $W \times [0,1]$ to W. According to Leray-Schauder's degree theory, we only need to prove that

(1°) $(u,v) = A_\lambda(u,v)$ has no solution on $\partial\Omega_\varepsilon$ for any $\lambda \in [0,1]$,
(2°) $d_{LS}[I - A_0(u,v), \Omega_\varepsilon, 0] \neq 0$,

then we can conclude that the system (P) has a solution on $\overline{\Omega_\varepsilon}$.

(1°) If there exists a $\lambda \in [0,1]$ and $(u,v) \in \partial\Omega_\varepsilon$ is a solution of (20), then (u,v) and λ satisfy

$$u'(t) = \varphi_{p_1}^{-1}\big[t, a_1 - F\big(\lambda\delta_1 N_{f_1}(u,v)\big)\big]$$

and

$$v'(t) = \varphi_{p_2}^{-1}\big[t, a_2 - F\big(\lambda\delta_2 N_{f_2}(u,v)\big)\big].$$

Since $(u,v) \in \partial\Omega_\varepsilon$, there exists an i such that $|u^i|_0 + |(u^i)'|_0 = \varepsilon$ or $|v^i|_0 + |(v^i)'|_0 = \varepsilon$.

(i) If $|u^i|_0 + |(u^i)'|_0 = \varepsilon$.

(i°) Suppose that $|u^i|_0 > 2\theta$, then $|(u^i)'|_0 < \varepsilon - 2\theta = \theta$. On the other hand, for any $t, t' \in J$, we have

$$\left|u^i(t) - u^i(t')\right| = \left|\int_{t'}^{t} (u^i)'(r)\,dr\right| \le \int_0^1 \left|(u^i)'(r)\right| dr < \theta.$$

This implies that $|u^i(t)| > \theta$ for each $t \in J$.

Note that $(u,v) \in \overline{\Omega_\varepsilon}$, then $|f_1(t,u,u',v,v')| \le \beta_{N\varepsilon}(t)$, holding $|F(N_{f_1})| \le \int_0^1 \beta_{N\varepsilon}(t)\,dt$. Since $P_1(\cdot)$ is continuous, when $0 < \delta_1$ is small enough, from (A_1), we have

$$\left|u(0)\right| = \left|P_1\left(\lambda\delta_1 N_{f_1}(u,v)\right)\right| < \theta.$$

It is a contradiction to $|u^i(t)| > \theta$ for each $t \in J$.

(ii°) Suppose that $|u^i|_0 \le 2\theta$, then $\theta \le |(u^i)'|_0 \le \varepsilon$. This implies that $|(u^i)'(t_2)| \ge \theta$ for some $t_2 \in J$, and we can find

$$\theta \le \left|(u^i)'(t_2)\right| \le \left|(u)'(t_2)\right| = \left|\varphi_{p_1}^{-1}\left[t_2, a_1 - F\left(\lambda\delta_1 N_{f_1}(u,v)\right)(t_2)\right]\right|. \tag{21}$$

Since $(u,v) \in \overline{\Omega_\varepsilon}$ and f_1 is Carathéodory, it is easy to see that

$$\left|f_1\left(t,u,u',v,v'\right)\right| \le \beta_{N\varepsilon}(t),$$

thus

$$\left|\delta_1 F(N_{f_1})\right| \le \delta_1 \int_0^1 \beta_{N\varepsilon}(t)\,dt.$$

From Lemma 2.2, $a_1(\cdot)$ is continuous, then we have

$$\left|a_1(\lambda\delta_1 N_{f_1})\right| \to \left|a_1(0)\right| \quad \text{as } \delta_1 \to 0.$$

When $0 < \delta_1$ is small enough, from (A_1) and (21), we can conclude that

$$\theta \le \left|\varphi_{p_1}^{-1}\left[t, a_1 - F\left(\lambda\delta_1 N_{f_1}(u,v)\right)(t)\right]\right| < \frac{\theta}{3}.$$

It is a contradiction. Thus $|u^i|_0 + |(u^i)'|_0 \ne \varepsilon$.

(ii) If $|v^i|_0 + |(v^i)'|_0 = \varepsilon$. Similar to the proof of (i), we get a contradiction. Thus $|v^i|_0 + |(v^i)'|_0 \ne \varepsilon$.

