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Approximate controllability of fractional integro-differential equations involving nonlocal initial conditions

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Abstract

We discuss the approximate controllability of nonlinear fractional integro-differential system under the assumptions that the corresponding linear system is approximately controllable. Using the fixed-point technique, fractional calculus and methods of controllability theory, a new set of sufficient conditions for approximate controllability of fractional integro-differential equations are formulated and proved. The results in this paper are generalization and continuation of the recent results on this issue. An example is provided to show the application of our result.

1 Introduction

Controllability is one of the fundamental concepts in mathematical control theory, which plays an important role in control systems. The controllability of nonlinear systems represented by evolution equations or inclusions in abstract spaces and qualitative theory of fractional differential equations has been extensively studied by several authors. An extensive list of these publications can be found in [1–44] and the references therein. Recently, the approximate controllability for various kinds of (fractional) differential equations has generated considerable interest. A pioneering work on the approximate controllability of deterministic and stochastic systems has been reported by Bashirov and Mahmudov [5], Dauer and Mahmudov [8] and Mahmudov [10]. Sakthivel *et al.* [28] studied the approximate controllability of nonlinear deterministic and stochastic evolution systems with unbounded delay in abstract spaces. On the other hand, the fractional differential equation has gained more attention due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Yan [45] derived a set of sufficient conditions for the controllability of fractional-order partial neutral functional integro-differential inclusions with infinite delay in Banach spaces. Debbouche and Baleanu [1] established the controllability result for a class of fractional evolution nonlocal impulsive quasi-linear delay integro-differential systems in a Banach space using the theory of fractional calculus and fixed point technique. However, there exists only a limited number of papers on the approximate controllability of the fractional nonlinear evolution systems. Sakthivel *et al.* [28] studied the approximate controllability of deterministic semilinear fractional differential equations in Hilbert spaces. Wang [40] investigated the nonlocal controllability of fractional evolution systems. Surendra Kumar and Sukavanam [33] obtained a new set of sufficient conditions for the approximate controllability of a class of semilinear delay control systems of fractional order using the contraction principle and the Schauder fixed-point theorem. More recently, Sakthivel *et al.* [27] derived a new set of sufficient conditions for approximate controllability of fractional stochastic differential equations.

In this paper, we discuss the approximate controllability of nonlinear fractional integro-differential system under the assumption that the corresponding linear system is approximately controllable. We consider the following fractional integro-differential control system involving nonlocal conditions,

$$\begin{aligned} &{}^{C}D_t^{\beta}x(t) = -Ax(t) + f\bigl(t,x(t)\bigr) + \int_0^t K(t-s)g\bigl(s,x(s)\bigr)\,ds + Bu(t), \\ &x(0) = x_0 + h(x), \end{aligned} \tag{1}$$

in X_α, where ${}^{C}D_t^{\beta}$, $0 < \beta < 1$, stands for the Caputo fractional derivative of order β, and $f : [0,T] \times X_\alpha \to X$, $g : [0,T] \times X_\alpha \to X$, $K : [0,T] \to R^+$, $h : C([0,T];X_\alpha) \to X_\alpha$ are given functions to be specified later. Here, $(-A, D(A))$ is the infinitesimal generator of a compact analytic semigroup of bounded linear operators $S(t)$, $t \geq 0$, on a real Hilbert space X. B is a linear bounded operator from a real Hilbert space U to X.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results on the fractional powers of the generator of an analytic compact semigroup and introduce the mild solution of system (1). In Section 3, we study the existence of mild solutions for system (1) under the feedback control $u_\varepsilon(t,x)$ defined in (5). We show that the control system (1) is approximately controllable on $[0,T]$ provided that the corresponding linear system is approximately controllable. Finally, an example is given to demonstrate the applicability of our result.

2 Preliminaries

In this section, we introduce some facts about the fractional powers of the generator of a compact analytic semigroup, the Caputo fractional derivative that are used throughout this paper.

We assume that X is a Hilbert space with norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. Let $C([0,T],X)$ be the Banach space of continuous functions from $[0,T]$ into X with the norm $\|x\| = \sup_{t\in[0,T]}\|x(t)\|$, here $x \in C([0,T],X)$. In this paper, we also assume that $-A : D(A) \subset X \to X$ is the infinitesimal generator of a compact analytic semigroup $S(t)$, $t > 0$, of uniformly bounded linear operator in X, that is, there exists $M > 1$ such that $\|S(t)\| \leq M$ for all $t \geq 0$. Without loss of generality, let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for any $\alpha > 0$, we can define $A^{-\alpha}$ by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}S(t)\,dt.$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of X. Hence we can define $A^\alpha := (A^{-\alpha})^{-1}$, which is a closed bijective linear operator in X. It can be shown that each A^α has dense domain and that $D(A^\beta) \subset D(A^\alpha)$ for $0 \leq \alpha \leq \beta$. Moreover, $A^{\alpha+\beta}x = A^\alpha A^\beta x = A^\beta A^\alpha x$ for every $\alpha, \beta \in R$ and $x \in D(A^\mu)$ with $\mu := \max(\alpha, \beta, \alpha+\beta)$, where $A^0 = I$, I is the identity in X. (For proofs of these facts, we refer to the literature [15, 20, 22].)

We denote by X_α the Hilbert space of $D(A^\alpha)$ equipped with norm $\|x\|_\alpha := \|A^\alpha x\| = \sqrt{\langle A^\alpha x, A^\alpha x\rangle}$ for $x \in D(A^\alpha)$, which is equivalent to the graph norm of A^α. Then we have $X_\beta \hookrightarrow X_\alpha$, for $0 \le \alpha \le \beta$ (with $X_0 = X$) and the embedding is continuous. Moreover, A^α has the following basic properties.

Lemma 1 [42] *A^α and $S(t)$ have the following properties.*

(i) *$S(t) : X \to X_\alpha$ for each $t > 0$ and $\alpha \ge 0$.*

(ii) *$A^\alpha S(t)x = S(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \ge 0$.*

(iii) *For every $t > 0$, $A^\alpha S(t)$ is bounded in X and there exists $M_\alpha > 0$ such that*

$$\|A^\alpha S(t)\| \le M_\alpha t^{-\alpha}.$$

(iv) *$A^{-\alpha}$ is a bounded linear operator for $0 \le \alpha \le 1$.*

Let us recall the following known definitions of fractional calculus. For more details, see [43, 44].

Definition 2 The fractional integral of order $\alpha > 0$ with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 3 The Caputo derivative of order $\alpha > 0$ with the lower limit 0 for a function f can be written as

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, 0 \le n-1 < \alpha < n.$$

The Caputo derivative of a constant is equal to zero. If f is an abstract function with values in X then the integrals which appear in Definitions 2 and 3 are taken in Bochner's sense.

