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Positive solutions for a fourth-order p-Laplacian boundary value problem with impulsive effects

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Abstract

This paper is devoted to study the existence and multiplicity of positive solutions for the fourth-order p-Laplacian boundary value problem involving impulsive effects

$$\begin{cases} (|y''|^{p-1}y'')'' = f(t,y), & t \in J, t \neq t_k, \\ \Delta y'|_{t=t_k} = -I_k(y(t_k)), & k = 1,2,\ldots,m, \\ y(0) = y(1) = y''(0) = y''(1) = 0, \end{cases}$$

where $J = [0,1]$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ := [0,\infty)$). Based on *a priori* estimates achieved by utilizing the properties of concave functions and Jensen's inequality, we adopt fixed point index theory to establish our main results.

MSC: 34B18; 47H07; 47H11; 45M20; 26D15

Keywords: p-Laplacian boundary value problem with impulsive effects; positive solution; fixed point index; concave function; Jensen inequality

1 Introduction

In this paper, we mainly investigate the existence and multiplicity of positive solutions for the fourth-order p-Laplacian boundary value problem with impulsive effects

$$\begin{cases} (|y''|^{p-1}y'')'' = f(t,y), & t \in J, t \neq t_k, \\ \Delta y'|_{t=t_k} = -I_k(y(t_k)), & k = 1,2,\ldots,m, \\ y(0) = y(1) = y''(0) = y''(1) = 0. \end{cases} \quad (1.1)$$

Here $J = [0,1]$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$. Let $0 < t_1 < \cdots < t_m < 1$ be fixed, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-)$, where $y'(t_k^+)$ and $y'(t_k^-)$ denote the right and left limit of $y'(t)$ at $t = t_k$, respectively.

Fourth-order boundary value problems, including those with the p-Laplacian operator, have their origin in beam theory [1, 2], ice formation [3, 4], fluids on lungs [5], brain warping [6, 7], designing special curves on surfaces [6, 8], *etc.* In beam theory, more specifically, a beam with a small deformation, a beam of a material which satisfies a nonlinear power-like stress and strain law, and a beam with two-sided links which satisfies a nonlinear power-like elasticity law can be described by fourth-order differential equations along

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with their boundary value conditions. For the case of $I_k = 0$, $k = 1, 2, \ldots, m$, and $p = 1$, problem (1.1) reduces to the differential equation $y^{(4)}(t) = f(t, y(t))$ subject to boundary value conditions $y(0) = y(1) = y''(0) = y''(1) = 0$, which can be used to model the deflection of elastic beams simply supported at the endpoints [9–11]. This explains the reason that the last two decades have witnessed an overgrowing interest in the research of such problems, with many papers in this direction published. We refer the interested reader to [12–26] and references therein devoted to the existence of solutions for the equations with p-Laplacian operator.

In [17], Zhang *et al.* studied the existence and nonexistence of symmetric positive solutions of the following fourth-order boundary value problem with integral boundary conditions:

$$\begin{cases} (\phi_p(u''(t)))'' = w(t)f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 g(s)u(s)\,\mathrm{d}s, \\ \phi_p(u''(0)) = \phi_p(u''(1)) = \int_0^1 h(s)\phi_p(u''(s))\,\mathrm{d}s, \end{cases} \tag{1.2}$$

where $w \in L^1[0,1]$ is nonnegative, symmetric on the interval $[0,1]$ (*i.e.*, $w(1-t) = w(t)$ for $t \in [0,1]$), $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $f(1-t, u) = f(t, u)$ for all $(t, x) \in [0,1] \times \mathbb{R}^+$, and $g, h \in L^1[0,1]$ are nonnegative, symmetric on $[0,1]$. The arguments are based upon a specially constructed cone and the fixed point theory for cones. Moreover, they also studied the nonexistence of a positive solution.

In [16], Luo and Luo considered the existence, multiplicity, and nonexistence of symmetric positive solutions for (1.2) with a ϕ-Laplacian operator and the term f involving the first derivative.

Except that, many researchers considered and studied the existence of positive solutions for a lot of impulsive boundary value problems; see, for example, [21–29] and the references therein.

In [21], Feng considered the problem (1.2) with impulsive effects and he obtained the existence and multiplicity of positive solutions. The fundamental tool in this paper is Guo-Krasnosel'skii fixed point theorem on a cone. Moreover, the nonlinearity f can be allowed to grow both sublinear and superlinear. Therefore, he improved and generalized the results of [17] to some degree. However, we can easily find that these papers do only simple promotion based on their original papers, and no substantial changes.