Summarizing this argument, for each $\lambda \in [0,1)$, $(u,v) = A_\lambda(u,v)$ has no solution on $\partial\Omega_\varepsilon$ when positive parameters δ_1 and δ_2 are small enough.

(2°) Since (u_0,v_0) (where (u_0,v_0) is defined in (18)) is the unique solution of $(u,v) = A_0(u,v)$, and (A_1) holds $(u_0,v_0) \in \Omega_\varepsilon$, we can see that the Leray-Schauder degree

$$d_{LS}\left[I - A_0(u,v), \Omega_\varepsilon, 0\right] \ne 0.$$

This completes the proof. □

As applications of Theorem 3.2, we have the following.

Corollary 3.3 *Assume that* $f_l(t,u,u',v,v') = \mu_l(t)|u|^{m_l(t)-2}u(t) + \gamma_l(t)|u'|^{n_l(t)-2}u'(t) + \widetilde{\mu}_l(t)|v|^{\widetilde{m}_l(t)-2}v(t) + \widetilde{\gamma}_l(t)|v'|^{\widetilde{n}_l(t)-2}v'(t)$, *where* $l = 1,2$; $m_l, n_l, \widetilde{m}_l, \widetilde{n}_l, \mu_l, \gamma_l, \widetilde{\mu}_l, \widetilde{\gamma}_l \in C(J,\mathbb{R})$ *satisfy* $\max_{t\in J} p_l(t) < m_l, n_l, \widetilde{m}_l, \widetilde{n}_l$, $\forall t \in J$. *If* $|e_1|$ *and* $|e_2|$ *are small enough, then the problem* (P) *possesses at least one solution.*

Proof It is easy to have

$$\left|f_l(t,u,u',v,v')\right| \le \left|\mu_l(t)\right||u|^{m_l(t)-1} + \left|\gamma_l(t)\right|\left|u'\right|^{n_l(t)-1} + \left|\widetilde{\mu}_l(t)\right||v|^{\widetilde{m}_l(t)-1} + \left|\widetilde{\gamma}_l(t)\right|\left|v'\right|^{\widetilde{n}_l(t)-1}.$$

From $\mu_l, \gamma_l, \widetilde{\mu}_l, \widetilde{\gamma}_l \in C(J,\mathbb{R})$ and the definition of Ω_ε, we have

$$\left|f_l(t,u,u',v,v')\right| \le C_{13}\varepsilon^{m_l(t)-1} + C_{14}\varepsilon^{n_l(t)-1} + C_{15}\varepsilon^{\widetilde{m}_l(t)-1} + C_{16}\varepsilon^{\widetilde{n}_l(t)-1}.$$

Since $\max_{t\in J} p_l(t) < m_l, n_l, \widetilde{m}_l, \widetilde{n}_l$, then there exists a small enough ε such that

$$\left|f_l(t,u,u',v,v')\right| \le \frac{1-\sigma_1}{4N} \cdot \left(\frac{\theta}{3}\right)^{p_l(t)-1}.$$

From Lemma 2.2 and the small enough $|e_2|$, we have

$$\left|a_1(\delta_1 f_1)\right| \le \frac{2N}{1-\sigma_1} \cdot \left(\|\delta_1 f_1\|_{L^1} + |e_2|\right) < \left(\frac{\theta}{3}\right)^{p_1(t)-1},$$

then $|a_1(0)| < \min_{t\in J}(\frac{\theta}{3})^{p_1(t)-1}$ is valid.

Similarly, we have $|a_2(0)| < \min_{t\in J}(\frac{\theta}{2})^{p_2(t)-1}$.

Obviously, it follows from $|a_1(0)| < \min_{t\in J}(\frac{\theta}{3})^{p_1(t)-1}$, $|a_2(0)| < \min_{t\in J}(\frac{\theta}{2})^{p_2(t)-1}$ and the small enough $|e_1|$ that $(u_0, v_0) \in \Omega_\varepsilon$, $|P_1(0)| < \theta$, and $|P_2(0)| < \theta$.