According to Definitions 2 and 3, it is suitable to rewrite the problem (1) in the equivalent integral equation

$$\begin{aligned} x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{\alpha-1} \\ \times \left[Ax(s) + Bu(s) + f\big(s, x(s)\big) + \int_0^s K(s-r) g\big(r, x(r)\big)\, dr \right] ds, \quad t \in [0,T], \end{aligned} \tag{2}$$

provided that the integral in (2) exists. Applying the Laplace transform

$$\begin{aligned} &v(\lambda) = \int_0^\infty e^{-\lambda s} x(s)\, ds, \qquad w(\lambda) = \int_0^\infty e^{-\lambda s} u(s)\, ds \quad \text{and} \\ &\omega(\lambda) = \int_0^\infty e^{-\lambda s} \left(f\big(s, x(s)\big) + \int_0^s K(s-r) g\big(r, x(r)\big)\, dr \right) ds, \quad \lambda > 0, \end{aligned}$$

to (2) and using the method similar to that used in [38] we get

$$x(t) = \int_0^\infty \Psi_\beta(\theta) S(t^\beta\theta) x_0 \, d\theta + \beta \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \Psi_\beta(\theta) S((t-s)^\beta \theta) \times \left[Bu(s) + \left(f(s, x(s)) + \int_0^s K(s-r) g(r, x(r)) \, dr \right) \right] d\theta \, ds,$$

where

$$\Psi_\beta(\theta) = \frac{1}{\beta} \theta^{-1-\frac{1}{\beta}} \bar{w}_q(\theta^{-\frac{1}{\beta}}) \geq 0,$$
$$\bar{w}_\beta(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\beta n-1} \frac{\Gamma(n\beta+1)}{n!} \sin(n\pi\beta), \quad \theta \in (0, \infty).$$

Here, Ψ_β is a probability density function defined on $(0,\infty)$, that is $\Psi_\beta(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \Psi_\beta(\theta)\, d\theta = 1$.

For $x \in X$, we define two families $\{S_\beta(t) : t \geq 0\}$ and $\{P_\beta(t) : t \geq 0\}$ of operators by

$$S_\beta(t) = \int_0^\infty \Psi_\beta(\theta) S(t^\beta\theta) \, d\theta,$$
$$P_\beta(t) = \beta \int_0^\infty \theta \Psi_\beta(\theta) S(t^\beta\theta) \, d\theta,$$

respectively.

The following lemma follows from the results given in [37–39].

Lemma 4 *The operators S_β and P_β have the following properties.*

(i) *For any fixed $t \geq 0$, and any $x \in X_\alpha$, we have the operators $S_\beta(t)$ and $P_\beta(t)$ are linear and bounded operators, i.e. for any $x \in X$,*

$$\|S_\beta(t)x\|_\alpha \leq M\|x\|_\alpha \quad \text{and} \quad \|P_\beta(t)x\|_\alpha \leq \frac{M}{\Gamma(\beta)}\|x\|_\alpha.$$

(ii) *The operators $S_\beta(t)$ and $P_\beta(t)$ are strongly continuous for all $t \geq 0$.*

(iii) *$S_\beta(t)$ and $P_\beta(t)$ are norm continuous in X for $t > 0$.*

(iv) *$S_\beta(t)$ and $P_\beta(t)$ are compact operators in X for $t > 0$.*

(v) *For every $t > 0$, the restriction of $S_\beta(t)$ to X_α and the restriction of $P_\beta(t)$ to X_α are norm continuous.*

(vi) *For every $t > 0$, the restriction of $S_\beta(t)$ to X_α and the restriction of $P_\beta(t)$ to X_α are compact operators in X_α.*

(vii) *For all $x \in X$ and $t \in (0,\infty)$,*

$$\|A^\alpha P_\beta(t)x\| \leq C_\alpha t^{-\alpha\beta}\|x\|, \quad \text{where } C_\alpha := \frac{M_\alpha \beta \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}.$$

In this paper, we adopt the following definition of mild solution of equation (1).

Definition 5 A function $x(\cdot;x_0,u)\in C([0,T],X_\alpha)$ is said to be a mild solution of (1) if for any $u\in L_2([0,T],U)$ the integral equation

$$\begin{aligned} x(t) &= S_\beta(t)\big(x_0+h(x)\big)+\int_0^t (t-s)^{\beta-1}P_\beta(t-s)Bu(s)\,ds \\ &\quad+\int_0^t (t-s)^{\beta-1}P_\beta(t-s)\left[f\big(s,x(s)\big)+\int_0^s K(s-r)g\big(r,x(r)\big)\,dr\right]ds, \end{aligned} \tag{3}$$

is satisfied.

It is clear that $L_0^t := \int_0^t (t-s)^{\beta-1}P_\beta(t-s)Bu(s)\,ds : L_2([0,T],U)\to C([0,T],X_\alpha)$ is bounded if $\frac{1}{2}<\beta\le 1$. In what follows, we assume that $\frac{1}{2}<\beta\le 1$.

3 Approximate controllability

In this section, we state and prove conditions for the approximate controllability of semilinear fractional control integro-differential systems. To do this, we first prove the existence of a fixed point of the operator Λ_ε defined below using Krasnoselskii's fixed-point theorem. Secondly, in Theorem 11, we show that under the uniform boundedness of f and g the approximate controllability of fractional systems (1) is implied by the approximate controllability of the corresponding linear system (4).

Let $x(T;x_0,u)$ be the state value of (1) at terminal time T corresponding to the control u and the initial value x_0. Introduce the set $\Re(T,x_0)=\{x(T;x_0,u): u\in L_2([0,T],U)\}$, which is called the reachable set of system (1) at terminal time T, its closure in X_α is denoted by $\overline{\Re(T,x_0)}$.

Definition 6 The system (1) is said to be approximately controllable on $[0,T]$ if $\overline{\Re(T,x_0)} = X_\alpha$, that is, given an arbitrary $\varepsilon>0$ it is possible to steer from the point x_0 to within a distance ε from all points in the state space X_α at time T.

Consider the following linear fractional differential system:

$$\begin{aligned} &D_t^\beta x(t) = Ax(t)+Bu(t), \quad t\in[0,T], \\ &x(0)=x_0. \end{aligned} \tag{4}$$

The approximate controllability for linear fractional system (4) is a natural generalization of approximate controllability of linear first order control system [9, 10, 12]. It is convenient at this point to introduce the controllability and resolvent operators associated with (4) as

$$\begin{aligned} &\Gamma_0^T = \int_0^T (T-s)^{\beta-1}P_\beta(T-s)BB^*P_\beta^*(T-s)\,ds : X\to X, \\ &R\big(\varepsilon,\Gamma_0^T\big) = \big(\varepsilon I+\Gamma_0^T\big)^{-1} : X\to X, \quad \varepsilon>0, \end{aligned}$$

respectively, where B^* denotes the adjoint of B and $P_\beta^*(t)$ is the adjoint of $P_\beta(t)$. It is straightforward that the operator Γ_0^T is a linear bounded operator.

