Zhao et al. *Boundary Value Problems* 2013, **2013**:129
http://www.boundaryvalueproblems.com/content/2013/1/129

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Periodic solutions for $N+2$-body problems with $N+1$ fixed centers

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Abstract

In this paper, we prove the existence of a new periodic solution for $N+2$-body problems with $N+1$ fixed centers and strong-force potentials. In this model, N particles with equal masses are fixed at the vertices of a regular N-gon and the $(N+1)$th particle is fixed at the center of the N-gon, the $(N+2)$th particle winding around N particles.

MSC: 34C15; 34C25; 70F10

Keywords: $N+2$-body problems with $N+1$-fixed centers; minimizing variational methods; strong-force potentials

1 Introduction and main results

In the eighteenth century, the 2-fixed center problem was studied by Euler [1–3]. Here, let us consider the $N+1$-fixed center problem: We assume N particles $q_1, q_2, \dots, q_N$ with equal masses 1 are fixed at the vertices $e^{\sqrt{-1}\frac{2\pi}{N}j} = (\cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N})$ $(j = 1, \dots, N)$ of a regular polygon and the $(N+1)$th particle q_{N+1} is fixed at the origin $(0,0)$, the $(N+2)$th particle with mass m_{N+2} is attracted by the other particles, and moves according to Newton's second law and a more general power law than the Newton's universal gravitational square law. In this system, the position $q(t)$ for the $(N+2)$th particle satisfies the following equation:

$$m_{N+2}\ddot{q}(t) = \sum_{i=1}^{N+1} \frac{m_i m_{N+2}(q(t) - q_i)}{|q(t) - q_i|^{\alpha+2}}. \tag{1.1}$$

Equivalently,

$$\ddot{q}(t) = \sum_{i=1}^{N} \frac{(q(t) - q_i)}{|q(t) - q_i|^{\alpha+2}} + \frac{m_{N+1}(q(t) - q_{N+1})}{|q(t) - q_{N+1}|^{\alpha+2}}, \tag{1.2}$$

$$\ddot{q}(t) = \frac{\partial U(q)}{\partial q}, \tag{1.3}$$

where

$$\alpha > 0 \quad \text{and} \quad U(q) = \sum_{i=1}^{N} \frac{1}{|q(t) - q_i|^{\alpha}} + \frac{m_{N+1}}{|q(t) - q_{N+1}|^{\alpha}}.$$

The type of system (1.2) is called a singular Hamiltonian system which attracts many researchers (see [1–10] and [11–16]).

Specially, Gordon [10] proved the Keplerian elliptical orbits are the minimizers of Lagrangian action defined on the space for non-zero winding numbers.

In this paper, we use a variational minimizing method to look for a periodic solution for the $(N+2)$th particle which winds around the q_i $(i = 1, \dots, N+1)$.

Definition 1.1 [10] Let $C : x(t) : [a,b] \to R^2$ be a given oriented closed curve, and $p \notin C$. Define $\varphi : C \to S^1$:

$$\varphi(t) = \frac{x(t) - p}{|x(t) - p|}.$$

When some point on C goes around the curve once, its image point $\varphi(x(t))$ will go around S^1 a number of times. This number is defined as the winding number of the curve C relative to the point p and is denoted by $\deg(x(t) - p)$.

Let

$$f(q) = \int_0^1 \left[\frac{1}{2}\left|\dot{q}(t)\right|^2 + U(q)\right] dt, \tag{1.4}$$

$$q \in \Lambda_1 = \left\{\begin{array}{l} q \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2), \quad q(t) \neq q_i, \quad \text{for } i = 1, \dots, N+1, \\ q(t + \frac{k}{N}) = \begin{pmatrix} \cos(\frac{2k\pi}{N}) & -\sin(\frac{2k\pi}{N}) \\ \sin(\frac{2k\pi}{N}) & \cos(\frac{2k\pi}{N}) \end{pmatrix} q(t), \\ \deg(q(t) - q_i) = 1, \quad \text{for } i = 1, \dots, N, \qquad \deg(q(t) - q_{N+1}) = -1 \end{array}\right\}, \tag{1.5}$$

$$q \in \Lambda_2 = \left\{\begin{array}{l} q \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2), \quad q(t) \neq q_i, \quad \text{for } i = 1, \dots, N+1, \\ q(t + \frac{k}{N}) = \begin{pmatrix} \cos(\frac{2k\pi}{N}) & -\sin(\frac{2k\pi}{N}) \\ \sin(\frac{2k\pi}{N}) & \cos(\frac{2k\pi}{N}) \end{pmatrix} q(t), \\ \deg(q(t) - q_i) = 0, \quad \text{for } i = 1, \dots, N, \qquad \deg(q(t) - q_{N+1}) = 1 \end{array}\right\}, \tag{1.6}$$