Thus, the conditions of (A_1) are satisfied, then the problem (P) possesses at least one solution. □

Corollary 3.4 *Assume that* $f_l(t,u,u',v,v') = \mu_l(t)|u|^{m_l(t)-2}u(t) + \gamma_l(t)|u'|^{n_l(t)-2}u'(t) + \widetilde{\mu}_l(t)|v|^{\widetilde{m}_l(t)-2}v(t) + \widetilde{\gamma}_l(t)|v'|^{\widetilde{n}_l(t)-2}v'(t)$, *where* $l = 1,2$; $m_l, n_l, \widetilde{m}_l, \widetilde{n}_l, \mu_l, \gamma_l, \widetilde{\mu}_l, \widetilde{\gamma}_l \in C(J,\mathbb{R})$ *satisfy* $\min_{t\in J} p_l(t) \le m_l, n_l, \widetilde{m}_l, \widetilde{n}_l \le \max_{t\in J} p_l(t)$. *If* $|e_1|$, $|e_2|$ *and* δ_l *are small enough, then the problem* (P) *possesses at least one solution.*

Proof From Lemma 2.2, we have

$$\left|a_1(\delta_1 f_1)\right| \le \frac{2N}{1-\sigma_1} \cdot \left(\|\delta_1 f_1\|_{L^1} + |e_2|\right).$$

Since $a_1(\delta_1 f_1)$ is dependent on the small enough δ_1 and $|e_2|$, then it follows from the continuity of a_1 that $|a_1(0)|$ is small enough, which implies that

$$\left|a_1(0)\right| < \min_{t\in J}\left(\frac{\theta}{3}\right)^{p_1(t)-1}.$$

Similarly, we have $|a_2(0)| < \min_{t\in J}(\frac{\theta}{2})^{p_2(t)-1}$.

From $|a_1(0)| < \min_{t\in J}(\frac{\theta}{3})^{p_1(t)-1}$, $|a_2(0)| < \min_{t\in J}(\frac{\theta}{2})^{p_2(t)-1}$ and the small enough $|e_1|$ and $|e_2|$, it is easy to have that $(u_0, v_0) \in \Omega_\varepsilon$, $|P_1(0)| < \theta$, and $|P_2(0)| < \theta$.

Thus, the conditions of (A_1) are satisfied, then the problem (P) possesses at least one solution. □

We denote

$$\Omega_{\varepsilon_1,\varepsilon_2} = \Big\{(u,v) \in W \mid \max_{1\le i\le N}\big(|u^i|_0 + |(u^i)'|_0\big) < \varepsilon_1 \text{ and } \max_{1\le i\le N}\big(|v^i|_0 + |(v^i)'|_0\big) < \varepsilon_2\Big\},$$
$$\theta_1 = \frac{\varepsilon_1}{3}, \qquad \theta_2 = \frac{\varepsilon_2}{3}.$$

Assume the following.

(A_2) Let positive constants ε_1 and ε_2 be such that $(u_0, v_0) \in \Omega_{\varepsilon_1,\varepsilon_2}$, $|P_1(0)| < \theta_1$, $|P_2(0)| < \theta_2$ and $|a_1(0)| < \min_{t\in J}(\frac{\theta_1}{3})^{p_1(t)-1}$, $|a_2(0)| < \min_{t\in J}(\frac{\theta_2}{2})^{p_2(t)-1}$, where (u_0, v_0) is defined in (18), $a_1(\cdot)$ and $a_2(\cdot)$ are defined in (5) and (13), respectively.

It is easy to see that $\Omega_{\varepsilon_1,\varepsilon_2}$ is an open bounded domain in W. We have the following.

Corollary 3.5 *Assume that*

$$\begin{aligned} f_1(t,u,u',v,v') &= \mu(t)|u|^{m(t)-2}u(t) + \gamma(t)|u'|^{n(t)-2}u'(t) \\ &\quad + \widetilde{\mu}(t)|v|^{\widetilde{m}(t)-2}v(t) + \widetilde{\gamma}(t)|v'|^{\widetilde{n}(t)-2}v'(t), \\ f_2(t,u,u',v,v') &= \varkappa(t)|u|^{\epsilon}|v|^{\varrho(t)-2}v(t) + \widetilde{\varkappa}(t)|u'|^{\widetilde{\epsilon}}|v'|^{\widetilde{\varrho}(t)-2}v'(t), \end{aligned}$$

where ϵ, $\widetilde{\epsilon}$ *are positive constants*; $m, n, \widetilde{m}, \widetilde{n}, \varrho, \widetilde{\varrho}, \mu, \gamma, \widetilde{\mu}, \widetilde{\gamma}, \varkappa, \widetilde{\varkappa} \in C(J,\mathbb{R})$ *satisfy* $1 < m, n, \widetilde{m}, \widetilde{n} < \min_{t\in J} p_1(t)$, *and* $\max_{t\in J} p_2(t) < \varrho, \widetilde{\varrho}$, $\forall t \in J$. *Then the problem* (P) *possesses at least one solution.*