Theorem 7 [10] *Let Z be a separable reflexive Banach space and let Z^* stands for its dual space. Assume that $\Gamma: Z^*\to Z$ is symmetric. Then the following two conditions are equivalent*:

1. $\Gamma : Z^* \to Z$ *is positive, that is,* $\langle z^*, \Gamma z^* \rangle > 0$ *for all nonzero* $z^* \in Z^*$.
2. *For all* $h \in Z z_\varepsilon(h) = \varepsilon(\varepsilon I + \Gamma J)^{-1}(h)$ *strongly converges to zero as* $\varepsilon \to 0^+$. *Here,* J *is the duality mapping of* Z *into* Z^*.

Lemma 8 *The linear fractional control system* (4) *is approximately controllable on* $[0, T]$ *if and only if* $\varepsilon R(\varepsilon, \Gamma_0^T) \to 0$ *as* $\varepsilon \to 0^+$ *in the strong operator topology.*

Proof The lemma is a straightforward consequence of Theorem 7. Indeed, the system (4) is approximately controllable on $[0, T]$ if and only if $\langle \Gamma_0^T x, x \rangle > 0$ for all nonzero $x \in X$, see [7]. By Theorem 7, $\|\varepsilon(\varepsilon I + \Gamma_0^T)^{-1}(h)\| \to 0$ as $\varepsilon \to 0^+$ for all $h \in X$. □

Remark 9 Notice that positivity of Γ_0^T is equivalent to $\langle \Gamma_0^T x, x \rangle = 0 \Longrightarrow x = 0$. In other words, since $\langle \Gamma_0^T x, x \rangle = \int_0^T (T-s)^{\beta-1} \|B^* P_\beta^*(T-s)x\|^2 \, ds$, approximate controllability of the linear system (4) is equivalent to $B^* P_\beta^*(T-s)x = 0$, $0 \le s < T \Longrightarrow x = 0$.

Before proving the main results, let us first introduce our basic assumptions.

(H_1) $f, g : [0, T] \times X_\alpha \times X_\alpha \to X$ are continuous and for each $r \in \mathbb{N}$, there exists a constant $\gamma \in [0, \beta(1-\alpha)]$ and functions $\varphi_r \in L^{1/\gamma}([0, T]; \mathbb{R}^+)$, $\psi_r \in L^\infty([0, T]; \mathbb{R}^+)$ such that

$$\sup\bigl\{\bigl\|f(t,x)\bigr\| : \|x\|_\alpha \le r\bigr\} \le \varphi_r \quad \text{and} \quad \liminf_{r\to\infty} \frac{\|\varphi_r\|_{L^{1/\gamma}}}{r} = \sigma_1 < \infty,$$
$$\sup\bigl\{\bigl\|g(t,x)\bigr\| : \|x\|_\alpha \le r\bigr\} \le \psi_r \quad \text{and} \quad \liminf_{r\to\infty} \frac{\|\psi_r\|_{L^\infty}}{r} = \sigma_2 < \infty.$$

(H_2) $h : C([0, T]; X_\alpha) \to X_\alpha$ is a Lipschitz function with Lipschitz constant L_h.
(H_c) The linear system (4) is approximately controllable on $[0, T]$.

Using the hypothesis (H_c), for an arbitrary function $x \in C([0, T]; X_\alpha)$, we choose the feedback control function as follows:

$$\begin{aligned} u_\varepsilon(t,x) &= B^* P_\beta^*(T-t)\bigl(\varepsilon I + \Gamma_0^T\bigr)^{-1}\biggl[S_\beta(T)\bigl(x_0 + h(x)\bigr) \\ &\quad - \int_0^T (T-s)^{\beta-1} P_\beta(T-s)\biggl[f\bigl(s, x(s)\bigr) + \int_0^s K(s,r) g\bigl(r, x(r)\bigr)\, dr\biggr] ds\biggr]. \end{aligned} \tag{5}$$

Let $B_r = \{x \in C([0, T]; X_\alpha) : \|x\|_\alpha \le r\}$, where r is a positive constant. Then B_r is clearly a bounded closed and convex subset in $C([0, T]; X_\alpha)$. We will show that when using the above control the operator $\Lambda_\varepsilon : B_k \to B_k$ defined by

$$(\Lambda_\varepsilon x)(t) := (\Phi_\varepsilon x)(t) + (\Pi_\varepsilon x)(t), \quad t \in [0, T],$$

where

$$\begin{aligned} (\Phi_\varepsilon x)(t) &:= S_\beta(t)\bigl(x_0 + h(x)\bigr), \\ (\Pi_\varepsilon x)(t) &:= \int_0^t (t-s)^{\beta-1} P_\beta(t-s)\biggl[f\bigl(s, x(s)\bigr) + \int_0^s K(s,r) g\bigl(r, x(r)\bigr)\, dr\biggr] ds \\ &\quad + \int_0^t (t-s)^{\beta-1} P_\beta(t-s) B u_\varepsilon(s,x)\, ds \end{aligned}$$

has a fixed point in $C([0, T]; X_\alpha)$.

Theorem 10 *Let the assumptions* (H_1) *and* (H_2) *be satisfied. Then for* $x_0 \in X_\alpha$, *the fractional Cauchy problem* (1) *with* $u = u_\varepsilon(t,x)$ *has at least one mild solution provided that*

$$L_C + \frac{C_\alpha T^{(1-\alpha)\beta}}{\varepsilon(1-\alpha)\beta} \frac{M}{\Gamma(\beta)} L_B^2 L_C < 1, \tag{6}$$

where

$$L_C := ML_h + C_\alpha \left(\frac{1-\gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta-\gamma}\sigma_1 + \frac{C_\alpha K T^{(1-\alpha)\beta}}{(1-\alpha)\beta}\sigma_2,$$
$$L_B := \|B\|, \qquad K := \max_{0 \le t \le b} \bigl|K(t)\bigr|.$$

Proof It is easy to see that for any $\varepsilon > 0$ the operator Λ_ε maps $C([0,T];X_\alpha)$ into itself.

