Zhao *Boundary Value Problems* 2013, **2013**:122
http://www.boundaryvalueproblems.com/content/2013/1/122

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

A logarithmically improved blow-up criterion for smooth solutions to the micropolar fluid equations in weak multiplier spaces

Juan Zhao*

*Correspondence:
jzhao79@163.com
School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou, 450011, China

Abstract
In this paper, we study the initial value problem for the three-dimensional micropolar fluid equations. A new logarithmically improved blow-up criterion for the three-dimensional micropolar fluid equations in a weak multiplier space is established.

MSC: 35K15; 35K45

Keywords: micropolar fluid equations; smooth solution; blow-up criterion

1 Introduction

In the paper, we consider the initial value problem for the micropolar fluid equations in $\mathbb{R}^3$

$$\begin{cases} \partial_t v - (\nu + \kappa)\Delta v + v \cdot \nabla v + \nabla p - 2\kappa \nabla \times w = 0, \\ \partial_t w - \gamma \Delta w - (\alpha + \beta)\nabla\nabla \cdot w + 4\kappa w + v \cdot \nabla w - 2\kappa \nabla \times v = 0, \\ \nabla \cdot v = 0 \end{cases} \tag{1.1}$$

with the initial value

$$t = 0: \qquad v = v_0(x), \quad w = w_0(x), \tag{1.2}$$

where $v(t,x)$, $w(t,x)$ and $p(t,x)$ represent the divergence free velocity field, non-divergence free micro-rotation field and the scalar pressure, respectively. $\nu > 0$ is the Newtonian kinetic viscosity and $\kappa > 0$ is the dynamics micro-rotation viscosity, $\alpha, \beta, \gamma > 0$ are the angular viscosity (see [1]).

The micropolar fluid equations were first proposed by Eringen [2]. The micropolar fluid equations are a generalization of the Navier-Stokes model. It takes into account the microstructure of the fluid, by which we mean the geometry and microrotation of particles. It is a type of fluids which exhibit the micro-rotational effects and micro-rotational inertia, and can be viewed as a non-Newtonian fluid. Physically, it may represent adequately the fluids consisting of bar-like elements. Certain anisotropic fluids, *e.g.*, liquid crystals that are made up of dumbbell molecules, are of the type. For more background, we refer to [1] and references therein.

Due to its importance in mathematics and physics, there is lots of literature devoted to the mathematical theory of the 3D micropolar fluid equations. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research and many interesting results have been established (see [3–12]). The regularity of weak solutions is examined by imposing some critical growth conditions only on the pressure field in the Lebesgue space, Morrey space, multiplier space, BMO space and Besov space, respectively (see [4]). A new logarithmically improved blow-up criterion for the 3D micropolar fluid equations in an appropriate homogeneous Besov space was obtained by Wang and Yuan [9]. A Serrin-type regularity criterion for the weak solutions to the micropolar fluid equations in $\mathbb{R}^3$ in the critical Morrey-Campanato space was built [10]. Wang and Zhao [12] established logarithmically improved blow-up criteria of a smooth solution to (1.1), (1.2) in the Morrey-Campanto space.

If $\kappa = 0$ and $w = 0$, then equations (1.1) reduce to be the Navier-Stokes equations. The Leray-Hopf weak solution was constructed by Leray [13] and Hopf [14], respectively. Later on, much effort has been devoted to establishing the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. Different criteria for regularity of the weak solutions have been proposed and many interesting results were established (see [15–24]).

Without loss of generality, we set $\nu = \kappa = \frac{1}{2}$, $\gamma = \alpha + \beta = 1$ in the rest of the paper. The purpose of this paper is to establish a new logarithmically improved blow-up criterion to (1.1), (1.2) in a weak multiplier space. Now we state our results as follows.

Theorem 1.1 *Assume that* $v_0, w_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$ *with* $\nabla \cdot v_0 = 0$. *Let* (v, w) *be a smooth solution to equations* (1.1), (1.2) *for* $0 \leq t < T$. *If* v *satisfies*

$$\int_0^T \frac{\|\nabla v\|^{1-r}_{\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})}}{1 + \ln(e + \|\nabla v\|_{L^\infty})}\,dt < \infty, \quad 0 \leq r < 1, \tag{1.3}$$

then the solution (v, w) *can be extended beyond* $t = T$.

We have the following corollary immediately.