Proof Similar to the proof of Theorem 3.2, we only need to prove that (A_2) is satisfied, then we can conclude that the problem (P) possesses at least one solution.

From $\mu, \gamma, \widetilde{\mu}, \widetilde{\gamma} \in C(J,\mathbb{R})$ and the definition of $\Omega_{\varepsilon_1,\varepsilon_2}$, it is easy to have that

$$\big|f_1(t,u,u',v,v')\big| \le C_{17}\varepsilon_1^{m(t)-1} + C_{18}\varepsilon_1^{n(t)-1} + C_{19}\varepsilon_2^{\widetilde{m}(t)-1} + C_{20}\varepsilon_2^{\widetilde{n}(t)-1},$$

where we suppose $\varepsilon_2 < 1 < \varepsilon_1$. Since $1 < m, n, \widetilde{m}, \widetilde{n} < \min_{t\in J} p_1(t)$, then there exists a big enough ε_1 such that

$$\big|f_1(t,u,u',v,v')\big| \le \frac{1-\sigma_1}{3N} \cdot \left(\frac{\theta_1}{3}\right)^{p_1(t)-1}.$$

From Lemma 2.2, we have

$$\big|a_1(\delta_1 f_1)\big| \le \frac{2N}{1-\sigma_1} \cdot \big(\|\delta_1 f_1\|_{L^1} + |e_2|\big) < \left(\frac{\theta_1}{3}\right)^{p_1(t)-1},$$

then $|a_1(0)| < \min_{t\in J}(\frac{\theta_1}{3})^{p_1(t)-1}$ is valid.

From $\varkappa, \widetilde{\varkappa} \in C(J, \mathbb{R})$ and the definition of $\Omega_{\varepsilon_1,\varepsilon_2}$, we have

$$\left|f_2\left(t,u,u',v,v'\right)\right| \le C_{21}\varepsilon_1^{\epsilon}\varepsilon_2^{\varrho(t)-1} + C_{22}\varepsilon_1^{\widetilde{\epsilon}}\varepsilon_2^{\widetilde{\varrho}(t)-1}.$$

Since $\max_{t\in J} p_2(t) < \varrho, \widetilde{\varrho}$, then there exists a ε_2 such that $\varepsilon_2 < (\frac{C_{23}}{C_{21}\varepsilon_1^{\epsilon}+C_{22}\varepsilon_1^{\widetilde{\epsilon}}})^{\frac{1}{\varrho^*-p_2^+}}$ (where $\varrho^* = \min\{\varrho^-, \widetilde{\varrho}^-\}$), which implies that

$$\left|f_2\left(t,u,u',v,v'\right)\right| \le \frac{1}{4N}\cdot\left(\frac{\theta_2}{2}\right)^{p_2(t)-1}.$$

From Lemma 2.3, we have

$$\left|a_2(\delta_2 f_2)\right| \le 3N\|\delta_2 f_2\|_0 < \left(\frac{\theta_2}{2}\right)^{p_2(t)-1},$$

then $|a_2(0)| < \min_{t\in J}(\frac{\theta_2}{2})^{p_2(t)-1}$ is valid.

Obviously, it follows from $|a_1(0)| < \min_{t\in J}(\frac{\theta_1}{3})^{p_1(t)-1}$ and $|a_2(0)| < \min_{t\in J}(\frac{\theta_2}{2})^{p_2(t)-1}$ that $(u_0, v_0) \in \Omega_{\varepsilon_1,\varepsilon_2}$, $|P_1(0)| < \theta_1$, and $|P_2(0)| < \theta_2$.