Motivated by the works mentioned above, in this paper, we study the existence and multiplicity of positive solutions for (1.1). Nevertheless, our methodology and results in this paper are different from those in the papers cited above. The main features of this paper are as follows. Firstly, we convert the boundary value problem (1.1) into an equivalent integral equation. Next, we consider impulsive effect as a perturbation to the corresponding problem without the impulsive terms, so that we can construct an integral operator for an appropriate linear Dirichlet boundary value problem and obtain its first eigenvalue and eigenfunction. Our main results are formulated in terms of spectral radii of the linear integral operator, and our *a priori* estimates for positive solutions are derived by developing some properties of positive concave functions and using Jensen's inequality. It is of interest to note that our nonlinearity f may grow superlinearly and sublinearly. The main tool used in the proofs is fixed point index theory, combined with the *a priori* estimates of positive solutions. Although our problem (1.1) merely involves Dirichlet boundary conditions, both our methodology and the results in this work improve and extend the corresponding ones from [21–29].

2 Preliminaries

Let $E := C[0,1]$, $\|u\| := \sup_{t\in[0,1]} |u(t)|$. Then $(E, \|\cdot\|)$ is a real Banach space. Let $J' := J \setminus \{t_1, t_2, \ldots, t_m\}$ and introduce the following space:

$$PC'[0,1] := \{y \in C[0,1], y'|_{(t_k,t_{k+1})} \in C(t_k, t_{k+1}), y'(t_k^-) = y'(t_k), \exists y'(t_k^+), k = 1,2,\ldots,m\}$$

with the norm $\|y\|_{PC'} = \max\{\|y\|, \|y'\|\}$. Then $(PC'[0,1], \|\cdot\|_{PC'})$ is also a Banach space.

A function $y \in PC'[0,1] \cap C^4(J')$ is called a solution of (1.1) if it satisfies the differential equation

$$\left(|y''|^{p-1}y''\right)'' = f(t,y), \quad t \in J',$$

and the function y satisfies the conditions $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-) = -I_k(y(t_k))$, and the Dirichlet boundary conditions $y(0) = y(1) = y''(0) = y''(1) = 0$.

Lemma 2.1 (see [21]) *If y is a solution of the integral equation*

$$y(t) = \int_0^1 G(t,s)\left(\int_0^1 G(s,\tau)f(\tau,y(\tau))\,d\tau\right)^{\frac{1}{p}} ds + \sum_{k=1}^m G(t,t_k)I_k(y(t_k)) := (Ay)(t), \qquad (2.1)$$

then y is a solution of (1.1), *where* $G(t,s) = \min\{t,s\}\min\{1-s,1-t\}$, $\forall t,s \in [0,1]$. *Note that if* $f \in C([0,1]\times\mathbb{R}^+, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, *then* $A : C[0,1] \to C[0,1]$ *is a completely continuous operator, and the existence of positive solutions for* (1.1) *is equivalent to that of positive fixed points of A.*

Remark 2.1 By (2.1), we easily find y is concave on $[0,1]$. Indeed,

$$y''(t) = -\left(\int_0^1 G(t,s)f(s,y(s))\,ds\right)^{\frac{1}{p}} \le 0$$

implies y is concave on $[0,1]$. Furthermore, $y(t_k) = 0$ $(k = 1,2,\ldots,m)$ leads to $y(t) \equiv 0$, $\forall t \in [0,1]$.

Let P be a cone in $C[0,1]$ which is defined as

$$P := \{y \in C[0,1] : y(t) \ge t(1-t)\|y\|, t \in J\}.$$

In what follows, we prove that $A(P) \subset P$.

Lemma 2.2 $A(P) \subset P$.

Proof We easily see that $t(1-t)G(s,s) \le G(t,s) \le G(s,s)$, $\forall t,s \in [0,1]$. Consequently, on the one hand, we find

$$(Ay)(t) \le \int_0^1 G(s,s)\left(\int_0^1 G(s,\tau)f(\tau,y(\tau))\,d\tau\right)^{\frac{1}{p}} ds + \sum_{k=1}^m G(t_k,t_k)I_k(y(t_k)).$$

On the other hand,

$$(Ay)(t) \geq t(1-t)\left[\int_0^1 G(s,s)\left(\int_0^1 G(s,\tau)f\bigl(\tau,y(\tau)\bigr)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^{m} G(t_k,t_k)I_k\bigl(y(t_k)\bigr)\right].$$

Therefore, $(Ay)(t) \geq t(1-t)\|Ay\|$, for any $t \in [0,1]$, as required. This completes the proof. □