Corollary 1.1 *Assume that* $v_0, w_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$ *with* $\nabla \cdot v_0 = 0$. *Let* (v, w) *be a smooth solution to equations* (1.1), (1.2) *for* $0 \leq t < T$. *Suppose that* T *is the maximal existence time, then*

$$\int_0^T \frac{\|\nabla v\|^{1-r}_{\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})}}{1 + \ln(e + \|\nabla v\|_{L^\infty})}\,dt = \infty, \quad 0 \leq r < 1. \tag{1.4}$$

The paper is organized as follows. We first state some preliminaries on functional settings and some important inequalities in Section 2, which play an important role in the proof of our main result. Then we prove the main result in Section 3.

2 Preliminaries

Definition 2.1 [25] For $0 < r < \frac{3}{2}$, $\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})$ is a Banach space of all distributions f on $\mathbb{R}^3$ such that there exists a constant C such that for all $u \in \mathcal{D}$, we have $fu \in \dot{H}^r$ and

$$\|fu\|_{\dot{H}^r} \leq C\|u\|_{\dot{H}^{-r}},$$

where we denote by $\dot{H}^r$ the completion of the space $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{\frac{r}{2}}u\|_{L^2}$.

The norm of $\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^{-r})} = \sup\bigl\{\|fu\|_{\dot{H}^r} : \|u\|_{\dot{H}^r} \le 1, u \in \mathcal{D}\bigr\}.$$

The following lemma comes from [26].

Lemma 2.2 *Assume that* $1 < p < \infty$. *For* $f, g \in W^{m,p}$, *and* $1 < q \le \infty$, $1 < r < \infty$, *we have*

$$\bigl\|\nabla^\alpha(fg) - f\nabla^\alpha g\bigr\|_{L^p} \le C\bigl(\|\nabla f\|_{L^{q_1}}\bigl\|\nabla^{\alpha-1}g\bigr\|_{L^{r_1}} + \|g\|_{L^{q_2}}\bigl\|\nabla^\alpha f\bigr\|_{L^{r_2}}\bigr), \tag{2.1}$$

where $1 \le \alpha \le m$ *and* $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$.

We also need the following interpolation inequalities in three space dimensions.

Lemma 2.3 *In three space dimensions*, *the following inequalities hold*:

$$\begin{cases} \|\nabla f\|_{L^4} \le C\|f\|_{L^2}^{\frac{1}{8}}\|\nabla^2 f\|_{L^2}^{\frac{7}{8}}, \\ \|f\|_{L^4} \le C\|f\|_{L^2}^{\frac{5}{8}}\|\nabla^2 f\|_{L^2}^{\frac{3}{8}}, \\ \|\nabla^2 f\|_{L^4} \le C\|f\|_{L^2}^{\frac{1}{12}}\|\nabla^3 f\|_{L^2}^{\frac{11}{12}}, \\ \|\nabla^2 f\|_{L^2} \le C\|f\|_{L^2}^{\frac{1}{3}}\|\nabla^3 f\|_{L^2}^{\frac{2}{3}}. \end{cases} \tag{2.2}$$

3 Proof of Theorem 1.1

Multiplying the first equation of (1.1) by v and integrating with x respect to on $\mathbb{R}^3$, using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\bigl\|v(t)\bigr\|_{L^2}^2 + \bigl\|\nabla v(t)\bigr\|_{L^2}^2 = \int_{\mathbb{R}^3}(\nabla \times w)\cdot v\,dx. \tag{3.1}$$

Similarly, we get

$$\frac{1}{2}\frac{d}{dt}\bigl\|w(t)\bigr\|_{L^2}^2 + \bigl\|\nabla w(t)\bigr\|_{L^2}^2 + \|\nabla\cdot w\|_{L^2}^2 + 2\|w\|_{L^2}^2 = \int_{\mathbb{R}^3}(\nabla \times v)\cdot w\,dx. \tag{3.2}$$

Summing up (3.1)-(3.2), we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\bigl(\bigl\|v(t)\bigr\|_{L^2}^2 + \bigl\|w(t)\bigr\|_{L^2}^2\bigr) + \bigl\|\nabla v(t)\bigr\|_{L^2}^2 + \bigl\|\nabla w(t)\bigr\|_{L^2}^2 + \|\nabla\cdot w\|_{L^2}^2 + 2\|w\|_{L^2}^2 \\ &\quad = \int_{\mathbb{R}^3}(\nabla \times w)\cdot v\,dx + \int_{\mathbb{R}^3}(\nabla \times v)\cdot w\,dx. \end{aligned} \tag{3.3}$$

We apply integration by parts and the Cauchy inequality. This yields

$$\int_{\mathbb{R}^3}(\nabla \times w)\cdot v\,dx + \int_{\mathbb{R}^3}(\nabla \times v)\cdot w\,dx \le \frac{1}{2}\|\nabla v\|_{L^2}^2 + 2\|w\|_{L^2}^2. \tag{3.4}$$