Thus, the conditions of (A_2) are satisfied, then the problem (P) possesses at least one solution. □

Corollary 3.6 *Assume that*

$$f_1\left(t,u,u',v,v'\right) = \varkappa(t)|u|^{\epsilon}|v|^{\varrho(t)-2}v(t) + \widetilde{\varkappa}(t)\left|u'\right|^{\widetilde{\epsilon}}\left|v'\right|^{\widetilde{\varrho}(t)-2}v'(t),$$
$$f_2\left(t,u,u',v,v'\right) = \mu(t)|u|^{m(t)-2}u(t) + \gamma(t)\left|u'\right|^{n(t)-2}u'(t) + \widetilde{\mu}(t)|v|^{\widetilde{m}(t)-2}v(t) + \widetilde{\gamma}(t)\left|v'\right|^{\widetilde{n}(t)-2}v'(t),$$

where ϵ, $\widetilde{\epsilon}$ *are positive constants*; $\varrho, \widetilde{\varrho}, m, n, \widetilde{m}, \widetilde{n}, \varkappa, \widetilde{\varkappa}, \mu, \gamma, \widetilde{\mu}, \widetilde{\gamma} \in C(J,\mathbb{R})$ *satisfy* $\max_{t\in J} p_1(t) < \varrho, \widetilde{\varrho}$, *and* $1 < m, n, \widetilde{m}, \widetilde{n} < \min_{t\in J} p_2(t)$, $\forall t \in J$. *If* $|e_1|$ *and* $|e_2|$ *are small enough, then the problem* (P) *possesses at least one solution.*

Proof Similar to the proof of Corollary 3.5, we conclude that (A_2) is satisfied. Then the problem (P) possesses at least one solution. □

4 Existence of nonnegative solutions

In the following, we deal with the existence of nonnegative solutions of (P). For any $x = (x^1, \ldots, x^N) \in \mathbb{R}^N$, the notation $x \ge 0$ ($x > 0$) means $x^j \ge 0$ ($x^j > 0$) for any $j = 1, \ldots, N$. For any $x, y \in \mathbb{R}^N$, the notation $x \ge y$ means $x - y \ge 0$, the notation $x > y$ means $x - y > 0$.

Theorem 4.1 *We assume that*

(1^0) $\delta_1 f_1(t,x,y,z,w) \le 0$, $\forall (t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;
(2^0) $\delta_2 f_2(t,x,y,z,w) \ge 0$, $\forall (t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;
(3^0) $e_1 \ge 0$;
(4^0) $e_2 \le 0$.

Then every solution of (P) *is nonnegative.*

Proof (i) We shall show that $u(t)$ is nonnegative.

If (u,v) is a solution of (P), from Lemma 2.5, we have

$$\varphi_{p_1}(t,u'(t)) = a_1(\delta_1 N_{f_1}(u,v)) - \int_0^t \delta_1 f_1(s,u,u',v,v')\,ds, \quad \forall t \in J,$$

which together with (5), (1^0) and (4^0) implies that

$$\begin{aligned}
&\varphi_{p_1}(t,u'(t)) \\
&\quad = a_1(\delta_1 N_{f_1}(u,v)) - F(\delta_1 N_{f_1}(u,v))(t) \\
&\quad = \frac{\int_0^1 \delta_1 N_{f_1}(u,v)(t)\,dt - \int_0^1 k(t)\int_0^t \delta_1 N_{f_1}(u,v)(s)\,ds\,dt + e_2}{1-\sigma_1} - \int_0^t \delta_1 N_{f_1}(u,v)(s)\,ds \\
&\quad = \frac{1}{1-\sigma_1}\left\{\int_0^1 k(t)\int_t^1 \delta_1 N_{f_1}(u,v)(s)\,ds\,dt + (1-\sigma_1)\int_t^1 \delta_1 N_{f_1}(u,v)(s)\,ds + e_2\right\} \le 0.
\end{aligned}$$

Thus $u'(t) \le 0$ for any $t \in J$. Holding $u(t)$ is decreasing, namely $u(t_1) \ge u(t_2)$ for any $t_1, t_2 \in J$ with $t_1 < t_2$.