Let $x \in B_r$ and $0 \le t \le T$. Using assumption (H_1) yield the following estimations,

$$\begin{aligned}
\bigl\|u_\varepsilon(s,x)\bigr\| &\le \frac{1}{\varepsilon}\frac{M}{\Gamma(\beta)} L_B \biggl[M\bigl(\|x_0\|_\alpha + L_h r + \bigl\|h(0)\bigr\|_\alpha\bigr) \\
&\quad + C_\alpha \left(\frac{1-\gamma}{(1-\alpha)\beta-\gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta-\gamma} \|\varphi_r\|_{L^{1/\gamma}} + \frac{C_\alpha K T^{(1-\alpha)\beta}}{(1-\alpha)\beta} \|\psi_r\|_{L^\infty} \biggr] \\
&\le \frac{1}{\varepsilon}\frac{M}{\Gamma(\beta)} L_B L_u(r), \\
L_u(r) &:= M\bigl(\|x_0\|_\alpha + L_h r + \bigl\|h(0)\bigr\|_\alpha\bigr) + C_\alpha \left(\frac{1-\gamma}{(1-\alpha)\beta-\gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta-\gamma} \|\varphi_r\|_{L^{1/\gamma}} \\
&\quad + \frac{C_\alpha K T^{(1-\alpha)\beta}}{(1-\alpha)\beta} \|\psi_r\|_{L^\infty},
\end{aligned}$$

and

$$\begin{aligned}
&\bigl\|(\Phi_\varepsilon y)(t) + (\Pi_\varepsilon x)(t)\bigr\|_\alpha \\
&\quad \le \bigl\|S_\beta(t)\bigl(x_0 - h(y)\bigr)\bigr\|_\alpha + \int_0^t (t-s)^{\beta-1} \bigl\|A^\alpha P_\beta(t-s)\bigr\|_{L(X)} \bigl\|Bu_\varepsilon(s,x)\bigr\|\,ds \\
&\qquad + \int_0^t (t-s)^{\beta-1} \bigl\|A^\alpha P_\beta(t-s)\bigr\|_{L(X)} \left\| f\bigl(s,x(s)\bigr) + \int_0^s K(s,r) g\bigl(r,x(r)\bigr)\,dr \right\| ds \\
&\quad \le M\bigl(\|x_0\|_\alpha + L_h r + \bigl\|h(0)\bigr\|_\alpha\bigr) + C_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1} \frac{1}{\varepsilon} L_B^2 \frac{M}{\Gamma(\beta)} L_u\,ds \\
&\qquad + C_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1} \bigl(\varphi_r(s) + K\|\psi_r\|_{L^\infty}\bigr)\,ds \\
&\quad \le M\bigl(\|x_0\|_\alpha + L_h r + \bigl\|h(0)\bigr\|_\alpha\bigr) + \frac{C_\alpha T^{(1-\alpha)\beta}}{\varepsilon(1-\alpha)\beta} \frac{M}{\Gamma(\beta)} L_B^2 L_u(r) \\
&\qquad + C_\alpha \left(\frac{1-\gamma}{(1-\alpha)\beta-\gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta-\gamma} \|\varphi_r\|_{L^{1/\gamma}} \\
&\qquad + \frac{C_\alpha K T^{(1-\alpha)\beta}}{(1-\alpha)\beta} \|\psi_r\|_{L^\infty}.
\end{aligned} \tag{7}$$

From (6) and the assumption (H_2), it follows that for any $\varepsilon > 0$ there exists $r(\varepsilon) > 0$ such that

$$\begin{aligned}
&M\bigl(\|x_0\|_\alpha + L_h r(\varepsilon) + \bigl\|h(0)\bigr\|_\alpha\bigr) + \frac{C_\alpha T^{(1-\alpha)\beta}}{\varepsilon(1-\alpha)\beta}\frac{M}{\Gamma(\beta)}L_B^2 L_u\bigl(r(\varepsilon)\bigr) \\
&\quad + C_\alpha\left(\frac{1-\gamma}{(1-\alpha)\beta-\gamma}\right)^{1-\gamma} T^{(1-\alpha)\beta-\gamma}\|\varphi_{r(\varepsilon)}\|_{L^{1/\gamma}} + \frac{C_\alpha K T^{(1-\alpha)\beta}}{(1-\alpha)\beta}\|\psi_{r(\varepsilon)}\|_{L^\infty} \le r(\varepsilon).
\end{aligned} \tag{8}$$

Therefore, from (7) and (8), it follows that for any $\varepsilon > 0$ there exists $r(\varepsilon) > 0$ such that $\Phi_\varepsilon y + \Pi_\varepsilon x \in B_{r(\varepsilon)}$ for every $x, y \in B_{r(\varepsilon)}$. Therefore, for any $\varepsilon > 0$ the fractional Cauchy problem (1) with the control (5) has a mild solution if and only if the operator $\Phi_\varepsilon + \Pi_\varepsilon$ has a fixed point in $B_{r(\varepsilon)}$.

In what follows, we will show that Φ_ε and Π_ε satisfy the conditions of Krasnoselskii's fixed-point theorem. From (H_2) and (6), we infer that Φ_ε is a contraction. Next, we show that Π_ε is completely continuous on $B_{r(\varepsilon)}$.

Step 1: We first prove that Π_ε is continuous on $B_{r(\varepsilon)}$. Let $\{x_n\}_{n=1}^\infty \subset B_{r(\varepsilon)}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $C([0,T]; X_\alpha)$. Therefore, it follows from the continuity of f, g and u_ε that for each $t \in [0,T]$,

$$\begin{aligned}
&f\bigl(s, x_n(s)\bigr) \to f\bigl(s, x(s)\bigr), \\
&g\bigl(s, x_n(s)\bigr) \to g\bigl(s, x(s)\bigr), \\
&u_\varepsilon\bigl(s, x_n(s)\bigr) \to Bu_\varepsilon\bigl(s, x(s)\bigr) \quad \text{as } n \to \infty.
\end{aligned}$$

Also, by (H_1), we see that

$$\begin{aligned}
&\int_0^t (t-s)^{\beta-1-\alpha\beta}\biggl(\bigl\|f\bigl(s, x_n(s)\bigr) - f\bigl(s, x(s)\bigr)\bigr\| \\
&\qquad + \int_0^s \bigl|K(s-r)\bigr|\bigl\|g\bigl(r, x_n(r)\bigr) - g\bigl(r, x(r)\bigr)\bigr\|\,dr\biggr)\,ds \\
&\qquad + \int_0^t (t-s)^{\beta-1-\alpha\beta}\bigl\|Bu_\varepsilon\bigl(s, x_n(s)\bigr) - Bu_\varepsilon\bigl(s, x(s)\bigr)\bigr\|\,ds \\
&\quad \le 2C_\alpha\left(\frac{1-\gamma}{(1-\alpha)\beta-\gamma}\right)^{1-\gamma} T^{(1-\alpha)\beta-\gamma}\|\varphi_k\|_{L^{1/\gamma}} + \frac{2C_\alpha K T^{(1-\alpha)\beta}}{(1-\alpha)\beta}\|\psi_k\|_{L^\infty} \\
&\qquad + C_\alpha\frac{1}{\varepsilon}\int_0^t (t-s)^{\beta(1-\alpha)-1}\frac{M}{\Gamma(\beta)}L_B^2 L_u\,ds.
\end{aligned}$$

Since

$$\begin{aligned}
&\bigl\|(\Pi_\varepsilon x_n)(t) - (\Pi_\varepsilon x)(t)\bigr\|_\alpha \\
&\quad \le C_\alpha\int_0^t (t-s)^{\beta-1-\alpha\beta} \\
&\qquad \times \biggl(\bigl\|f\bigl(s, x_n(s)\bigr) - f\bigl(s, x(s)\bigr)\bigr\| + \int_0^s \bigl|K(s-r)\bigr|\bigl\|g\bigl(r, x_n(r)\bigr) - g\bigl(r, x(r)\bigr)\bigr\|\,dr\biggr)\,ds,
\end{aligned}$$

using the Lebesgue dominated convergence theorem that for all $t \in [0,T]$, we conclude

$$\left\|(\Pi_\varepsilon x_n)(t)-(\Pi_\varepsilon x)(t)\right\|_\alpha \to 0, \quad \text{as } n\to\infty,$$

implying that $\|\Pi_\varepsilon x_n - \Pi_\varepsilon x\|_\alpha \to 0$ as $n\to\infty$. This proves that Π_ε is continuous on $B_{r(\varepsilon)}$.