Substituting (3.3) into (3.4) yields

$$\frac{1}{2}\frac{d}{dt}\bigl(\bigl\|v(t)\bigr\|_{L^2}^2 + \bigl\|w(t)\bigr\|_{L^2}^2\bigr) + \frac{1}{2}\bigl\|\nabla v(t)\bigr\|_{L^2}^2 + \bigl\|\nabla w(t)\bigr\|_{L^2}^2 + \|\nabla\cdot w\|_{L^2}^2 \le 0.$$

Integrating with respect to t, we have

$$\begin{aligned}&\|v(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2+\int_0^t\left(\|\nabla v(\tau)\|_{L^2}^2+2\|\nabla w(\tau)\|_{L^2}^2\right)d\tau+2\int_0^t\|\nabla\cdot w(\tau)\|_{L^2}^2\,d\tau\\&\quad\le\|v_0\|_{L^2}^2+\|w_0\|_{L^2}^2.\end{aligned}\tag{3.5}$$

Multiplying the first equation of (1.1) by $|v|^2v$, then integrating the resulting equation with respect to x over $\mathbb{R}^3$ and using integrating by parts, we obtain

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\|v\|_{L^4}^4+\big\||v|\nabla v\big\|_{L^2}^2+\frac{1}{2}\big\|\nabla|v|^2\big\|_{L^2}^2\\&\quad=\int_{\mathbb{R}^3}w\nabla\times\left(|v|^2v\right)dx-\int_{\mathbb{R}^3}(v\cdot\nabla p)|v|^2\,dx.\end{aligned}\tag{3.6}$$

We multiply the second equation of (1.1) by $|w|^2w$, then integrate the resulting equation with respect to x over $\mathbb{R}^3$ and use integrating by parts. This yields

$$\frac{1}{4}\frac{d}{dt}\|w\|_{L^4}^4+\big\||w|\nabla w\big\|_{L^2}^2+\frac{1}{2}\big\||w|\nabla\cdot w\big\|_{L^2}^2+2\|w\|_{L^4}^4\le\int_{\mathbb{R}^3}v\nabla\times\left(|w|^2w\right)dx,\tag{3.7}$$

where we have used

$$\begin{aligned}\int_{\mathbb{R}^3}(\nabla\cdot w)\nabla\cdot\left(w|w|^2\right)dx&=\int_{\mathbb{R}^3}|\nabla\cdot w|^2|w|^2\,dx+\int_{\mathbb{R}^3}(\nabla\cdot w)w\cdot\nabla|w|^2\,dx\\&\ge\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\cdot w|^2|w|^2\,dx-\frac{1}{2}\int_{\mathbb{R}^3}\big|\nabla|w|^2\big|^2\,dx.\end{aligned}$$

Equations (3.6) and (3.7) give

$$\begin{aligned}&\frac{1}{4}\frac{d}{dt}\left(\|v\|_{L^4}^4+\|w\|_{L^4}^4\right)+\big\||v|\nabla v\big\|_{L^2}^2+\frac{1}{2}\big\|\nabla|v|^2\big\|_{L^2}^2\\&\qquad+\big\||w|\nabla w\big\|_{L^2}^2+\frac{1}{2}\big\||w|\nabla\cdot w\big\|_{L^2}^2+2\|w\|_{L^4}^4\\&\quad\le\int_{\mathbb{R}^3}w\nabla\times\left(|v|^2v\right)dx-\int_{\mathbb{R}^3}(v\cdot\nabla p)|v|^2\,dx+\int_{\mathbb{R}^3}v\nabla\times\left(|w|^2w\right)dx.\end{aligned}\tag{3.8}$$

Making use of the Young inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}w\nabla\times\left(|v|^2v\right)dx+\int_{\mathbb{R}^3}v\nabla\times\left(|w|^2w\right)dx\\&\quad\le C\|v\|_{L^4}\|w\|_{L^4}\left(\big\||v|\nabla v\big\|_{L^2}+\big\||w|\nabla w\big\|_{L^2}\right)\\&\quad\le\frac{1}{4}\big\||v|\nabla v\big\|_{L^2}^2+\frac{1}{4}\big\||w|\nabla w\big\|_{L^2}^2+C\|v\|_{L^4}^4+C\|w\|_{L^4}^4.\end{aligned}\tag{3.9}$$

Applying the divergence operator $\nabla\cdot$ to the first equation of (1.1) produces the expression of the pressure

$$p=(-\Delta)^{-1}\nabla\cdot(v\cdot\nabla v).\tag{3.10}$$

It follows from (3.10) that

$$\|p\|_{L^2} \le C\|v\|_{L^4}^2, \qquad \|\nabla p\|_{L^2} \le C\big\||v|\nabla v\big\|_{L^2}. \tag{3.11}$$

