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Existence of subharmonic solutions for non-quadratic second-order Hamiltonian systems

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Abstract

In this paper, some existence theorems are obtained for subharmonic solutions of second-order Hamiltonian systems with linear part under non-quadratic conditions. The approach is the minimax principle. We consider some new cases and obtain some new existence results.

MSC: 34C25; 58E50; 70H05

Keywords: second-order Hamiltonian systems; subharmonic solution; critical point; linking theorem

1 Introduction and main results

Consider the second-order Hamiltonian system

$$\ddot{u}(t) + Au(t) + \nabla F\big(t, u(t)\big) = 0 \quad \text{a.e. } t \in \mathbb{R}, \tag{1.1}$$

where A is an $N \times N$ symmetric matrix and $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t and satisfies the following assumption:

Assumption (A)′ *$F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ which is T-periodic and $b \in L^p(0,T;\mathbb{R}^+)$ with $p > 1$ such that*

$$\big|F(t,x)\big| \le a\big(|x|\big)b(t), \qquad \big|\nabla F(t,x)\big| \le a\big(|x|\big)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

When $A = 0$, system (1.1) reduces to the second-order Hamiltonian system

$$\ddot{u}(t) + \nabla F\big(t, u(t)\big) = 0 \quad \text{a.e. } t \in \mathbb{R}. \tag{1.2}$$

There have been many existence results for system (1.2) (for example, see [1–7] and references therein). In 1978, Rabinowitz [6] obtained the nonconstant periodic solutions for system (1.2) under the following AR-condition: there exist $\mu > 2$ and $L > 0$ such that

$$0 < \mu F(t,x) \le \big(\nabla F(t,x), x\big), \quad \forall |x| \ge L, t \in [0,T].$$

From then on, the condition has been used extensively in the literature; see [8–12] and the references therein. In [13], Fei also obtained the existence of nonconstant solutions for system (1.2) under a kind of new superquadratic condition. Subsequently, Tao and Tang [14] gave the following more general one than Fei's: there exist $\theta > 2$ and $\mu > \theta - 2$ such that

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^\theta} < \infty \quad \text{uniformly for a.e. } t \in [0,T], \tag{1.3}$$

$$\liminf_{|x|\to\infty} \frac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^\mu} > 0 \quad \text{uniformly for a.e. } t \in [0,T]. \tag{1.4}$$

They also considered the existence of subharmonic solutions and obtained the following result.

Theorem A (See [14], Theorem 2) *Suppose that F satisfies*

(A) *$F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that*

$$\left|F(t,x)\right| \le a\left(|x|\right)b(t), \qquad \left|\nabla F(t,x)\right| \le a\left(|x|\right)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Assume that (1.3), (1.4) *and the following conditions hold*:

$$F(t,x) \ge 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N, \tag{1.5}$$

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0 \quad \textit{uniformly for a.e. } t \in [0,T], \tag{1.6}$$

$$\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} > \frac{2\pi^2}{T^2} \quad \textit{uniformly for a.e. } t \in [0,T]. \tag{1.7}$$

Then system (1.2) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

Recently, Ma and Zhang [15] considered the following p-Laplacian system:

$$\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + \nabla F\left(t,u(t)\right) = 0 \quad \text{a.e. } t \in [0,T], \tag{1.8}$$

where $p > 1$. By using some techniques, they obtained the following more general result than Theorem A.

Theorem B (See [15], Theorem 1) *Suppose that F satisfies* (A), (1.3) *and* (1.4) *with* 2 *replaced by p*, (1.5) *and the following condition*:

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^p} = 0 < \lim_{|x|\to\infty} \frac{F(t,x)}{|x|^p} \quad \textit{uniformly for a.e. } t \in [0,T]. \tag{1.9}$$

Then system (1.8) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

When $A = m^2\omega^2 I_N$, where $\omega = 2\pi/T$ and I_N is the unit matrix of order N. Ye and Tang [16] obtained the following result.