We denote $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

Lemma 2.3 (see [30]) *Suppose $A : P \to P$ is a completely continuous operator and has no fixed points on $\partial B_\rho \cap P$.*

1. *If $\|Ay\| \leq \|y\|$ for all $y \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 1$, where i is fixed point index on P.*
2. *If $\|Ay\| \geq \|y\|$ for all $y \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 0$.*

Lemma 2.4 (see [30]) *If $A : \overline{B}_\rho \cap P \to P$ is a completely continuous operator. If there exists $y_0 \in P \setminus \{0\}$ such that $y - Ay \neq \lambda y_0$, $\forall \lambda \geq 0$, $y \in \partial B_\rho \cap P$, then $i(A, B_\rho \cap P, P) = 0$.*

Lemma 2.5 (see [30]) *If $0 \in B_\rho$ and $A : \overline{B}_\rho \cap P \to P$ is a completely continuous operator. If $y \neq \lambda Ay$, $\forall y \in \partial B_\rho \cap P$, $0 \leq \lambda \leq 1$, then $i(A, B_\rho \cap P, P) = 1$.*

Lemma 2.6 *Let $\psi(t) := \sin(\pi t)$. Then*

$$\int_0^1 G(t,s)\psi(t)\,\mathrm{d}t = \frac{1}{\pi^2}\psi(s), \qquad \int_0^1 G(t,s)\psi(s)\,\mathrm{d}s = \frac{1}{\pi^2}\psi(t). \tag{2.2}$$

Lemma 2.7 (Jensen's inequalities) *Let $\theta > 0$, $n \geq 1$, $a_i \geq 0$ $(i = 1,2,\ldots,n)$, and $\varphi \in C([0,1],\mathbb{R}^+)$. Then*

$$\left(\int_0^1 \varphi(t)\,\mathrm{d}t\right)^{\theta} \leq \int_0^1 \bigl(\varphi(t)\bigr)^{\theta}\,\mathrm{d}t \quad \text{and} \quad \left(\sum_{i=1}^{n} a_i\right)^{\theta} \leq 2^{(n-1)(\theta-1)}\sum_{i=1}^{n} a_i^{\theta}, \quad \forall \theta \geq 1,$$

$$\left(\int_0^1 \varphi(t)\,\mathrm{d}t\right)^{\theta} \geq \int_0^1 \bigl(\varphi(t)\bigr)^{\theta}\,\mathrm{d}t \quad \text{and} \quad \left(\sum_{i=1}^{n} a_i\right)^{\theta} \geq 2^{(n-1)(\theta-1)}\sum_{i=1}^{n} a_i^{\theta}, \quad \forall 0 < \theta \leq 1.$$

3 Main results

Let $p^* := \max\{1,p\}$, $p_* := \min\{1,p\}$, $\kappa_1 := 2^{p_*-1}$, $\kappa_2 := 2^{m(p_*-1)}$, $\kappa_3 := 2^{p^*-1}$, $\kappa_4 := 2^{m(p^*-1)}$, $\kappa_5 := 2^{\frac{p^*}{p}+p^*-2}$, $\kappa_6 := 2^{(m+1)(p^*-1)}$. We now list our hypotheses.

(H1) There is a $\rho > 0$ such that $0 \leq y < \rho$ and $0 \leq t \leq 1$ imply

$$f(t,y) \leq \eta^p\rho^p, \qquad I_k(y) \leq \eta_k\rho,$$

where $\eta, \eta_k \geq 0$ satisfy

$$\eta + \sum_{k=1}^{m}\eta_k > 0, \qquad \eta\int_0^1 G(s,s)\left(\int_0^1 G(s,\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^{m} G(t_k,t_k)\eta_k < 1.$$

(H2) There exist $0 < r_0 < \rho$ and $a_1 \geq 0$, $a_2 \geq 0$ satisfying

$$a_1^{\frac{p_*}{p}} \kappa_1 + \frac{\pi^3}{2} \sigma^{p_*} a_2^{p_*} \kappa_2 \sum_{k=1}^{m} \sin(\pi t_k) > \pi^4$$

such that

$$f(t,y) \geq a_1 y^p, \qquad I_k(y) \geq a_2 y, \quad \forall t \in [0,1], 0 < y < r_0, \tag{3.1}$$

where $\sigma := \min_{t\in[t_1,t_m]} t(1-t) > 0$.