$$q \in \Lambda_3 = \left\{\begin{array}{l} q \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2), \quad q(t) \neq q_i, \quad \text{for } i = 1, \dots, N+1, \\ q(t + \frac{k}{N}) = \begin{pmatrix} \cos(\frac{2k\pi}{N}) & -\sin(\frac{2k\pi}{N}) \\ \sin(\frac{2k\pi}{N}) & \cos(\frac{2k\pi}{N}) \end{pmatrix} q(t), \\ \deg(q(t) - q_i) = 1, \quad \text{for } i = 1, \dots, N, \qquad \deg(q(t) - q_{N+1}) = 1 \end{array}\right\}, \tag{1.7}$$

$$q \in \Lambda_4 = \left\{\begin{array}{l} q \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2), \quad q(t) \neq q_i, \quad \text{for } i = 1, \dots, N+1, \\ q(t + \frac{k}{N}) = \begin{pmatrix} \cos(\frac{2k\pi}{N}) & -\sin(\frac{2k\pi}{N}) \\ \sin(\frac{2k\pi}{N}) & \cos(\frac{2k\pi}{N}) \end{pmatrix} q(t), \\ \deg(q(t) - q_i) = 1, \quad \text{for } i = 1, \dots, N, \qquad \deg(q(t) - q_{N+1}) = N-1 \end{array}\right\}. \tag{1.8}$$

We have the following theorem.

Theorem 1.1 *For $\alpha \geq 2$, the minimizer of $f(q)$ on $\overline{\Lambda}_i$ $(i = 1,2,3,4)$ exists and it is a non-collision periodic solution of* (1.1) *or* (1.2)-(1.3) (*please see Figures* 1-4 *for* $N = 4$).

2 The proof of Theorem 1.1

We recall the following famous lemmas, which we need to prove Theorem 1.1.

Lemma 2.1 [9] *If $x \in W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$, $\alpha \geq 2$, $a > 0$, and there exists $t_0 \in [0,1]$ such that $x(t_0) = 0$, then $\int_0^1 [\frac{1}{2}|\dot{x}(t)|^2 + \frac{a}{|x(t)|^\alpha}]\, dt = +\infty$.*

If $x_n \rightharpoonup x$ in $W^{1,2}(R/Z, R^2)$ and $\exists t_0$, s.t. $x(t_0) = 0$, $\alpha \geq 2$, then $\int_0^1 \frac{1}{|x_n(t)|^\alpha}\, dt \to +\infty$.

Figure 1 $q \in \Lambda_1$.

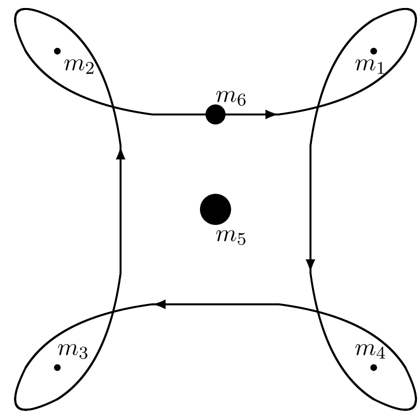


Figure 2 $q \in \Lambda_2$.

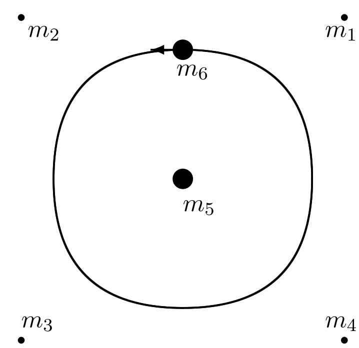
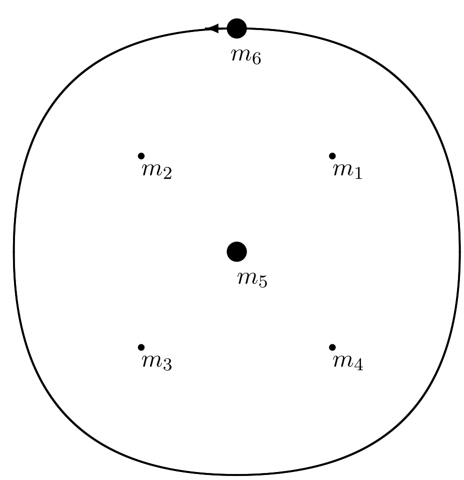


Figure 3 $q \in \Lambda_3$.