Theorem C (See [16], Theorem 2) *Suppose that* $A = m^2\omega^2 I_N$, F *satisfies* (A), (1.3), (1.4), (1.5), (1.6) *and the following conditions*:

$$\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} > \frac{1+2m}{2}\omega^2 \quad \textit{uniformly for a.e. } t \in [0,T].$$

Then system (1.1) *has a sequence of distinct periodic solutions with period* $k_j T$ *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Recently, in [17], we considered a more general case than that in [16]. We considered the case that A only has 0 or $l_i^2\omega^2$ as its eigenvalues, where $\omega = 2\pi/T$, $l_i \in \mathbb{N}$, $i = 1,\dots,r$ and $0 \le r \le N$. In [17], we used the following condition which presents some advantages over (1.3) and (1.4):

(H) *there exist positive constants* m, ζ, η *and* $\nu \in [0,2)$ *such that*

$$\left(2 + \frac{1}{\zeta + \eta|x|^\nu}\right)F(t,x) \le \big(\nabla F(t,x), x\big), \quad x \in \mathbb{R}^N, |x| > m \text{ a.e. } t \in [0,T].$$

In this paper, we consider some new cases which can be seen as a continuance of our work in [17].

Next, we state our main results. Assume that $r \in \mathbb{N} \cup \{0\}$ and $r \le N$. Let $\lambda_i > 0$ ($i \in \{1,\dots,r\}$) and $-\lambda_i < 0$ ($i \in \{r+s+1,\dots,N\}$) be the positive and negative eigenvalues of A, respectively, where r and s denote the number of positive eigenvalues and zero eigenvalues of A (counted by multiplicity), respectively. Moreover, we denote by q the number of negative eigenvalues of A (counted by multiplicity). We make the following assumption:

Assumption (A0) *A has at least one nonzero eigenvalue and all positive eigenvalues are not equal to $l^2\omega^2$ for all $l \in \mathbb{N}$, where $\omega = 2\pi/T$, that is, $\lambda_i \ne l^2\omega^2$ ($i = 1,\dots,r$) for all $l \in \mathbb{N}$.*

The Assumption (A0) implies that one can find $l_i \in \mathbb{Z}^+ := \{0,1,2,\dots\}$ such that

$$l_i^2\omega^2 < \lambda_i < (l_i+1)^2\omega^2, \quad i = 1,\dots,r. \tag{1.10}$$

For the sake of convenience, we set

$$\lambda_{i^+} = \max\{\lambda_i | i = 1,\dots,r\}, \qquad \lambda_{i^-} = \min\{\lambda_i | i = 1,\dots,r\},$$
$$\lambda_{i_+} = \max\{\lambda_i | i = r+s+1,\dots,N\}, \qquad \lambda_{i_-} = \min\{\lambda_i | i = r+s+1,\dots,N\}.$$

Then

$$i^+, i^- \in \{1,\dots,r\}, \qquad i_+, i_- \in \{r+s+1,\dots,N\}.$$

Corresponding to (1.10), we know that there exist $l_{i^+}, l_{i^-} \in \mathbb{Z}^+$ such that

$$l_{i^+}^2\omega^2 < \lambda_{i^+} < (l_{i^+}+1)^2\omega^2, \qquad l_{i^-}^2\omega^2 < \lambda_{i^-} < (l_{i^-}+1)^2\omega^2.$$

Moreover, set

$$h_i = (l_i + 1)^2\omega^2 - \lambda_i, \quad i = 1, \dots, r,$$

and let $h_{i_0} = \min_{i\in\{1,\dots,r\}}\{h_i\}$. Then $i_0 \in \{1,\dots,r\}$. Corresponding to (1.10), there exists $l_{i_0} \in \mathbb{Z}^+$ such that

$$l_{i_0}^2\omega^2 < \lambda_{i_0} < (l_{i_0} + 1)^2\omega^2. \tag{1.11}$$

Theorem 1.1 *Assume that* (A0) *holds and F satisfies* (A)′, (1.5) *and the following conditions*.