(H3) There exist $c > 0$ and $a_3 \geq 0$, $a_4 \geq 0$ satisfying

$$a_3^{\frac{p_*}{p}} \kappa_1 + \frac{\pi^3}{2} \sigma^{p_*} a_4^{p_*} \kappa_2 \sum_{k=1}^{m} \sin(\pi t_k) > \pi^4$$

such that

$$f(t,y) \geq a_3 y^p - c, \qquad I_k(y) \geq a_4 y - c, \quad \forall t \in [0,1], y \geq 0. \tag{3.2}$$

(H4) There is a $\rho > 0$ such that $\sigma\rho \leq y \leq \rho$ and $0 \leq t \leq 1$ imply

$$f(t,y) \geq \xi^p \rho^p, \qquad I_k(y) \geq \xi_k \rho,$$

where $\xi, \xi_k \geq 0$ satisfy

$$\xi + \sum_{k=1}^{m} \xi_k > 0, \qquad \xi \int_{t_1}^{t_m} G\left(\frac{1}{2}, s\right) \left(\int_0^1 G(s,\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}} \mathrm{d}s + \sum_{k=1}^{m} G\left(\frac{1}{2}, t_k\right) \xi_k > 1.$$

(H5) There exist $0 < r_0 < \rho$ and $b_1 \geq 0$, $b_2 \geq 0$ satisfying

$$b_1^2 + b_2^2 \neq 0, \qquad b_1^{\frac{p^*}{p}} \kappa_3 + \frac{\pi^2 b_2^{p^*} \kappa_4 \sum_{k=1}^{m} \sin(\pi t_k)}{\int_0^1 (t(1-t))^{p^*} \sin(\pi t)\,\mathrm{d}t} < \pi^4$$

such that

$$f(t,y) \leq b_1 y^p, \qquad I_k(y) \leq b_2 y, \quad \forall t \in [0,1], 0 < y < r_0. \tag{3.3}$$

(H6) There exist $c > 0$ and $b_3 \geq 0$, $b_4 \geq 0$ satisfying

$$b_3^2 + b_4^2 \neq 0, \qquad b_3^{\frac{p^*}{p}} \kappa_5 + \frac{\pi^2 b_4^{p^*} \kappa_6 \sum_{k=1}^{m} \sin(\pi t_k)}{\int_0^1 (t(1-t))^{p^*} \sin(\pi t)\,\mathrm{d}t} < \pi^4$$

such that

$$f(t,y) \leq b_3 y^p + c, \qquad I_k(y) \leq b_4 y + c, \quad \forall t \in [0,1], y \geq 0. \tag{3.4}$$

Theorem 3.1 *Suppose that* (H1)-(H3) *are satisfied. Then* (1.1) *has at least two positive solutions.*

Proof If $y \in \partial B_\rho \cap P$, it follows from (H1) that

$$\begin{aligned}\|Ay\| &\le \int_0^1 G(s,s)\left(\int_0^1 G(s,\tau)f\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^m G(t_k,t_k)I_k\big(y(t_k)\big)\\ &\le \rho\left(\eta\int_0^1 G(s,s)\left(\int_0^1 G(s,\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^m G(t_k,t_k)\eta_k\right) < \rho = \|y\|.\end{aligned}$$

Now Lemma 2.3 yields

$$i(A, B_\rho \cap P, P) = 1. \tag{3.5}$$

Let $r \in (0, r_0)$. Then for $y \in \partial B_r \cap P$, we find

$$y(t) \ge t(1-t)\|y\| \ge \sigma r, \quad \forall t \in [t_1, t_m], \tag{3.6}$$

where $\sigma = \min_{t\in[t_1,t_m]} t(1-t) > 0$. Let $\mathcal{M}_1 := \{y \in P : y = Ay + \lambda\psi \text{ for some } \lambda \ge 0\}$, where $\psi(t) = \sin(\pi t)$. Next, from (H2), we prove $\mathcal{M}_1 \subset \{0\}$. Indeed, $y \in \mathcal{M}_1$ implies $y(t) \ge (Ay)(t)$. Lemma 2.6, together with this, leads to

$$\begin{aligned}y^{p_*}(t) &\ge \left[\int_0^1 G(t,s)\left(\int_0^1 G(s,\tau)f\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^m G(t,t_k)I_k\big(y(t_k)\big)\right]^{p_*}\\ &\ge \kappa_1\left[\int_0^1 G(t,s)\left(\int_0^1 G(s,\tau)f\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s\right]^{p_*} + \kappa_1\left[\sum_{k=1}^m G(t,t_k)I_k\big(y(t_k)\big)\right]^{p_*}\\ &\ge \kappa_1\int_0^1\int_0^1 G(t,s)G(s,\tau)f^{\frac{p_*}{p}}\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s + \kappa_2\sum_{k=1}^m G(t,t_k)I_k^{p_*}\big(y(t_k)\big).\end{aligned} \tag{3.7}$$