By integration by parts and (3.11), we obtain

$$\begin{aligned}
-\int_{\mathbb{R}^3}(v\cdot\nabla p)|v|^2\,dx &\le C\big\||v|^2\nabla v\big\|_{\dot H^{-r}}\|p\|_{\dot H^r} \\
&\le C\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}\big\||v|^2\big\|_{\dot H^r}\|p\|_{L^2}^{1-r}\|\nabla p\|_{L^2}^r \\
&\le C\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}\big\||v|^2\big\|_{L^2}^{1-r}\big\|\nabla|v|^2\big\|_{L^2}^r\|p\|_{L^2}^{1-r}\|\nabla p\|_{L^2}^r \\
&\le C\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}\|v\|_{L^4}^{4(1-r)}\big\||v|\nabla v\big\|_{L^2}^{2r} \\
&\le \epsilon\big\||v|\nabla v\big\|_{L^2}^2 + C\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}\|v\|_{L^4}^4.
\end{aligned} \tag{3.12}$$

Combining (3.8), (3.9), (3.12) and (3.5) yields

$$\begin{aligned}
&\frac{d}{dt}\big(\|v\|_{L^4}^4+\|w\|_{L^4}^4\big)+\big\||v|\nabla v\big\|_{L^2}^2+\big\|\nabla|v|^2\big\|_{L^2}^2+\big\||w|\nabla w\big\|_{L^2}^2+\big\||w|\nabla\cdot w\big\|_{L^2}^2+\|w\|_{L^4}^4 \\
&\quad\le C\|v\|_{L^4}^4+C\|w\|_{L^4}^4+C\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}\|v\|_{L^4}^4 \\
&\quad\le C\|v\|_{L^4}^4+C\|w\|_{L^4}^4+C\frac{\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}}{1+\ln(e+\|\nabla v\|_{L^\infty})}\|v\|_{L^4}^4\big[1+\ln\big(e+\|\nabla v\|_{L^\infty}\big)\big] \\
&\quad\le C\|v\|_{L^4}^4+C\|w\|_{L^4}^4+C\frac{\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}}{1+\ln(e+\|\nabla v\|_{L^\infty})}\|v\|_{L^4}^4\big[1+\ln\big(e+\|\nabla v\|_{H^2}\big)\big] \\
&\quad\le C\|v\|_{L^4}^4+C\|w\|_{L^4}^4+C\frac{\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}}{1+\ln(e+\|\nabla v\|_{L^\infty})}\|v\|_{L^4}^4\big[1+\ln\big(e+\|v\|_{H^3}\big)\big] \\
&\quad\le C\|v\|_{L^4}^4+C\|w\|_{L^4}^4+C\frac{\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}}{1+\ln(e+\|\nabla v\|_{L^\infty})}\|v\|_{L^4}^4\big[1+\ln\big(e+\|v\|_{H^3}^2\big)\big] \\
&\quad\le C\|v\|_{L^4}^4+C\|w\|_{L^4}^4+C\frac{\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}}{1+\ln(e+\|\nabla v\|_{L^\infty})}\|v\|_{L^4}^4\big[1+\ln\big(e+\big\|\nabla^3 v\big\|_{L^2}^2\big)\big] \\
&\quad\le C\|v\|_{L^4}^4+C\|w\|_{L^4}^4 \\
&\qquad+C\frac{\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}}{1+\ln(e+\|\nabla v\|_{L^\infty})}\|v\|_{L^4}^4\big[1+\ln\big(e+\big\|\nabla^3 v\big\|_{L^2}^2+\big\|\nabla^3 w\big\|_{L^2}^2\big)\big],
\end{aligned} \tag{3.13}$$

where we have used

$$H^2 \hookrightarrow L^\infty.$$

By (1.3), we know that for any small constant $\varepsilon > 0$, there exists $T_* < T$ such that

$$\int_{T_*}^T \frac{\|\nabla v\|_{\mathcal{M}(\dot H^r\to\dot H^{-r})}^{1-r}}{1+\ln(e+\|\nabla v\|_{L^\infty})}\,dt \le \varepsilon. \tag{3.14}$$

Let

$$A(t) = \sup_{T_*\le\tau\le t}\big(\big\|\nabla^3 v\big\|_{L^2}^2+\big\|\nabla^3 w\big\|_{L^2}^2\big), \quad T_* \le t < T. \tag{3.15}$$