Step 2. Π_ε is compact on $B_{r(\varepsilon)}$.

For the sake of brevity, we write

$$N\bigl(x(s)\bigr) := f\bigl(s,x(s)\bigr) + \int_0^s K(s,r)g\bigl(r,x(r)\bigr)\,dr + Bu_\varepsilon(s,x).$$

Let $t\in[0,T]$ be fixed and $\delta,\eta>0$ be small enough. For $x\in B_{r(\varepsilon)}$, we define the map

$$\begin{aligned}\bigl(\Pi_\varepsilon^{\delta\eta}x\bigr)(t) &= \int_0^\delta\int_\eta^\infty \beta r(t-s)^{\beta-1}\Psi_\beta(r)S\bigl((t-s)^\beta r\bigr)N\bigl(x(s)\bigr)\,dr\,ds\\ &= S\bigl(\delta^\beta\eta\bigr)\int_0^\delta\int_\eta^\infty \beta r(t-s)^{\beta-1}\Psi_\beta(r)S\bigl((t-s)^\beta r-\delta^\beta\eta\bigr)N\bigl(x(s)\bigr)\,dr\,ds.\end{aligned}$$

Therefore, from Lemma 4, we see that for each $t\in(0,T]$, the set $\{(\Pi_\varepsilon^{\delta\eta}x)(t): x\in B_{r(\varepsilon)}\}$ is relatively compact in X_α. Since

$$\begin{aligned}&\left\|(\Pi_\varepsilon x)(t)-\bigl(\Pi_\varepsilon^{\delta\eta}x\bigr)(t)\right\|_\alpha\\
&\quad\le \left\|\int_0^t\int_0^\eta \beta r(t-s)^{\beta-1}\Psi_\beta(r)S\bigl((t-s)^\beta r\bigr)N\bigl(x(s)\bigr)\,dr\,ds\right\|_\alpha\\
&\qquad+\left\|\int_{t-\delta}^t\int_\eta^\infty \beta r(t-s)^{\beta-1}\Psi_\beta(r)S\bigl((t-s)^\beta r\bigr)N\bigl(x(s)\bigr)\,dr\,ds\right\|_\alpha\\
&\quad\le \beta M_\alpha\biggl[\int_0^t(t-s)^{\beta(1-\alpha)-1}\biggl(\varphi_{r(\varepsilon)}(s)+K\|\psi_{r(\varepsilon)}\|_{L^\infty}+\frac{1}{\varepsilon}\frac{M}{\Gamma(\beta)}L_B^2L_u\biggr)\,ds\int_0^\eta\tau^{1-\alpha}\Psi_\beta(\tau)\,d\tau\\
&\qquad+\int_{t-\delta}^t(t-s)^{\beta(1-\alpha)-1}\biggl(\varphi_{r(\varepsilon)}(s)+K\|\psi_{r(\varepsilon)}\|_{L^\infty}+\frac{1}{\varepsilon}\frac{M}{\Gamma(\beta)}L_B^2L_u\biggr)\,ds\int_\eta^\infty\tau^{1-\alpha}\Psi_\beta(\tau)\,d\tau\biggr]\\
&\quad\le \beta M_\alpha\biggl[\biggl(\frac{1-\gamma}{(1-\alpha)\beta-\gamma}\biggr)^{1-\gamma}T^{(1-\alpha)\beta-\gamma}\|\varphi_{r(\varepsilon)}\|_{L^{1/\gamma}}+\frac{KT^{(1-\alpha)\beta}}{(1-\alpha)\beta}\|\psi_{r(\varepsilon)}\|_{L^\infty}\\
&\qquad+\frac{T^{(1-\alpha)\beta}}{(1-\alpha)\beta}\frac{1}{\varepsilon}\frac{M}{\Gamma(\beta)}L_B^2L_u\biggr]\int_0^\eta\tau^{1-\alpha}\Psi_\beta(\tau)\,d\tau\\
&\qquad+\frac{\beta M_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\biggl[\biggl(\frac{1-\gamma}{(1-\alpha)\beta-\gamma}\biggr)^{1-\gamma}\eta^{(1-\alpha)\beta-\gamma}\|\varphi_{r(\varepsilon)}\|_{L^{1/\gamma}}+\frac{K\eta^{(1-\alpha)\beta}}{(1-\alpha)\beta}\|\psi_{r(\varepsilon)}\|_{L^\infty}\\
&\qquad+\frac{\eta^{(1-\alpha)\beta}}{(1-\alpha)\beta}\frac{1}{\varepsilon}\frac{M}{\Gamma(\beta)}L_B^2L_u\biggr]\end{aligned}$$

approaches to zero as $\eta\to0^+$, using the total boundedness, we conclude that for each $t\in[0,T]$, the set $\{(\Pi_\varepsilon^{\delta\eta}x)(t): x\in B_{r(\varepsilon)}\}$ is relatively compact in X_α.

On the other hand, for $0<t_1<t_2\le T$ and $\delta>0$ small enough, we have

$$\left\|(\Pi_\varepsilon x)(t_1)-(\Pi_\varepsilon x)(t_2)\right\|_\alpha \le I_1+I_2+I_3+I_4,$$

where

$$
\begin{aligned}
I_1 &:= \int_{t_1}^{t_2} (t_2 - s)^{\beta-1-\alpha\beta} \| N(x(s)) \| \, ds, \\
I_2 &:= \int_0^{t_1-\delta} (t_1 - s)^{\beta-1} \| A^\alpha P_\beta(t_2 - s) - A^\alpha P_\beta(t_1 - s) \|_{L(X)} \| N(x(s)) \| \, ds, \\
I_3 &:= \int_{t_1-\delta}^{t_1} (t_1 - s)^{\beta-1} \big((t_2 - s)^{-\alpha\beta} + (t_1 - s)^{-\alpha\beta} \big) \| N(x(s)) \| \, ds, \\
I_4 &:= \int_0^{t_1} (t_2 - s)^{-\alpha\beta} \big| (t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \big| \| N(x(s)) \| \, ds.
\end{aligned}
$$