Multiply both sides of the above by $\sin(\pi t)$ and integrate over $[0,1]$ and use (2.2) to obtain

$$\begin{aligned}\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t &\ge \kappa_1\int_0^1 \sin(\pi t)\int_0^1\int_0^1 G(t,s)G(s,\tau)f^{\frac{p_*}{p}}\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t\\ &\quad + \kappa_2\sum_{k=1}^m\int_0^1 \sin(\pi t)G(t,t_k)I_k^{p_*}\big(y(t_k)\big)\,\mathrm{d}t\\ &\ge \frac{\kappa_1}{\pi^4}\int_0^1 f^{\frac{p_*}{p}}\big(t,y(t)\big)\sin(\pi t)\,\mathrm{d}t + \frac{\kappa_2}{\pi^2}\sum_{k=1}^m I_k^{p_*}\big(y(t_k)\big)\sin(\pi t_k).\end{aligned} \tag{3.8}$$

Combining this and (3.1), we get

$$\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t \ge \frac{a_1^{\frac{p_*}{p}}\kappa_1}{\pi^4}\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t + \frac{a_2^{p_*}\kappa_2}{\pi^2}\sum_{k=1}^m y^{p_*}(t_k)\sin(\pi t_k). \tag{3.9}$$

In what follows, we will distinguish three cases.

Case 1. $a_1^{\frac{p_*}{p}}\kappa_1 = \pi^4$. By (H2), we know $a_2 > 0$. (3.9) implies

$$\frac{a_2^{p_*}\kappa_2}{\pi^2}\sum_{k=1}^{m} y^{p_*}(t_k)\sin(\pi t_k) \le 0.$$

Therefore, $y(t_k) = 0$ ($k = 1,2,\dots,m$), and then $y(t) \equiv 0$, $\forall t \in [0,1]$ by Remark 2.1, which contradicts $y \in \partial B_r \cap P$.

Case 2. $a_1^{\frac{p_*}{p}}\kappa_1 > \pi^4$. Equation (3.9) implies

$$\left(\frac{a_1^{\frac{p_*}{p}}\kappa_1}{\pi^4} - 1\right)\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t + \frac{a_2^{p_*}\kappa_2}{\pi^2}\sum_{k=1}^{m} y^{p_*}(t_k)\sin(\pi t_k) \le 0,$$

and thus $y(t) \equiv 0$, $\forall t \in [0,1]$, which also contradicts $y \in \partial B_r \cap P$.

Case 3. $a_1^{\frac{p_*}{p}}\kappa_1 < \pi^4$. Since $\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t \le \frac{2r^{p_*}}{\pi}$, we have by (3.6) and (3.9),

$$\frac{2[\pi^4 - a_1^{\frac{p_*}{p}}\kappa_1]r^{p_*}}{\pi} \ge \left[\pi^4 - a_1^{\frac{p_*}{p}}\kappa_1\right]\int_0^1 y^{p_*}(t)\sin(\pi t) \ge \pi^2\sigma^{p_*}r^{p_*}a_2^{p_*}\kappa_2\sum_{k=1}^{m}\sin(\pi t_k).$$

Therefore,

$$a_1^{\frac{p_*}{p}}\kappa_1 + \frac{\pi^3}{2}\sigma^{p_*}a_2^{p_*}\kappa_2\sum_{k=1}^{m}\sin(\pi t_k) \le \pi^4,$$

which contradicts (H2). So, we have $y - Ay \ne \lambda\psi$ for all $y \in \partial B_r \cap P$ and $\lambda \ge 0$. Now, by virtue of Lemma 2.4, we obtain

$$i(A, B_r \cap P, P) = 0. \tag{3.10}$$

On the other hand, by (H3), we prove $\mathcal{M}_1$ is bounded in P. By (3.2) together with (3.8), we obtain

$$\begin{aligned}
&\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t \\
&\quad\ge \frac{\kappa_1}{\pi^4}\int_0^1 \left[a_3 y^p(t) - c\right]^{\frac{p_*}{p}}\sin(\pi t)\,\mathrm{d}t + \frac{\kappa_2}{\pi^2}\sum_{k=1}^{m}\left[a_4 y(t_k) - c\right]^{p_*}\sin(\pi t_k) \\
&\quad\ge \frac{a_3^{\frac{p_*}{p}}\kappa_1}{\pi^4}\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t + \frac{a_4^{p_*}\kappa_2}{\pi^2}\sum_{k=1}^{m} y^{p_*}(t_k)\sin(\pi t_k) - c_1,
\end{aligned} \tag{3.11}$$

where $c_1 := \frac{2\kappa_1 c^{\frac{p_*}{p}}}{\pi^5} + \frac{\kappa_2 c^{p_*}}{\pi^2}\sum_{k=1}^{m}\sin(\pi t_k)$. Now we distinguish the following two cases.