Lemma 2.2 (Palais's symmetry principle [17]) *Let σ be an orthogonal representation of a finite or compact group G on a real Hilbert space H, and let $f : H \to R$ be such that for $\forall\sigma \in G, f(\sigma \cdot x) = f(x)$. Set $H^G = \{x \in H : \sigma \cdot x = x, \forall\sigma \in G\}$. Then the critical point of f in H^G is also a critical point of f in H.*

Lemma 2.3 [5] *If X is a reflexive Banach space, M is a weakly closed subset of X, and $f : M \to R \cup \{+\infty\}, f \not\equiv +\infty$ is weakly lower semi-continuous and coercive, then f attains its infimum on M.*

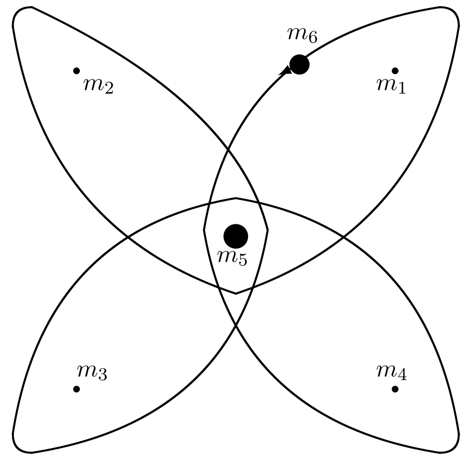


Figure 4 $q \in \Lambda_4$.

Lemma 2.4 (Poincare-Wirtinger inequality) *Let* $q \in W^{1,2}(\mathbb{R}/\mathbb{ZT}, \mathbb{R}^d)$ *and* $\int_0^T q(t)\,dt = 0$, *then* $\int_0^T |\dot{q}(t)|^2\,dt \geq (\frac{2\pi}{T})^2 \int_0^T q(t)^2\,dt$. *And the inequality takes the equality if and only if* $q(t) = \alpha \cos \frac{2\pi}{T} t + \beta \sin \frac{2\pi}{T} t$, $\alpha, \beta \in R^d$.

We now prove Theorem 1.1.

Proof By the symmetry of Λ_i, we know for $\forall x \in \Lambda_i$,

$$\int_0^T q(t)\,dt = 0. \tag{2.1}$$

If $q_n(t) \rightharpoonup q(t)$ in $\overline{\Lambda_i}$, then by Sobolev's compact embedding theorem, we have $q_n(t) \to q(t)$ in $C[0,1]$.

(i) If $q(t) \in \Lambda_i$, then $\lim_{n\to+\infty} \int_0^1 U(q_n(t))\,dt = \int_0^1 U(q_n(t))\,dt$. Since $\int_0^1 q_n\,dt = 0$, $\frac{1}{2}\int_0^1 |\dot{q}_n|^2\,dt$ can be regarded as the square of an equivalent norm for $W^{1,2}$, so it is weakly lower semi-continuous, so $\underline{\lim} f(q_n(t)) \geq f(q)$.

(ii) If $q(t) \in \partial\Lambda_i$, then by Lemma 2.1, $f(q) = +\infty$, we have $\int_0^1 U(q_n(t))\,dt \to +\infty$. So, $\underline{\lim}_{n\to+\infty} f(q_n) = +\infty \geq f(q)$. Hence f is w.l.s.c.

Using (2.1), we know that $f(q)$ is coercive on $\overline{\Lambda_i}$. Lemma 2.3 guarantees that $f(q)$ attains its infimum on $\overline{\Lambda_i}$. Let the minimizer be $\widetilde{q}$, then

$$f(\widetilde{q}) = \inf_{q\in\overline{\Lambda_i}} f(q) < +\infty. \tag{2.2}$$

If $\widetilde{q}$ is a collision periodic solution, then there exist $t_0 \in [0,1]$ and $j \in \{1,2,\dots,N,N+1\}$ such that $\widetilde{q}(t_0) = q_j$. Let $x(t) = \widetilde{q}(t) - q_j$ and note $x(t_0) = 0$. By Lemma 2.1, we have

$$\begin{aligned} f(\widetilde{q}) &= \int_0^1 \left[\frac{1}{2}\left|\dot{\widetilde{q}}(t)\right|^2 + \frac{m_j}{|\widetilde{q}(t) - q_j|^\alpha} + \sum_{i\neq j}^{N+1} \frac{m_i}{|\widetilde{q}(t) - q_i|^\alpha}\right] dt \\ &\geq \int_0^1 \left[\frac{1}{2}\left|\dot{x}(t)\right|^2 + \frac{m_j}{|x(t)|^\alpha}\right] dt = +\infty, \end{aligned} \tag{2.3}$$

which contradicts the inequality in (2.2). By Lemma 2.2, $\widetilde{q}(t)$ is the critical point of f in $W^{1,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2)$; therefore, $\widetilde{q}(t)$ is a non-collision periodic solution.

This completes the proof of Theorem 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read, checked and approved the final manuscript.

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Acknowledgements
The authors sincerely thank the referees for their many helpful comments and suggestions and also express their sincere gratitude to Professor Zhang Shiqing for his discussions and corrections. This work is supported by NSF of China and Youth Fund of Mianyang Normal University.

Received: 26 January 2013 Accepted: 2 May 2013 Published: 17 May 2013

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doi:10.1186/1687-2770-2013-129
Cite this article as: Zhao et al.: **Periodic solutions for $N+2$-body problems with $N+1$ fixed centers.** *Boundary Value Problems* 2013 **2013**:129.