(H1) *For some $k \in \mathbb{N}$, assume that k satisfies*

$$\left(l_i + 1 - \frac{1}{k}\right)^2\omega^2 \le \lambda_i < (l_i + 1)^2\omega^2 \quad \textit{for all } i \in \{1,\dots,r\}. \tag{1.12}$$

(H2) *There exist positive constants m, ζ, η and $\nu \in [0,2)$ such that*

$$\left(2 + \frac{1}{\zeta + \eta|x|^\nu}\right)F(t,x) \le \bigl(\nabla F(t,x), x\bigr), \quad x \in \mathbb{R}^N, |x| > m, \textit{ a.e. } t \in [0,T].$$

(H3) *Assume that one of the following cases holds*:

(1) *when $r > 0$, $s > 0$ and $r + s = N$, there exist $L_k > 0$ and $\beta_k > \min\{\frac{(l_{i_0}+1)^2\omega^2-\lambda_{i_0}}{2}, \frac{\omega^2}{2k^2}\}$ such that*

$$F(t,x) \ge \beta_k|x|^2, \quad \forall x \in \mathbb{R}^N, |x| > L_k, \textit{ a.e. } t \in [0,T], \tag{1.13}$$

where l_{i_0} and λ_{i_0} are defined by (1.11);

(2) *when $r > 0$, $s > 0$ and $r + s < N$, there exist $L_k > 0$ and $\beta_k > \min\{\frac{(l_{i_0}+1)^2\omega^2-\lambda_{i_0}}{2}, \frac{\omega^2}{2k^2}, \frac{\lambda_{i_-}}{2}\}$ such that* (1.13) *holds*;

(3) *when $r > 0$, $s = 0$ and $r + s < N$, there exist $L_k > 0$ and $\beta_k > \min\{\frac{(l_{i_0}+1)^2\omega^2-\lambda_{i_0}}{2}, \frac{\lambda_{i_-}}{2}\}$ such that* (1.13) *holds*;

(4) *when $r > 0$, $s = 0$ and $r = N$, there exist $L_k > 0$ and $\beta_k > \frac{(l_{i_0}+1)^2\omega^2-\lambda_{i_0}}{2}$ such that* (1.13) *holds*;

(5) *when $r = 0$, $s > 0$ and $s < N$, there exist $L_k > 0$ and $\beta_k > \min\{\frac{\omega^2}{2k^2}, \frac{\lambda_{i_-}}{2}\}$ such that* (1.13) *holds*;

(6) *when $r = 0$, $s = 0$ and $q = N$, there exist $L_k > 0$ and $\beta_k > \frac{\lambda_{i_-}}{2}$ such that* (1.13) *holds*;

(H4) *there exist $l_k > 0$ and $\alpha_k < \frac{\sigma_k}{2}$ such that*

$$F(t,x) \le \alpha_k|x|^2 \quad \textit{for all } |x| \le l_k \textit{ and a.e. } t \in [0,T],$$

where

$$\sigma_k = \min\left\{\min_{i\in\{1,\dots,r\}}\left\{\frac{(l_i+1)^2\omega^2 - \lambda_i}{(l_i+1)^2\omega^2 + 1}\right\}, \frac{\omega^2}{\omega^2 + k^2}\right\} \quad \textit{if } \text{(H3) (1)} \textit{ holds};$$

$$\sigma_k = \min\left\{\min_{i\in\{1,\dots,r\}}\left\{\frac{(l_i+1)^2\omega^2 - \lambda_i}{(l_i+1)^2\omega^2 + 1}\right\}, \frac{\omega^2}{\omega^2 + k^2}, \frac{\lambda_{i_-}}{1 + \lambda_{i_+}}\right\} \quad \textit{if } \text{(H3) (2)} \textit{ holds};$$

$$\sigma_k \equiv \sigma = \min\left\{\min_{i\in\{1,\dots,r\}}\left\{\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}\right\}, \frac{\lambda_{i_-}}{1+\lambda_{i_+}}\right\} \quad \textit{if } (\mathrm{H3})\ (3)\ \textit{holds};$$
$$\sigma_k \equiv \sigma = \min_{i\in\{1,\dots,N\}}\left\{\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}\right\} \quad \textit{if } (\mathrm{H3})\ (4)\ \textit{holds};$$
$$\sigma_k = \min\left\{\frac{\omega^2}{\omega^2+k^2}, \frac{\lambda_{i_-}}{1+\lambda_{i_+}}\right\} \quad \textit{if } (\mathrm{H3})\ (5)\ \textit{holds};$$
$$\sigma_k \equiv \sigma = \frac{\lambda_{i_-}}{1+\lambda_{i_+}} \quad \textit{if } (\mathrm{H3})\ (6)\ \textit{holds},$$