Case 1. $a_3^{\frac{p_*}{p}}\kappa_1 \ge \pi^4$. (H3) implies

$$\left(a_3^{\frac{p_*}{p}}\kappa_1 - \pi^4\right)\int_0^1 t^{p_*}(1-t)^{p_*}\sin(\pi t)\,\mathrm{d}t + \pi^2\sigma^{p_*}a_4^{p_*}\kappa_2\sum_{k=1}^{m}\sin(\pi t_k) > 0.$$

Combining this and (3.11), we have

$$\bigl(a_3^{\frac{p_*}{p}}\kappa_1-\pi^4\bigr)\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t+\pi^2 a_4^{p_*}\kappa_2\sum_{k=1}^m y^{p_*}(t_k)\sin(\pi t_k)\le\pi^4 c_1.$$

Therefore,

$$\|y\|^{p_*}\le\frac{\pi^4 c_1}{(a_3^{\frac{p_*}{p}}\kappa_1-\pi^4)\int_0^1 t^{p_*}(1-t)^{p_*}\sin(\pi t)\,\mathrm{d}t+\pi^2\sigma^{p_*}a_4^{p_*}\kappa_2\sum_{k=1}^m\sin(\pi t_k)}:=\mathcal{N}_1.$$

Case 2. $a_3^{\frac{p_*}{p}}\kappa_1<\pi^4$. (3.11) implies

$$\begin{aligned}
&\frac{2[\pi^4-a_3^{\frac{p_*}{p}}\kappa_1]\|y\|^{p_*}}{\pi}+\pi^4 c_1\\
&\quad\ge\bigl[\pi^4-a_3^{\frac{p_*}{p}}\kappa_1\bigr]\int_0^1 y^{p_*}(t)\sin(\pi t)\,\mathrm{d}t+\pi^4 c_1\\
&\quad\ge\pi^2 a_4^{p_*}\kappa_2\sum_{k=1}^m y^{p_*}(t_k)\sin(\pi t_k)\\
&\quad\ge\pi^2\sigma^{p_*}a_4^{p_*}\|y\|^{p_*}\kappa_2\sum_{k=1}^m\sin(\pi t_k),
\end{aligned}$$

and thus

$$\|y\|^{p_*}\le\frac{\pi^5 c_1}{2a_3^{\frac{p_*}{p}}\kappa_1+\pi^3\sigma^{p_*}a_4^{p_*}\kappa_2\sum_{k=1}^m\sin(\pi t_k)-2\pi^4}:=\mathcal{N}_2.$$

Therefore, we obtain the boundedness of $\mathcal{M}_1$, as claimed. Taking $R>\sup\{\rho,\sqrt[p_*]{\mathcal{N}_1},\sqrt[p_*]{\mathcal{N}_2}\}$, we have $y-Ay\neq\lambda\psi$ for all $y\in\partial B_R\cap P$ and $\lambda\ge 0$. Now, by virtue of Lemma 2.4, we obtain

$$i(A,B_R\cap P,P)=0.\tag{3.12}$$

Combining (3.5), (3.10), and (3.12), we arrive at

$$i\bigl(A,(B_R\backslash\overline{B}_\rho)\cap P,P\bigr)=0-1=-1,\qquad i\bigl(A,(B_\rho\backslash\overline{B}_r)\cap P,P\bigr)=1-0=1.$$

Now A has at least two fixed points, one on $(B_R\backslash\overline{B}_\rho)\cap P$ and the other on $(B_\rho\backslash\overline{B}_r)\cap P$. Hence (1.1) has at least two positive solutions. The proof is completed. □

Theorem 3.2 *Suppose that* (H4)-(H6) *are satisfied. Then* (1.1) *has at least two positive solutions.*