According to the boundary value condition (2) and condition (3^0), we have

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1 \ge \sum_{i=1}^{m-2} \alpha_i u(1) + e_1,$$

then

$$u(1) \ge \frac{e_1}{1-\sum_{i=1}^{m-2}\alpha_i} \ge 0.$$

Thus $u(t)$ is nonnegative.

(ii) We shall show that $v(t)$ is nonnegative.

If (u,v) is a solution of (P), From Lemma 2.5, we have

$$v(t) = v(0) + F\{\varphi_{p_2}^{-1}[t, a_2 - F(\delta_2 N_{f_2}(u,v))]\}(t).$$

We claim that $a_2(\delta_2 N_{f_2}(u,v)) \ge 0$. If it is false, then there exists some $j \in \{1,\dots,N\}$ such that $a_2^j(\delta_2 N_{f_2}(u,v)) < 0$, which together with condition (2^0) implies that

$$[a_2(\delta_2 N_{f_2}(u,v)) - F(\delta_2 N_{f_2}(u,v))(t)]^j < 0, \quad \forall t \in J. \tag{22}$$

Similar to the proof of Lemma 2.3, the boundary value condition (8) implies

$$\begin{aligned}
0 ={}& \frac{k_1\varphi_{p_2}^{-1}(0,a_2) + \int_0^1 e(t)\int_0^t \varphi_{p_2}^{-1}[s,a_2 - F(\delta_2 N_{f_2}(u,v))(s)]\,ds\,dt}{1-\sigma_2} \\
&+ \frac{\sum_{i=1}^{m-2}\beta_i \int_{\eta_i}^1 \varphi_{p_2}^{-1}[t,a_2 - F(\delta_2 N_{f_2}(u,v))(t)]\,dt + k_2\varphi_{p_2}^{-1}[1,a_2 - F(\delta_2 N_{f_2}(u,v))(1)]}{1-\sum_{i=1}^{m-2}\beta_i} \\
&+ \int_0^1 \varphi_{p_2}^{-1}[t,a_2 - F(\delta_2 N_{f_2}(u,v))(t)]\,dt.
\end{aligned} \tag{23}$$

From (22) and $a_2^j(\delta_2 N_{f_2}(u,v)) < 0$, we get a contradiction to (23).
Thus $a_2(\delta_2 N_{f_2}(u,v)) \geq 0$.
We claim that

$$a_2\big(\delta_2 N_{f_2}(u,v)\big) - F\big(\delta_2 N_{f_2}(u,v)\big)(1) \leq 0. \tag{24}$$

If it is false, then there exists some $j \in \{1,\dots,N\}$ such that

$$\big[a_2\big(\delta_2 N_{f_2}(u,v)\big) - F\big(\delta_2 N_{f_2}(u,v)\big)(1)\big]^j > 0,$$

which together with condition (2^0) implies

$$\big[a_2\big(\delta_2 N_{f_2}(u,v)\big) - F\big(\delta_2 N_{f_2}(u,v)\big)(t)\big]^j > 0, \quad \forall t \in J. \tag{25}$$

From (25) and $a_2(\delta_2 N_{f_2}(u,v)) \geq 0$, we get a contradiction to (23). Thus (24) is valid.
Denote

$$\Gamma(t) = a_2\big(\delta_2 N_{f_2}(u,v)\big) - F\big(\delta_2 N_{f_2}(u,v)\big)(t), \quad \forall t \in J.$$

Obviously, $\Gamma(0) = a_2(\delta_2 N_{f_2}(u,v)) \geq 0$, $\Gamma(1) \leq 0$, and $\Gamma(t)$ is decreasing, *i.e.*, $\Gamma(t') \leq \Gamma(t'')$ for any $t', t'' \in J$ with $t' \geq t''$. For any $j = 1,\dots,N$, there exist $\zeta_j \in J$ such that

$$\Gamma^j(t) \geq 0, \quad \forall t \in [0,\zeta_j] \quad \text{and} \quad \Gamma^j(t) \leq 0, \quad \forall t \in [\zeta_j, T).$$