The Gronwall inequality and (3.13)-(3.15) give

$$\begin{aligned}\|v\|_{L^4}^4 + \|w\|_{L^4}^4 &\le C\exp\bigl\{C\epsilon\bigl(1+\ln\bigl(e+A(t)\bigr)\bigr)\bigr\}\\ &\le C\exp\bigl\{2C\epsilon\ln\bigl(e+A(t)\bigr)\bigr\}\\ &\le C\bigl(e+A(t)\bigr)^{2C\varepsilon},\quad T_*\le t<T.\end{aligned}\tag{3.16}$$

Applying ∇ to the first equation, then multiplying the resulting equation by ∇v and using integration by parts, the Hölder inequality, (2.2) and the Young inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2 + \bigl\|\nabla^2 v\bigr\|_{L^2}^2 &= \int_{\mathbb{R}^3}\nabla v\cdot\nabla v\nabla v\,dx + \int_{\mathbb{R}^3}(\nabla\times w)\nabla^2 v\,dx\\ &\le C\|v\|_{L^4}\|\nabla v\|_{L^4}\bigl\|\nabla^2 v\bigr\|_{L^2} + C\|\nabla w\|_{L^2}\bigl\|\nabla^2 v\bigr\|_{L^2}\\ &\le C\|v\|_{L^4}\|v\|_{L^2}^{\frac{1}{8}}\bigl\|\nabla^2 v\bigr\|_{L^2}^{\frac{15}{8}} + \frac{1}{8}\bigl\|\nabla^2 v\bigr\|_{L^2}^2 + C\|w\|_{L^2}\bigl\|\nabla^2 w\bigr\|_{L^2}\\ &\le \frac{1}{4}\bigl\|\nabla^2 v\bigr\|_{L^2}^2 + \frac{1}{4}\bigl\|\nabla^2 w\bigr\|_{L^2}^2 + C\|v\|_{L^4}^{16}\|v\|_{L^2}^2 + C\|w\|_{L^2}^2.\end{aligned}\tag{3.17}$$

Similarly, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla w\|_{L^2}^2 + \bigl\|\nabla^2 w\bigr\|_{L^2}^2 + \|\nabla\nabla\cdot w\|_{L^2}^2 + 2\|\nabla w\|_{L^2}^2\\ &\quad= \int_{\mathbb{R}^3}\nabla v\cdot\nabla w\nabla w\,dx + \int_{\mathbb{R}^3}(\nabla\times v)\nabla^2 w\,dx\\ &\quad\le C\|v\|_{L^4}\|\nabla w\|_{L^4}\bigl\|\nabla^2 w\bigr\|_{L^2} + C\|\nabla v\|_{L^2}\bigl\|\nabla^2 w\bigr\|_{L^2}\\ &\quad\le C\|v\|_{L^4}\|w\|_{L^2}^{\frac{1}{8}}\bigl\|\nabla^2 w\bigr\|_{L^2}^{\frac{15}{8}} + \frac{1}{8}\bigl\|\nabla^2 w\bigr\|_{L^2}^2 + C\|v\|_{L^2}\bigl\|\nabla^2 v\bigr\|_{L^2}\\ &\quad\le \frac{1}{4}\bigl\|\nabla^2 w\bigr\|_{L^2}^2 + \frac{1}{4}\bigl\|\nabla^2 v\bigr\|_{L^2}^2 + C\|v\|_{L^4}^{16}\|w\|_{L^2}^2 + C\|v\|_{L^2}^2.\end{aligned}\tag{3.18}$$

Adding (3.17) and (3.18), we arrive at

$$\begin{aligned}&\frac{d}{dt}\bigl(\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\bigr) + \bigl\|\nabla^2 v\bigr\|_{L^2}^2 + \bigl\|\nabla^2 w\bigr\|_{L^2}^2 + \|\nabla\nabla\cdot w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\\ &\quad\le C\|v\|_{L^4}^{16}\bigl(\|v\|_{L^2}^2 + \|w\|_{L^2}^2\bigr) + C\bigl(\|v\|_{L^2}^2 + \|w\|_{L^2}^2\bigr).\end{aligned}\tag{3.19}$$

Equations (3.5), (3.16), (3.19) and the Gronwall inequality give

$$\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \le C\bigl(e+A(t)\bigr)^{8C\varepsilon},\quad T_*\le t<T.\tag{3.20}$$

Applying ∇^m to the first equation in (1.1), then taking L^2 inner product of the resulting equation with $\nabla^m v$ and using integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\bigl\|\nabla^m v\bigr\|_{L^2}^2 + \bigl\|\nabla^{m+1} v\bigr\|_{L^2}^2\\ &\quad= -\int_{\mathbb{R}^3}\nabla^m(v\cdot\nabla v)\nabla^m v\,dx + \int_{\mathbb{R}^3}\nabla^m(\nabla\times w)\nabla^m v\,dx.\end{aligned}\tag{3.21}$$