Therefore, it follows from (H_1) and Lemma 4 that

$$
\begin{aligned}
I_1 &\le C_\alpha \int_{t_1}^{t_2} (t_2 - s)^{\beta-1-\alpha\beta} \left(\big(\varphi_{r(\varepsilon)}(s) + K \|\psi_{r(\varepsilon)}\|_{L^\infty} \big) + \frac{1}{\varepsilon} L_B^2 \frac{M}{\Gamma(\beta)} L_u\big(r(\varepsilon)\big) \right) ds \\
&\le C_\alpha \left(\left(\frac{1-\gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} (t_2 - t_1)^{(1-\alpha)\beta-\gamma} \|\varphi_{r(\varepsilon)}\|_{L^{1/\gamma}} \right. \\
&\quad + \frac{C_\alpha K (t_2 - t_1)^{(1-\alpha)\beta}}{(1-\alpha)\beta} \|\psi_{r(\varepsilon)}\|_{L^\infty} \\
&\quad \left. + \frac{C_\alpha^{(1-\alpha)\beta}(t_2 - t_1)}{\varepsilon(1-\alpha)\beta} \frac{M}{\Gamma(\beta)} L_B^2 L_u\big(r(\varepsilon)\big) \right), \\
I_2 &\le \sup_{0 \le s \le t_1 - \delta} \| A^\alpha P_\beta(t_2 - s) - A^\alpha P_\beta(t_1 - s) \|_{L(X)} \\
&\quad \times \int_0^{t_1-\delta} (t_2 - s)^{\beta-1-\alpha\beta} \left(\big(\varphi_{r(\varepsilon)}(s) + K \|\psi_{r(\varepsilon)}\|_{L^\infty} \big) + \frac{1}{\varepsilon} \frac{M}{\Gamma(\beta)} L_B^2 L_u\big(r(\varepsilon)\big) \right) ds \\
&\le \sup_{0 \le s \le t_1 - \delta} \| A^\alpha P_\beta(t_2 - s) - A^\alpha P_\beta(t_1 - s) \|_{L(X)} \\
&\quad \times \left[\left(\frac{1-\gamma}{\beta - \gamma} \right)^{1-\gamma} \|\varphi_{r(\varepsilon)}\|_{L^{1/\gamma}} \big(t_1^{\frac{\beta-\gamma}{1-\gamma}} - \delta^{\frac{\beta-\gamma}{1-\gamma}} \big)^{1-\gamma} + \frac{K \|\psi_{r(\varepsilon)}\|_{L^\infty}}{\beta} \big(t_1^\beta - \delta^\beta \big) \right. \\
&\quad \left. + \frac{1}{\varepsilon\beta} \frac{M}{\Gamma(\beta)} L_B^2 L_u\big(r(\varepsilon)\big) \big(t_1^\beta - \delta^\beta \big) \right], \\
I_3 &\le 2C_\alpha \int_{t_1-\delta}^{t_1} (t_1 - s)^{\beta-1-\alpha\beta} \left(\big(\varphi_k(s) + K \|\psi_k\|_{L^\infty} \big) + \frac{1}{\varepsilon} \frac{M}{\Gamma(\beta)} L_B^2 L_u \right) ds \\
&\le 2C_\alpha \left(\left(\frac{1-\gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} \delta^{(1-\alpha)\beta-\gamma} \|\varphi_k\|_{L^{1/\gamma}} + \frac{C_\alpha K \|\psi_k\|_{L^\infty}}{(1-\alpha)\beta} \delta^{(1-\alpha)\beta} \right. \\
&\quad \left. + \frac{C_\alpha L_B^2 L_u}{\varepsilon(1-\alpha)\beta} \frac{M}{\Gamma(\beta)} \delta^{(1-\alpha)\beta} \right),
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &\le C_\alpha \int_0^{t_1} \big| (t_1 - s)^{\beta-1-\alpha\beta} - (t_2 - s)^{\beta-1-\alpha\beta} \big| \\
&\quad \times \left(\big(\varphi_{r(\varepsilon)}(s) + K \|\psi_{r(\varepsilon)}\|_{L^\infty} \big) + \frac{1}{\varepsilon} \frac{M}{\Gamma(\beta)} L_B^2 L_u\big(r(\varepsilon)\big) \right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_\alpha \left(\frac{1-\gamma}{(1-\alpha)\beta-\gamma}\right)^{1-\gamma} \|\varphi_{r(\varepsilon)}\|_{L^{1/\gamma}} \left[t_1^{(1-\alpha)\beta-\gamma} - \left(t_2^{\frac{(1-\alpha)\beta-\gamma}{1-\gamma}} - (t_2-t_1)^{\frac{(1-\alpha)\beta-\gamma}{1-\gamma}}\right)^{1-\gamma}\right] \\
&\quad + C_\alpha \frac{2K}{(1-\alpha)\beta} \|\psi_{r(\varepsilon)}\|_{L^\infty} \left[t_1^{(1-\alpha)\beta} - t_2^{(1-\alpha)\beta} - (t_2-t_1)^{(1-\alpha)\beta}\right] \\
&\quad + C_\alpha \frac{2L_B^2 L_u(r(\varepsilon))}{(1-\alpha)\beta} \frac{M}{\Gamma(\beta)} \|\psi_{r(\varepsilon)}\|_{L^\infty} \left[t_1^{(1-\alpha)\beta} - t_2^{(1-\alpha)\beta} - (t_2-t_1)^{(1-\alpha)\beta}\right],
\end{aligned}
$$

from which it is easy to see that all I_i, $i = 1,2,3,4$, tend to zero independent of $x \in B_k$ as $t_2 - t_1 \to 0$ and $\delta \to 0$. Thus, we can conclude that

$$\left\|(\Pi_\varepsilon x)(t_1) - (\Pi_\varepsilon x)(t_2)\right\|_\alpha \to 0 \quad \text{as } t_2 - t_1 \to 0,$$

and the limit is independent of $x \in B_{r(\varepsilon)}$. The case $t_1 = 0$ is trivial. Consequently, the set $\{(\Pi_\varepsilon x)(t) : t \in [0,T], x \in B_{r(\varepsilon)}\}$ is equicontinuous. Now applying the Arzela-Ascoli theorem, it results that Π_ε is compact on $B_{r(\varepsilon)}$.

Therefore, applying Krasnoselskii's fixed-point theorem, we conclude that Λ_ε has a fixed point, which gives rise to a mild solution of Cauchy problem (1) with control given in (5). This completes the proof. □

Theorem 11 *Let the assumptions* (H_1), (H_2) *and* (H_c) *be satisfied. Moreover, assume the functions* $f, g : [0,T] \times X_\alpha \times X_\alpha \to X$ *and* $h : C([0,T]; X_\alpha) \to X_\alpha$ *are bounded and* $ML_h < 1$. *Then the semilinear fractional system* (3) *is approximately controllable on* $[0,T]$.