Proof If $y\in\partial B_\rho\cap P$, then we find

$$y(t)\ge t(1-t)\|y\|=\sigma\rho,\quad\forall t\in[t_1,t_m].\tag{3.13}$$

By (H4),

$$
\begin{aligned}
(Ay)\left(\frac{1}{2}\right) &\geq \int_{t_1}^{t_m} G\left(\frac{1}{2},s\right)\left(\int_0^1 G(s,\tau)f\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^{m} G\left(\frac{1}{2},t_k\right)I_k\big(y(t_k)\big) \\
&\geq \rho\left(\xi\int_{t_1}^{t_m} G\left(\frac{1}{2},s\right)\left(\int_0^1 G(s,\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^{m} G\left(\frac{1}{2},t_k\right)\xi_k\right) > \rho = \|y\|,
\end{aligned}
$$

so that

$$\|Ay\| > \|y\|, \quad \forall y \in \partial B_\rho \cap P.$$

Now Lemma 2.3 yields

$$i(A, B_\rho \cap P, P) = 0. \tag{3.14}$$

Let $r \in (0, r_0)$. Then for $y \in \partial B_r \cap P$, we find

$$y(t) \geq t(1-t)\|y\| = t(1-t)r, \quad \forall t \in [0,1]. \tag{3.15}$$

Let $\mathcal{M}_2 := \{y \in P : y = \lambda Ay \text{ for some } \lambda \in [0,1]\}$. Next, from (H5), we prove $\mathcal{M}_2 = \{0\}$. Indeed, if $y \in \mathcal{M}_2$, we have

$$
\begin{aligned}
y^{p^*}(t) \leq (Ay)^{p^*}(t) &= \left[\int_0^1 G(t,s)\left(\int_0^1 G(s,\tau)f\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\mathrm{d}s + \sum_{k=1}^{m} G(t,t_k)I_k\big(y(t_k)\big)\right]^{p^*} \\
&\leq \kappa_3\int_0^1\int_0^1 G(t,s)G(s,\tau)f^{\frac{p^*}{p}}\big(\tau,y(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s + \kappa_4\sum_{k=1}^{m} G(t,t_k)I_k^{p^*}\big(y(t_k)\big).
\end{aligned}
$$

Multiply both sides of the above by $\sin(\pi t)$ and integrate over $[0,1]$ and use (2.2) to obtain

$$
\begin{aligned}
&\int_0^1 y^{p^*}(t)\sin(\pi t)\,\mathrm{d}t \\
&\quad\leq \frac{\kappa_3}{\pi^4}\int_0^1 f^{\frac{p^*}{p}}\big(t,y(t)\big)\sin(\pi t)\,\mathrm{d}t + \frac{\kappa_4}{\pi^2}\sum_{k=1}^{m} I_k^{p^*}\big(y(t_k)\big)\sin(\pi t_k).
\end{aligned} \tag{3.16}
$$

Combining this and (3.3), we have

$$\int_0^1 y^{p^*}(t)\sin(\pi t)\,\mathrm{d}t \leq \frac{b_1^{\frac{p^*}{p}}\kappa_3}{\pi^4}\int_0^1 y^{p^*}(t)\sin(\pi t)\,\mathrm{d}t + \frac{b_2^{p^*}\kappa_4}{\pi^2}\sum_{k=1}^{m} y^{p^*}(t_k)\sin(\pi t_k).$$

Consequently,

$$
\begin{aligned}
&r^{p^*}\big(\pi^4 - b_1^{\frac{p^*}{p}}\kappa_3\big)\int_0^1 \big(t(1-t)\big)^{p^*}\sin(\pi t)\,\mathrm{d}t \\
&\quad\leq \big(\pi^4 - b_1^{\frac{p^*}{p}}\kappa_3\big)\int_0^1 y^{p^*}(t)\sin(\pi t)\,\mathrm{d}t \leq r^{p^*}\pi^2 b_2^{p^*}\kappa_4\sum_{k=1}^{m}\sin(\pi t_k),
\end{aligned}
$$

which contradicts (H5). This implies $\mathcal{M}_2 = \{0\}$, and thus $y \neq \lambda Ay$ for all $y \in \partial B_r \cap P$ and $\lambda \in [0,1]$. Now Lemma 2.5 yields

$$i(A, B_r \cap P, P) = 1. \tag{3.17}$$

On the other hand, by (H6), we prove $\mathcal{M}_2$ is bounded in P. By (3.4) together with (3.16), we obtain

$$\begin{aligned}\int_0^1 y^{p^*}(t)\sin(\pi t)\,\mathrm{d}t &\le \frac{\kappa_3}{\pi^4}\int_0^1 \bigl(b_3 y^p(t) + c\bigr)^{\frac{p^*}{p}}\sin(\pi t)\,\mathrm{d}t + \frac{\kappa_4}{\pi^2}\sum_{k=1}^m \bigl(b_4 y(t_k) + c\bigr)^{p^*}\sin(\pi t_k)\\ &\le \frac{b_3^{\frac{p^*}{p}}\kappa_5}{\pi^4}\int_0^1 y^{p^*}(t)\sin(\pi t)\,\mathrm{d}t + \frac{b_4^{p^*}\kappa_6}{\pi^2}\sum_{k=1}^m y^{p^*}(t_k)\sin(\pi t_k) + c_2,\end{aligned}$$