Similarly, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^m w\|_{L^2}^2 + \|\nabla^{m+1} w\|_{L^2}^2 + \|\nabla^m \nabla\cdot w\|_{L^2}^2 + 2\|\nabla^m w\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \nabla^m (v\cdot\nabla w)\nabla^m w\,dx + \int_{\mathbb{R}^3} \nabla^m(\nabla\times v)\nabla^m w\,dx.
\end{aligned} \tag{3.22}$$

Summing (3.21), (3.22) and using $\nabla\cdot v = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^m v\|_{L^2}^2 + \|\nabla^m w\|_{L^2}^2\right) + \|\nabla^{m+1} v\|_{L^2}^2 + \|\nabla^{m+1} w\|_{L^2}^2 + \|\nabla^m \nabla\cdot w\|_{L^2}^2 + 2\|\nabla^m w\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^3} \left[\nabla^m(v\cdot\nabla v) - v\cdot\nabla^m\nabla v\right]\nabla^m v\,dx - \int_{\mathbb{R}^3}\left[\nabla^m(v\cdot\nabla w) - v\cdot\nabla^m\nabla w\right]\nabla^m w\,dx \\
&\qquad + \int_{\mathbb{R}^3} \nabla^m(\nabla\times w)\nabla^m v\,dx + \int_{\mathbb{R}^3}\nabla^m(\nabla\times v)\nabla^m w\,dx \\
&\quad \triangleq I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.23}$$

In what follows, for simplicity, we set $m = 3$.

By the Hölder inequality, (2.1), (2.2) and the Young inequality, we obtain

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\left[\nabla^3(v\cdot\nabla v) - v\cdot\nabla\nabla^3 v\right]\nabla^3 v\,dx \\
&\quad \le \|\nabla^3(v\cdot\nabla v) - v\cdot\nabla\nabla^3 v\|_{L^2}\|\nabla^3 v\|_{L^2} \\
&\quad \le C\|\nabla v\|_{L^4}\|\nabla^3 v\|_{L^4}\|\nabla^3 v\|_{L^2} \\
&\quad \le C\|\nabla v\|_{L^2}^{\frac{5}{8}}\|\nabla^3 v\|_{L^2}^{\frac{3}{8}}\|\nabla v\|_{L^2}^{\frac{1}{12}}\|\nabla^4 v\|_{L^2}^{\frac{11}{12}}\|\nabla v\|_{L^2}^{\frac{1}{3}}\|\nabla^4 v\|_{L^2}^{\frac{2}{3}} \\
&\quad \le C\|\nabla v\|_{L^2}^{\frac{25}{24}}\|\nabla^3 v\|_{L^2}^{\frac{3}{8}}\|\nabla^4 v\|_{L^2}^{\frac{19}{12}} \\
&\quad \le \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^5\|\nabla^3 v\|_{L^2}^{\frac{9}{5}} \\
&\quad \le \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + C\big(e + A(t)\big)^{20C\varepsilon + \frac{9}{10}}
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^3}\left[\nabla^3(v\cdot\nabla w) - v\cdot\nabla\nabla^3 w\right]\nabla^3 w\,dx \\
&\quad \le \|\nabla^3(v\cdot\nabla w) - v\cdot\nabla\nabla^3 w\|_{L^2}\|\nabla^3 w\|_{L^2} \\
&\quad \le C\|\nabla v\|_{L^4}\|\nabla^3 w\|_{L^4}\|\nabla^3 w\|_{L^2} + \|\nabla w\|_{L^4}\|\nabla^3 v\|_{L^4}\|\nabla^3 w\|_{L^2} \\
&\quad \le C\|\nabla v\|_{L^2}^{\frac{5}{8}}\|\nabla^3 v\|_{L^2}^{\frac{3}{8}}\|\nabla w\|_{L^2}^{\frac{1}{12}}\|\nabla^4 w\|_{L^2}^{\frac{11}{12}}\|\nabla w\|_{L^2}^{\frac{1}{3}}\|\nabla^4 w\|_{L^2}^{\frac{2}{3}} \\
&\qquad + C\|\nabla w\|_{L^2}^{\frac{5}{8}}\|\nabla^3 w\|_{L^2}^{\frac{3}{8}}\|\nabla v\|_{L^2}^{\frac{1}{12}}\|\nabla^4 v\|_{L^2}^{\frac{11}{12}}\|\nabla w\|_{L^2}^{\frac{1}{3}}\|\nabla^4 w\|_{L^2}^{\frac{2}{3}} \\
&\quad \le C\|\nabla v\|_{L^2}^{\frac{5}{8}}\|\nabla^3 v\|_{L^2}^{\frac{3}{8}}\|\nabla w\|_{L^2}^{\frac{5}{12}}\|\nabla^4 w\|_{L^2}^{\frac{19}{12}} \\
&\qquad + C\|\nabla w\|_{L^2}^{\frac{23}{24}}\|\nabla^3 w\|_{L^2}^{\frac{3}{8}}\|\nabla v\|_{L^2}^{\frac{1}{12}}\|\nabla^4 v\|_{L^2}^{\frac{11}{12}}\|\nabla^4 w\|_{L^2}^{\frac{2}{3}} \\
&\quad \le \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + \frac{1}{8}\|\nabla^4 w\|_{L^2}^2 + C\|\nabla v\|_{L^2}^3\|\nabla w\|_{L^2}^2\|\nabla^3 v\|_{L^2}^{\frac{9}{5}}
\end{aligned}$$