Proof It is clear that all assumptions of Theorem 10 are satisfied with $\sigma_1 = \sigma_2 = 0$. Let x_ε be a fixed point of F_ε in B_r. Any fixed point of F_ε is a mild solution of (3) under the control

$$
\begin{aligned}
u_\varepsilon(t, x_\varepsilon) = B^* P_\beta^*(T-t) R\left(\varepsilon, \Gamma_0^T\right) \Bigg(& h - S_\beta(T)\left(x_0 + h(x_\varepsilon)\right) \\
& - \int_0^T (T-s)^{\beta-1} P_\beta(T-s) \left[f\left(s, x_\varepsilon(s)\right) + \int_0^s K(s-\tau) g\left(\tau, x_\varepsilon(\tau)\right) d\tau\right] ds \Bigg)
\end{aligned}
$$

and satisfies the equality

$$x_\varepsilon(T) = h - \varepsilon R\left(\varepsilon, \Gamma_0^T\right) p(x_\varepsilon), \tag{9}$$

where

$$
\begin{aligned}
p(x_\varepsilon) = \Bigg(& h - S_\beta(T)\left(x_0 + h(x_\varepsilon)\right) \\
& - \int_0^T (T-s)^{\beta-1} P_\beta(T-s) \left[f\left(s, x_\varepsilon(s)\right) + \int_0^s K(s-\tau) g\left(\tau, x_\varepsilon(\tau)\right) d\tau\right] ds \Bigg).
\end{aligned}
$$

Moreover, by the boundedness of the functions f and g and Dunford-Pettis theorem, we have that the sequences $\{f(s, x_\varepsilon(s))\}$ and $\{g(s, x_\varepsilon(s))\}$ are weakly compact in $L^2([0,T]; X)$, so there are subsequences still denoted by $\{f(s, x_\varepsilon(s))\}$ and $\{g(s, x_\varepsilon(s))\}$, that weakly converge to, say, f and g in $L^2([0,T]; X)$. On the other hand, there exists $\widetilde{h} \in X_\alpha$ such that $h(x_\varepsilon)$ converges to $\widetilde{h}$ weakly in X_α. Denote

$$w = h - S_\beta\left(x_0 + \widetilde{h}\right) - \int_0^T (T-s)^{\beta-1} P_\beta(T-s) \left[f(s) + \int_0^s K(s-\tau) g(\tau)\, d\tau\right] ds.$$

It follows that

$$\begin{aligned}\|p(x_\varepsilon)-w\|_\alpha &\le \|S_\beta(T)h(x_\varepsilon)-S_\beta(T)\widetilde{h}\|_\alpha\\ &\quad+\left\|\int_0^T(T-s)^{\beta-1}P_\beta(T-s)\big(f\big(s,x_\varepsilon(s)\big)-f(s)\big)\,ds\right\|_\alpha\\ &\quad+\left\|\int_0^T(T-s)^{\beta-1}P_\beta(T-s)\int_0^s K(s-\tau)\big(g\big(\tau,x_\varepsilon(\tau)-g(\tau)\big)\,d\tau\,ds\right\|_\alpha\to 0\end{aligned}$$

as $\varepsilon\to 0^+$ because of compactness of the operator

$$l(\cdot)\to\int_0^{\cdot}(\cdot-s)^{\beta-1}P_\beta(\cdot-s)l(s)\,ds: L_2\big([0,T],X\big)\to C\big([0,T],X_\alpha\big).$$

Then from (9), we obtain

$$\begin{aligned}\|x_\varepsilon(T)-h\|_\alpha &\le \|\varepsilon R(\varepsilon,\Gamma_0^T)(w)\|_\alpha+\|\varepsilon R(\varepsilon,\Gamma_0^T)\|\,\|p(x_\varepsilon)-w\|_\alpha\\ &\le \|\varepsilon R(\varepsilon,\Gamma_0^T)(w)\|_\alpha+\|p(x_\varepsilon)-w\|_\alpha\to 0\end{aligned}\tag{10}$$

as $\varepsilon\to 0^+$. This proves the approximate controllability of (1). □

4 Applications

Example 1 As an application to Theorem 11, we study the following simple example. Consider a control system governed by the fractional partial differential equation of the form

$$\begin{cases}{}^c\partial_t^{\frac{3}{4}}x(t,z)=\partial_z^2x(t,z)+u(t,z)+F(t,z,x(t,z))\\ \quad+\int_0^t K(t,s)G(s,z,x(s,z))\,ds,\quad t\in[0,T],z\in[0,\pi],\\ x(t,0)=x(t,\pi)=0,\\ x(0,z)=x_0(z)+\sum_{k=1}^p\int_0^\pi k(z,r)\cos(x(t_k,r))\,dr,\end{cases}\tag{11}$$

where $f,g:[0,T]\times[0,\pi]\times R\to R$, $k:[0,\pi]\times[0,\pi]\to R$, $0<t_1<\cdots<t_p<T$.

Let us take $X=U=L^2[0,\pi]$ and define the operator A by $Aw=-w''$ with the domain $D(A)=\{w(\cdot)\in L^2[0,\pi], w, w'$ are absolutely continuous, $w''\in L^2[0,\pi], w(0)=w(\pi)=0\}$. Then

$$Aw=\sum_{n=1}^\infty n^2\langle w,e_n\rangle e_n,\quad w\in D(A),$$

where $e_n(z)=\sqrt{\dfrac{2}{\pi}}\sin nz$, $0\le z\le\pi$, $n=1,2,\ldots$. Clearly $-A$ generates a compact analytic semigroup $S(t)$, $t>0$ in X and it is given by

$$S(t)w=\sum_{n=1}^\infty e^{-n^2t}\langle w,e_n\rangle e_n,\quad w\in X.$$

Clearly, the assumption (H_1) is satisfied. On the other hand, it can be easily seen that the deterministic linear system corresponding to (11) is approximately controllable on $[0,T]$; see [12].

The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}w = \sum_{n=1}^{\infty} n\langle w, e_n\rangle e_n, \quad w \in D\bigl(A^{\frac{1}{2}}\bigr),$$

where $D(A^{\frac{1}{2}}) = \{w \in X : \sum_{n=1}^{\infty} n\langle w, e_n\rangle e_n \in X\}$ and $\|A^{-\frac{1}{2}}\| = 1$.