We can conclude that $v^j(t)$ is increasing on $[0,\zeta_j]$, and $v^j(t)$ is decreasing on $[\zeta_j, T]$. Thus

$$\min\big\{v^j(0), v^j(1)\big\} = \inf_{t\in I} v^j(t), \quad j = 1,\dots,N.$$

For any fixed $j \in \{1,\dots,N\}$, if

$$v^j(0) = \inf_{t\in I} v^j(t),$$

which together with (8) implies that

$$v^j(0) = \int_0^1 e(t)v^j(t)\,dt + k_1\big(v'\big)^j(0) \geq \int_0^1 e(t)v^j(0)\,dt + k_1\big(v'\big)^j(0). \tag{26}$$

From $a_2(\delta_2 N_{f_2}(u,v)) \geq 0$, we have

$$\big(v'\big)^j(0) = \big(\varphi_{p_2}^{-1}[0,a_2]\big)^j \geq 0. \tag{27}$$

It follows from (26) and (27) that

$$v^j(0) \geq \frac{k_1(v')^j(0)}{1-\sigma_2} \geq 0.$$

If

$$v^j(1) = \inf_{t\in I} v^j(t), \tag{28}$$

from (8) and (28), we have

$$v^j(1) = \sum_{i=1}^{m-2} \beta_i v^j(\eta_i) - k_2\big(v'\big)^j(1) \geq \sum_{i=1}^{m-2} \beta_i v^j(1) - k_2\big(v'\big)^j(1). \tag{29}$$

Since $a_2(\delta_2 N_{f_2}(u,v)) - F(\delta_2 N_{f_2}(u,v))(1) \leq 0$, we have

$$\big(v'\big)^j(1) = \big(\varphi_{p_2}^{-1}\big[1, a_2 - F\big(\delta_2 N_{f_2}(u,v)\big)(1)\big]\big)^j \leq 0. \tag{30}$$

Combining (29) and (30), we have

$$v^j(1) \geq \frac{-k_2(v')^j(1)}{1 - \sum_{i=1}^{m-2} \beta_i} \geq 0.$$

Thus $v(t)$ is nonnegative.

Combining (i) and (ii), we find that every solution of (P) is nonnegative. □

Corollary 4.2 *We assume that*

(1^0) $\delta_1 f_1(t,x,y,z,w) \leq 0$, $\forall (t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ *with* $x, z \geq 0$;
(2^0) $\delta_2 f_2(t,x,y,z,w) \geq 0$, $\forall (t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ *with* $x, z \geq 0$;
(3^0) $e_1 \geq 0$;
(4^0) $e_2 \leq 0$.

Then we have

(a) *Under the conditions of Theorem* 3.1, (P) *has at least one nonnegative solution* (u,v);
(b) *Under the conditions of Theorem* 3.2, (P) *has at least one nonnegative solution* (u,v).

Proof (a) Define

$$L_1(u) = \big(L_*\big(u^1\big), \ldots, L_*\big(u^N\big)\big), \qquad L_2(v) = \big(L_*\big(v^1\big), \ldots, L_*\big(v^N\big)\big),$$

where

$$L_*(t) = \begin{cases} t, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Denote

$$\widetilde{f}_l\big(t,u,u',v,v'\big) = f_l\big(t, L_1(u), u', L_2(v), v'\big), \quad \forall \big(t,u,u',v,v'\big) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N,$$

where $l = 1,2$, then $\widetilde{f}_l(t,u,u',v,v')$ satisfies the Carathéodory condition, $\widetilde{f}_1(t,u,u',v,v') \leq 0$ and $\widetilde{f}_2(t,u,u',v,v') \geq 0$.

We assume the following.

(A_2) $\lim_{|u|+|v| \to +\infty} (\widetilde{f}_l(t,u,u',v,v')/(|u|+|v|)^{q_l(t)-1}) = 0$ for $t \in J$ uniformly, where $q_l(t) \in C(I,\mathbb{R})$ and $1 < q_l^- \leq q_l^+ < p_l^-$.

Obviously, $\widetilde{f}_l(t,\cdot,\cdot,\cdot,\cdot)$ satisfies a sub-$(p_l^- - 1)$ growth condition.