where σ implies that σ_k is independent of k. Then system (1.1) *has a nonzero kT-periodic solution. Especially, for cases* (H3)(1) *and* (H3)(4), *system* (1.1) *has a nonconstant kT-periodic solution.*

Remark 1.1 For cases (H3)(1)-(H3)(4), from (1.10) and (1.12), it is easy to see that the number of $k \in \mathbb{N}$ satisfying (1.12) is finite. Let $m \in K$ be the maximum integer satisfying (1.12), where

$$K = \{k \in \mathbb{N} \mid k \text{ satisfies } (1.12)\}.$$

Then $K = \{1, 2, \dots, m\}$. Hence, Theorem 1.1 implies that system (1.1) has nonzero kT-periodic solutions ($k = 1, 2, \dots, m$). For cases (H3)(5) and (H3)(6), since $r = 0$, (1.12) holds for every $k \in \mathbb{N}$. Hence, Theorem 1.1 implies that system (1.11) has nonzero kT-periodic solutions for every $k \in \mathbb{N}$.

Remark 1.2 In [18], Costa and Magalhães studied the first-order Hamiltonian system

$$-J\dot{u}(t) + Au + \nabla H(t,u) = 0 \quad \text{a.e. } t \in [0,T]. \tag{1.14}$$

They obtained that system (1.14) has a $T = 2\pi$ periodic solution under the following non-quadraticity conditions:

$$\liminf_{|x|\to\infty} \frac{(x, \nabla H(t,x)) - 2H(t,x)}{|x|^\mu} \ge a > 0 \quad \text{uniformly for a.e. } t \in [0,2\pi], \tag{1.15}$$

and the so-called asymptotic noncrossing conditions

$$\lambda_{k-1} < \liminf_{|x|\to\infty} \frac{2H(t,x)}{|x|^2} \le \limsup_{|x|\to\infty} \frac{2H(t,x)}{|x|^2} \le \lambda_k \quad \text{uniformly for a.e. } t \in [0,2\pi],$$

where $\lambda_{k-1} < \lambda_k$ are consecutive eigenvalues of the operator $L = -Jd/dt - A$. Moreover, they also obtained system (1.14) has a nonzero $T = 2\pi$ periodic solution under (1.15) and the called crossing conditions

$$H(t,u) \ge \frac{1}{2}\lambda_{k-1}|x|^2 \quad \text{for all } (t,u) \in [0,2\pi] \times \mathbb{R}^{2N},$$
$$\limsup_{|x|\to 0} \frac{2H(t,x)}{|x|^2} \le \alpha < \lambda_k < \beta \le \liminf_{|x|\to\infty} \frac{2H(t,x)}{|x|^2} \quad \text{uniformly for } t \in [0,2\pi].$$

One can also establish the similar results for the second-order Hamiltonian system (1.1). Some related contents can be seen in [19]. It is worth noting that in [18] and [19], $\lambda_{k-1} < \lambda_k$ are consecutive eigenvalues of the operator $L = -Jd/dt - A$ or $-d^2/dt^2 + A$. In our Theorem 1.1 and Theorem 1.2, we study the existence of subharmonic solutions for system (1.1) from a different perspective. λ_i ($i \in \{1, \dots, r\}$) in our theorems are the eigenvalues of the matrix A. Obviously, it is much easier to seek the eigenvalue of a matrix. In Section 4, we present an interesting example satisfying our Theorem 1.1 but not satisfying the theorem in [19].