$$\begin{aligned}&+ C\|\nabla v\|_{L^2}^{\frac{2}{5}}\|\nabla w\|_{L^2}^{\frac{23}{5}}\|\nabla^3 w\|_{L^2}^{\frac{9}{5}}\\ &\leq \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + \frac{1}{8}\|\nabla^4 w\|_{L^2}^2 + C\big(e + A(t)\big)^{20C\varepsilon + \frac{9}{10}}.\end{aligned} \tag{3.25}$$

From the Young inequality and (2.2), we deduce that

$$\begin{aligned} I_3 &\leq C\|\nabla^4 v\|_{L^2}\|\nabla^3 w\|_{L^2}\\ &\leq \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + C\|\nabla^3 w\|_{L^2}^2\\ &\leq \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + C\|\nabla w\|_{L^2}^{\frac{2}{3}}\|\nabla^4 w\|_{L^2}^{\frac{4}{3}}\\ &\leq \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + \frac{1}{8}\|\nabla^4 w\|_{L^2}^2 + C\|\nabla w\|_{L^2}^2\\ &\leq \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + \frac{1}{8}\|\nabla^4 w\|_{L^2}^2 + C\big(e + A(t)\big)^{8C\varepsilon}, \quad T_* \leq t < T. \end{aligned} \tag{3.26}$$

Similarly, we have

$$I_4 \leq \frac{1}{8}\|\nabla^4 v\|_{L^2}^2 + \frac{1}{8}\|\nabla^4 w\|_{L^2}^2 + C\big(e + A(t)\big)^{8C\varepsilon}, \quad T_* \leq t < T. \tag{3.27}$$

Inserting (3.24)-(3.27) into (3.23) and taking ε small enough such that $20C\varepsilon < \frac{1}{10}$, we obtain

$$\frac{d}{dt}\big(\|\nabla^3 v\|_{L^2}^2 + \|\nabla^3 w\|_{L^2}^2\big) \leq C\big(e + A(t)\big), \quad T_* \leq t < T, \tag{3.28}$$

for all $T_* \leq t < T$.

Integrating (3.28) with respect to time from T_* to τ, we have

$$\begin{aligned}&e + \|\nabla^3 v(\tau)\|_{L^2}^2 + \|\nabla^3 w(\tau)\|_{L^2}^2\\ &\quad\leq e + \|\nabla^3 v(T_*)\|_{L^2}^2 + \|\nabla^3 w(T_*)\|_{L^2}^2 + C_2\int_{T_*}^{\tau}\big(e + A(s)\big)\,ds.\end{aligned} \tag{3.29}$$

We get from (3.29)

$$e + A(t) \leq e + \|\nabla^3 v(T_*)\|_{L^2}^2 + \|\nabla^3 w(T_*)\|_{L^2}^2 + C_2\int_{T_*}^{t}\big(e + A(\tau)\big)\,d\tau. \tag{3.30}$$

For all $T_* \leq t < T$, with the help of Gronwall inequality and (3.30), we have

$$e + \|\nabla^3 v(t)\|_{L^2}^2 + \|\nabla^3 w(t)\|_{L^2}^2 \leq C, \tag{3.31}$$

where C depends on $\|\nabla v(T_*)\|_{L^2}^2 + \|\nabla w(T_*)\|_{L^2}^2$. (3.31) and (3.5) imply $(v, w) \in L^\infty(0, T; H^3(\mathbb{R}^3))$. Thus, (v, w) can be extended smoothly beyond $t = T$. We have completed the proof of Theorem 1.1.

Competing interests

The author declares that she has no competing interests.

Authors' contributions

The author completed the paper herself. The author read and approved the final manuscript.