Let us consider the existence of solutions of the following system:

$$\left.\begin{array}{l} -\triangle_{p_1(t)}u = \delta_1\widetilde{f}_1(t,u,u',v,v'), \quad t\in(0,1), \\ -\triangle_{p_2(t)}v = \delta_2\widetilde{f}_2(t,u,u',v,v'), \quad t\in(0,1), \\ u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i) + e_1, \\ \lim_{t\to 1^-}|u'|^{p_1(t)-2}u'(t) = \int_0^1 k(t)|u'|^{p_1(t)-2}u'(t)\,dt + e_2, \\ v(0) - k_1v'(0) = \int_0^1 e(t)v(t)\,dt, \qquad v(1)+k_2v'(1) = \sum_{i=1}^{m-2}\beta_i v(\eta_i). \end{array}\right\} \tag{31}$$

According to Theorem 3.1, (31) has at least a solution (u,v). From Theorem 4.1, we can see that (u,v) is nonnegative. Thus, (u,v) is a nonnegative solution of (P).

(b) It is similar to the proof of (a).

This completes the proof. □

5 Examples

Example 5.1 Consider the following problem:

$$(S_1)\quad \begin{cases} -\triangle_{p_1(t)}u = f_1(t,u,u',v,v') = e^{-2t}(|u|^{q(t)-2}u + |u'|^{q(t)-2}u') \\ \quad + |v|^{q(t)-2}v + |v'|^{q(t)-2}v' + (t+1)^{-2}, \quad t\in(0,1), \\ -\triangle_{p_2(t)}v = f_2(t,u,u',v,v') = |u|^{q(t)-2}u + |u'|^{q(t)-2}u' \\ \quad + t^2(|v|^{q(t)-2}v + |v'|^{q(t)-2}v') + (t+2)^2, \quad t\in(0,1), \\ u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i) + e_1, \\ \lim_{t\to 1^-}|u'|^{p_1(t)-2}u'(t) = \int_0^1 \frac{1}{1+t}|u'|^{p_1(t)-2}u'(t)\,dt + e_2, \\ v(0) - k_1v'(0) = \int_0^1 e^{-t}v(t)\,dt, \qquad v(1)+k_2v'(1) = \sum_{i=1}^{m-2}\beta_i v(\eta_i), \end{cases}$$

where $p_1(t) = 7 + 3^{-t}\cos 3t$, $p_2(t) = 7 + 3^{-t}\sin 3t$, $q(t) = 3 + 2^{-t}\cos t$.

Obviously, f_1 and f_2 are Caratheodory, $q(t) \le 4 < 5 \le \min\{p_1(t), p_2(t)\}$, $\sum_{i=1}^{m-2}\alpha_i < 1$, $\sum_{i=1}^{m-2}\beta_i < 1$, then the conditions of Theorem 3.1 are satisfied, then (S_1) has a solution.

Example 5.2 Consider the following problem

$$(S_2)\quad \begin{cases} -\triangle_{p_1(t)}u = f_1(t,u,u',v,v') = -e^{-2t}(|u|^{q(t)-2}u + |u'|^{q(t)-1}) \\ \quad - |v|^{q(t)-1} - |v'|^{q(t)-1} - (t+1)^{-2}, \quad t\in(0,1), \\ -\triangle_{p_2(t)}v = f_2(t,u,u',v,v') = |u|^{q(t)-2}u + |u'|^{q(t)-1} \\ \quad + t^2(|v|^{q(t)-2}v + |v'|^{q(t)-1}) + (t+2)^2, \quad t\in(0,1), \\ u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i) + 1, \\ \lim_{t\to 1^-}|u'|^{p_1(t)-2}u'(t) = \int_0^1 \frac{1}{1+t}|u'|^{p_1(t)-2}u'(t)\,dt - 2, \\ v(0) - k_1v'(0) = \int_0^1 e^{-t}v(t)\,dt, \qquad v(1)+k_2v'(1) = \sum_{i=1}^{m-2}\beta_i v(\eta_i), \end{cases}$$

where $N = 1$, $p_1(t) = 7 + 3^{-t}\cos 3t$, $p_2(t) = 7 + 3^{-t}\sin 3t$, $q(t) = 4 + e^{-2t}\sin 2t$.

Obviously, f_1 and f_2 are Caratheodory, $q(t) \le 5 < 6 \le \min\{p_1(t), p_2(t)\}$, $\sum_{i=1}^{m-2}\alpha_i < 1$, $\sum_{i=1}^{m-2}\beta_i < 1$, the conditions of Corollary 4.2 are satisfied, then (S_2) has a nonnegative solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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