Theorem 1.2 *Suppose that* (A0) *holds and* F *satisfies* (A)$'$, (1.5), (H2) *and the following conditions*:

(H3)$'$ *when* $r = 0$, $s > 0$ *and* $s < N$, *there exist* $L > 0$ *and* $\beta > \frac{\omega^2}{2}$ *such that*

$$F(t,x) \geq \beta |x|^2, \quad \forall x \in \mathbb{R}^N, |x| > L, \text{ a.e. } t \in [0,T]; \tag{1.16}$$

(H4)$'$

$$\lim_{|x| \to 0} \frac{F(t,x)}{|x|^2} = 0 \quad \text{uniformly for a.e. } t \in [0,T].$$

Then system (1.1) *has a sequence of distinct periodic solutions with period* $k_j T$ *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

In the final theorem, we present a result about the existence of subharmonic solutions for system (1.8). Using a condition like (H2) and similar to the argument of Remark 1.1 in [17], we can improve Theorem B.

Theorem 1.3 *Suppose that* F *satisfies* (A), (1.5) *and the following conditions*:

(H5) *there exist positive constants* m, ζ, η *and* $\nu \in [0,p)$ *such that*

$$\left(p + \frac{1}{\zeta + \eta |x|^\nu}\right) F(t,x) \leq \big(\nabla F(t,x), x\big), \quad x \in \mathbb{R}^N, |x| > m \text{ a.e. } t \in [0,T];$$

(H6)

$$\lim_{|x| \to 0} \frac{F(t,x)}{|x|^p} = 0 < \lim_{|x| \to \infty} \frac{F(t,x)}{|x|^p} \quad \text{uniformly for a.e. } t \in [0,T].$$

Then system (1.8) *has a sequence of distinct nonconstant periodic solutions with period* $k_j T$ *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

2 Some preliminaries

Let

$$H^1_{kT} = \big\{u : \mathbb{R} \to \mathbb{R}^N | u \text{ be absolutely continuous}, u(t) = u(t + kT) \text{ and } \dot{u} \in L^2\big([0,kT]\big)\big\}.$$

Then H^1_{kT} is a Hilbert space with the inner product and the norm defined by

$$\langle u, v \rangle = \int_0^{kT} \big(u(t), v(t)\big)\, dt + \int_0^{kT} \big(\dot{u}(t), \dot{v}(t)\big)\, dt$$

and

$$\|u\| = \left[\int_0^{kT} |u(t)|^2\,dt + \int_0^{kT} |\dot{u}(t)|^2\,dt\right]^{1/2}$$

for each $u, v \in H^1_{kT}$. Let

$$\bar{u} = \frac{1}{kT}\int_0^{kT} u(t)\,dt \quad \text{and} \quad \tilde{u}(t) = u(t) - \bar{u}.$$

Then one has

$$\|\tilde{u}\|^2_\infty \le \frac{kT}{12}\int_0^{kT} |\dot{u}(t)|^2\,dt \quad \text{(Sobolev's inequality)},$$
$$\|\tilde{u}\|^2_{L^2} \le \frac{k^2T^2}{4\pi^2}\int_0^{kT} |\dot{u}(t)|^2\,dt \quad \text{(Wirtinger's inequality)}$$

(see Proposition 1.3 in [1]).

Lemma 2.1 *If* $u \in H^1_{kT}$, *then*

$$\|u\|_\infty \le \sqrt{\frac{12 + k^2T^2}{12kT}}\,\|u\|,$$

where $\|u\|_\infty = \max_{t\in[0,kT]} |u(t)|$.

Proof Fix $t \in [0, kT]$. For every $\tau \in [0, kT]$, we have

$$u(t) = u(\tau) + \int_\tau^t \dot{u}(s)\,ds. \tag{2.1}$$

Set

$$\phi(s) = \begin{cases} s - t + \frac{kT}{2}, & t - kT/2 \le s \le t, \\ t + \frac{kT}{2} - s, & t \le s \le t + kT/2. \end{cases}$$