Let $X_{\frac{1}{2}} := (D(A^{\frac{1}{2}}), \|\cdot\|_{1/2})$, where $\|x\|_{1/2} := \|A^{\frac{1}{2}}x\|_X$ for $x \in D(A^{\frac{1}{2}})$. Assume that $F, G : [0,T] \times [0,\pi] \times R \to R$ satisfies the following conditions:

1. The functions $F(\cdot,\cdot,\cdot)$, $G(\cdot,\cdot,\cdot)$ are continuous and uniformly bounded.
2. $F(0,\cdot,\cdot) = F(\pi,\cdot,\cdot) = G(0,\cdot,\cdot) = G(\pi,\cdot,\cdot) = 0$.
3. $k : [0,\pi] \times [0,\pi] \to R$ is continuously differentiable, $k(0,\cdot) = k(\pi,\cdot) = 0$ and

$$\int_0^\pi \int_0^\pi \left| \frac{\partial^2}{\partial \xi^2} k(\xi, y) \right|^2 dy\, d\xi < \infty.$$

Denote by $E_{\beta,\zeta}$, the Mittag-Leffler special function defined by

$$E_{\beta,\zeta} = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\zeta k + \beta)}, \quad \zeta, \beta > 0, t \in R.$$

Therefore,

$$S_\beta(t)x = \sum_{k=0}^{\infty} E_{\beta,1}\bigl(-n^2 t^\beta\bigr)\langle x, e_n\rangle e_n, \qquad \bigl\|S_\beta(t)\bigr\|_{L(X)} \le 1,$$

$$P_\beta(t)x = \sum_{k=0}^{\infty} E_{\beta,\beta}\bigl(-n^2 t^\beta\bigr)\langle x, e_n\rangle e_n, \qquad \bigl\|P_\beta(t)\bigr\|_{L(X)} \le \frac{1}{\Gamma(\beta)}, \quad x \in X, t \ge 0.$$

Define

$$f\bigl(t, x(t)\bigr)(z) = F\bigl(t, z, x(t,z)\bigr),$$

$$g\bigl(t, x(t)\bigr)(z) = G\bigl(s, z, x(s,z)\bigr),$$

$$h(x)(z) = \sum_{k=1}^{p} \int_0^\pi k(z,y) \cos\bigl(x(t_k, y)\bigr)\, dy.$$

Then, for each $x, y \in C([0,T], X_{1/2})$ we have

$$\begin{aligned}
\bigl\|h(x)\bigr\|_{1/2}^2 &= \bigl\|A^{1/2}h(x)(\cdot)\bigr\|_{L^2[0,\pi]}^2 = \sum_{n=1}^{\infty} n^2 \|e_n\|_{L^2[0,\pi]}^2 \bigl|\bigl\langle h(x)(\cdot), e_n\bigr\rangle\bigr|^2 \\
&= \frac{2}{\pi} \sum_{n=1}^{\infty} n^2 \left| \int_0^\pi h(x)(\xi)\sin(n\xi)\, d\xi \right|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \left| \int_0^\pi \frac{\partial^2}{\partial \xi^2} h(x)(\xi) e_n(\xi)\, d\xi \right|^2 \\
&\le \frac{\pi^2}{6} \left\| \frac{\partial^2}{\partial \xi^2} h(x)(\xi) \right\|_{L^2[0,\pi]}^2 = \frac{\pi^2}{6} \left\| \frac{\partial^2}{\partial \xi^2} \sum_{k=0}^{p} \int_0^\pi k(\xi, y)\cos\bigl(x(t_k, y)\bigr)\, dy \right\|_{L^2[0,\pi]}^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi^2}{6}\int_0^\pi \left|\sum_{k=1}^p \int_0^\pi \frac{\partial^2}{\partial\xi^2}k(\xi,y)\cos\bigl(x(t_k,y)\bigr)\,dy\right|^2 d\xi \\
&\leq \frac{p\pi^3}{6}\int_0^\pi\int_0^\pi \left|\frac{\partial^2}{\partial\xi^2}k(\xi,y)\right|^2 dy\,d\xi = \frac{p\pi^3}{6}\left\|\frac{\partial^2}{\partial\xi^2}k(\xi,y)\right\|^2_{L^2[0,\pi]\times[0,\pi]}
\end{aligned}
$$

and

$$
\begin{aligned}
\bigl\|h(x)-h(y)\bigr\|^2_{1/2} &= \bigl\|A^{1/2}h(x)(\cdot)-A^{1/2}h(y)(\cdot)\bigr\|^2_{L^2[0,\pi]} \\
&\leq \frac{\pi^2}{6}\left\|\frac{\partial^2}{\partial\xi^2}\sum_{k=0}^p\int_0^\pi k(\xi,r)\bigl[\cos\bigl(x(t_k,r)\bigr)-\cos\bigl(y(t_k,r)\bigr)\bigr]\,dr\right\|^2_{L^2[0,\pi]} \\
&= \frac{\pi^2}{6}\int_0^\pi\left|\sum_{k=0}^p\int_0^\pi \frac{\partial^2}{\partial\xi^2}k(\xi,r)\bigl[\cos\bigl(x(t_k,r)\bigr)-\cos\bigl(y(t_k,r)\bigr)\bigr]\,dr\right|^2 d\xi \\
&\leq \frac{p\pi^2}{6}\int_0^\pi\int_0^\pi\left|\frac{\partial^2}{\partial\xi^2}k(\xi,r)\right|^2 dr\,d\xi \sup_{0\leq t\leq\pi}\int_0^\pi \bigl|x(t,r)-y(t,r)\bigr|^2\,dr.
\end{aligned}
$$

It follows that $h: C([0,T];X_{1/2}) \to X_{1/2}$ is bounded and Lipschitz continuous. On the other hand, it is not difficult to verify that $f, g: [0,T]\times X_{1/2}\to X$ are continuous.

Next, we show that the linear system corresponding to (11) is approximately controllable on $[0,T]$. It is clear that $P_\beta(t): X_{\frac{1}{2}} \to X_{\frac{1}{2}}$ is defined as follows:

$$
\begin{aligned}
&P_\beta(t) = \beta\int_0^\infty \theta\Psi_\beta(\theta)S\bigl(t^\beta\theta\bigr)\,d\theta, \\
&B^*P^*_\beta(T-t)x \\
&\quad = \beta\sum_{n=1}^\infty n\int_0^\infty \theta\Psi_\beta(\theta)E_{\beta,\beta}\bigl(-n^2(T-t)^\beta\theta\bigr)\,d\theta\,\langle x,e_n\rangle e_n, \quad x\in X_{\frac{1}{2}}, 0\leq t<T.
\end{aligned}
$$

By Remark 9, the linear system corresponding to (11) is approximately controllable on $[0,T]$ if and only if $B^*P^*_\beta(T-t)x = 0$, $0\leq t<T$ implies that $x=0$. This follows from the representation of $B^*P^*_\beta(T-t)x$.

Now, we note that the problem (11) can be reformulated as the abstract problem. Thus, by Theorem 11, the system (11) is approximately controllable on $[0,T]$, provided that

$$
ML_h = \frac{p\pi^2}{6}\int_0^\pi\int_0^\pi\left|\frac{\partial^2}{\partial\xi^2}k(\xi,r)\right|^2 dr\,d\xi < 1.
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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