where $c_2 := \frac{2\kappa_5 c^{\frac{p^*}{p}}}{\pi^5} + \frac{\kappa_6 c^{p^*}}{\pi^2}\sum_{k=1}^m \sin(\pi t_k)$. Therefore,

$$\begin{aligned}&\|y\|^{p^*}\bigl(\pi^4 - b_3^{\frac{p^*}{p}}\kappa_5\bigr)\int_0^1 \bigl(t(1-t)\bigr)^{p^*}\sin(\pi t)\,\mathrm{d}t\\ &\quad\le \bigl(\pi^4 - b_3^{\frac{p^*}{p}}\kappa_5\bigr)\int_0^1 y^{p^*}(t)\sin(\pi t)\,\mathrm{d}t \le \|y\|^{p^*}\pi^2 b_4^{p^*}\kappa_6\sum_{k=1}^m \sin(\pi t_k) + \pi^4 c_2,\end{aligned}$$

namely,

$$\|y\|^{p^*} \le \frac{\pi^4 c_2}{(\pi^4 - b_3^{\frac{p^*}{p}}\kappa_5)\int_0^1 (t(1-t))^{p^*}\sin(\pi t)\,\mathrm{d}t - \pi^2 b_4^{p^*}\kappa_6\sum_{k=1}^m \sin(\pi t_k)} := \mathcal{N}_2.$$

This proves the boundedness of $\mathcal{M}_2$, as required. Choosing $R > \sqrt[p^*]{\mathcal{N}_2}$ and $R > \rho$, we have $y \neq \lambda Ay$ for all $y \in \partial B_R \cap P$ and $\lambda \in [0,1]$. Now Lemma 2.5 yields

$$i(A, B_R \cap P, P) = 1. \tag{3.18}$$

Combining (3.14), (3.17), and (3.18), we obtain

$$i\bigl(A, (B_R \backslash \overline{B}_\rho) \cap P, P\bigr) = 1 - 0 = 1, \qquad i\bigl(A, (B_\rho \backslash \overline{B}_r) \cap P, P\bigr) = 0 - 1 = -1.$$

Hence A has at least two fixed points, one on $(B_R \setminus \overline{B}_\rho) \cap P$ and the other on $(B_\rho \setminus \overline{B}_r) \cap P$, and thus (1.1) has at least two positive solutions. The proof is completed. □

4 An example

Let us consider the problem

$$\begin{cases}(|y''|^{p-1}y'')'' = y^\alpha + y^\beta, & t \in J', 0 < \alpha < p < \beta,\\ \Delta y'|_{t=t_k} = -c_k y(t_k), & c_k \ge 0, k = 1,2,\ldots,m,\\ y(0) = y(1) = y''(0) = y''(1) = 0.\end{cases} \tag{4.1}$$

Taking $\rho = 1$ in (H1), $\sum_{k=1}^{m} G(t_k,t_k)c_k < \frac{2}{3}$, and $\eta > 0$ is chosen such that $2 < \eta < 2 \cdot 6^{\frac{1}{p}}$. Set $f(t,y) = y^\alpha + y^\beta$, $0 < \alpha < p < \beta$, $\eta_k = c_k$. Therefore, $f(t,y) \le \rho^\alpha + \rho^\beta = 2 < \eta^p$, $I_k(y) = c_k y \le c_k \rho = \eta_k$, and

$$\eta + \sum_{k=1}^{m} \eta_k > 0, \qquad \eta \int_0^1 G(s,s)\left(\int_0^1 G(s,\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}} \mathrm{d}s + \sum_{k=1}^{m} G(t_k,t_k)\eta_k < 1.$$

As a result, (H1) holds. On the other hand, by simple computation, we have

$$\liminf_{y\to 0^+} \min_{t\in[0,1]} \frac{f(t,y)}{y^p} = +\infty, \qquad \liminf_{y\to +\infty} \min_{t\in[0,1]} \frac{f(t,y)}{y^p} = +\infty.$$

Therefore,

(i) There exist $0 < r_0 < \rho$ and $a_1 > 0$, $a_2 > 0$ such that (H2) holds.

(ii) There exist $c > 0$ and $a_3 > 0$, $a_4 > 0$ such that (H3) holds.

Consequently, the problem (4.1) has at least two positive solutions by Theorem 3.1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

KZ and JX obtained the results in a joint research. All the authors read and approved the final manuscript.

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