Received: 11 March 2013 Accepted: 26 April 2013 Published: 10 May 2013

References

1. Lukaszewicz, G: Micropolar Fluids. Theory and Applications. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Boston (1999)
2. Eringen, A: Theory of micropolar fluids. J. Math. Mech. **16**, 1-18 (1966)
3. Dong, B, Chen, Z: Regularity criteria of weak solutions to the three-dimensional micropolar flows. J. Math. Phys. **50**, 103525 (2009)
4. Dong, B, Jia, Y, Chen, Z: Pressure regularity criteria of the three-dimensional micropolar fluid flows. Math. Methods Appl. Sci. **34**, 595-606 (2011)
5. He, X, Fan, J: A regularity criterion for 3D micropolar fluid flows. Appl. Math. Lett. **25**, 47-51 (2012)
6. Jia, Y, Zhang, W, Dong, B: Remarks on the regularity criterion of the 3D micropolar fluid flows in terms of the pressure. Appl. Math. Lett. **24**, 199-203 (2011)
7. Ortega-Torres, E, Rojas-Medar, M: On the regularity for solutions of the micropolar fluid equations. Rend. Semin. Mat. Univ. Padova **122**, 27-37 (2009)
8. Wang, Y, Chen, Z: Regularity criterion for weak solution to the 3D micropolar fluid equations. J. Appl. Math. **2011**, Article ID 456547 (2011)
9. Wang, Y, Yuan, H: A logarithmically improved blow up criterion of smooth solutions to the 3D micropolar fluid equations. Nonlinear Anal., Real World Appl. **13**, 1904-1912 (2012)
10. Gala, S: On regularity criteria for the three-dimensional micropolar fluid equations in the critical Morrey-Campanato space. Nonlinear Anal., Real World Appl. **12**, 2142-2150 (2011)
11. Gala, S, Yan, J: Two regularity criteria via the logarithmic of the weak solutions to the micropolar fluid equations. J. Partial Differ. Equ. **25**, 32-40 (2012)
12. Wang, Y, Zhao, H: Logarithmically improved blow up criterion of smooth solutions to the 3D micropolar fluid equations. J. Appl. Math. **2012**, Article ID 541203 (2012)
13. Leray, J: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**, 183-248 (1934)
14. Hopf, E: Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. Math. Nachr. **4**, 213-231 (1951)
15. Fan, J, Ozawa, T: Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the gradient of the pressure. J. Inequal. Appl. **2008**, Article ID 412678 (2008)
16. Fan, J, Jiang, S, Nakamura, G, Zhou, Y: Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations. J. Math. Fluid Mech. (Printed Ed.) **13**, 557-571 (2011)
17. He, C: New sufficient conditions for regularity of solutions to the Navier-Stokes equations. Adv. Math. Sci. Appl. **12**, 535-548 (2002)
18. Serrin, J: On the interior regularity of weak solutions of the Navier-Stokes equations. Arch. Ration. Mech. Anal. **9**, 187-195 (1962)
19. Zhang, Z, Chen, Q: Regularity criterion via two components of vorticity on weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. J. Differ. Equ. **216**, 470-481 (2005)
20. Zhou, Y, Gala, S: Logarithmically improved regularity criteria for the Navier-Stokes equations in multiplier spaces. J. Math. Anal. Appl. **356**, 498-501 (2009)
21. Zhou, Y, Pokorný, M: On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component. J. Math. Phys. **50**, 123514 (2009)
22. Zhou, Y, Pokorný, M: On the regularity of the solutions of the Navier-Stokes equations via one velocity component. Nonlinearity **23**, 1097-1107 (2010)
23. Guo, Z, Gala, S: Remarks on logarithmical regularity criteria for the Navier-Stokes equations. J. Math. Phys. **52**, 063503 (2011)
24. He, X, Gala, S: Regularity criterion for weak solutions to the Navier-Stokes equations in terms of the pressure in the class $L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$. Nonlinear Anal., Real World Appl. **12**, 3602-3607 (2011)
25. Jiang, Z, Gala, S, Ni, L: On the regularity criterion for the solutions of 3D Navier-Stokes equations in weak multiplier spaces. Math. Methods Appl. Sci. **34**, 2060-2064 (2011)
26. Kato, T, Ponce, G: Commutator estimates and the Euler and Navier-Stokes equations. Commun. Pure Appl. Math. **41**, 891-907 (1988)

doi:10.1186/1687-2770-2013-122
Cite this article as: Zhao: **A logarithmically improved blow-up criterion for smooth solutions to the micropolar fluid equations in weak multiplier spaces.** *Boundary Value Problems* 2013 **2013**:122.