Integrating (2.1) over $[t - kT/2, t + kT/2]$ and using the Hölder inequality, we obtain

$$\begin{aligned} kT|u(t)| &= \left|\int_{t-kT/2}^{t+kT/2} u(\tau)\,d\tau + \int_{t-kT/2}^{t+kT/2}\int_\tau^t \dot{u}(s)\,ds\,d\tau\right| \\ &\le \int_{t-kT/2}^{t+kT/2} |u(\tau)|\,d\tau + \int_{t-kT/2}^{t}\int_\tau^t |\dot{u}(s)|\,ds\,d\tau + \int_t^{t+kT/2}\int_t^\tau |\dot{u}(s)|\,ds\,d\tau \\ &= \int_{t-kT/2}^{t+kT/2} |u(\tau)|\,d\tau + \int_{t-kT/2}^{t}\left(s - t + \frac{kT}{2}\right)|\dot{u}(s)|\,ds \\ &\quad + \int_t^{t+kT/2}\left(t + \frac{kT}{2} - s\right)|\dot{u}(s)|\,ds \\ &= \int_{t-kT/2}^{t+kT/2} |u(\tau)|\,d\tau + \int_{t-kT/2}^{t+kT/2} \phi(s)|\dot{u}(s)|\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq (kT)^{1/2}\left(\int_{t-kT/2}^{t+kT/2}\left|u(\tau)\right|^{2}d\tau\right)^{1/2} \\
&\quad+\left(\int_{t-kT/2}^{t+kT/2}\left[\phi(s)\right]^{2}ds\right)^{1/2}\left(\int_{t-kT/2}^{t+kT/2}\left|\dot{u}(s)\right|^{2}ds\right)^{1/2} \\
&= (kT)^{1/2}\left(\int_{t-kT/2}^{t+kT/2}\left|u(\tau)\right|^{2}d\tau\right)^{1/2}+\frac{(kT)^{3/2}}{2\sqrt{3}}\left(\int_{t-kT/2}^{t+kT/2}\left|\dot{u}(s)\right|^{2}ds\right)^{1/2} \\
&\leq \left(kT+\frac{(kT)^{3}}{12}\right)^{1/2}\left(\int_{t-kT/2}^{t+kT/2}\left|u(\tau)\right|^{2}d\tau+\int_{t-kT/2}^{t+kT/2}\left|\dot{u}(s)\right|^{2}ds\right)^{1/2} \\
&= \left(kT+\frac{(kT)^{3}}{12}\right)^{1/2}\left(\int_{0}^{kT}\left|u(\tau)\right|^{2}d\tau+\int_{0}^{kT}\left|\dot{u}(s)\right|^{2}ds\right)^{1/2}.
\end{aligned}
$$

Hence, we have

$$
\|u\|_{\infty}\leq\left(\frac{1}{kT}+\frac{kT}{12}\right)^{1/2}\left(\int_{0}^{kT}\left|u(s)\right|^{2}ds+\int_{0}^{kT}\left|\dot{u}(s)\right|^{2}ds\right)^{1/2}.
$$

The proof is complete. □

Lemma 2.2 (see [17, Lemma 2.2]) *Assume that $F = F(t,x) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is T-periodic in t, $F(t,x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$. If there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b \in L^p([0,T],\mathbb{R}^+)$ $(p > 1)$ such that*

$$
\left|\nabla F(t,x)\right|\leq a\big(|x|\big)b(t),\quad \forall x\in\mathbb{R}^N,\ a.e.\ t\in[0,T], \tag{2.2}
$$

then

$$
c(u)=\int_{0}^{kT}F\big(t,u(t)\big)\,dt
$$

is weakly continuous and uniformly differentiable on bounded subsets of H^1_{kT}.

Remark 2.1 In [17, Lemma 2.2], $F \in C^1(\mathbb{R},\mathbb{R}^N)$. In fact, in its proof, it is not essential that F is continuously differentiable in t.

We use Lemma 2.3 below due to Benci and Rabinowitz [20] to prove our results.

Lemma 2.3 (see [20] or [5, Theorem 5.29]) *Let E be a real Hilbert space with $E = E_1 \oplus E_2$ and $E_2 = E_1^{\perp}$. Suppose that $\varphi \in C^1(E,\mathbb{R})$ satisfies (PS)-condition, and*

(I_1) *$\varphi(u) = 1/2(\Phi u,u) + b(u)$, where $\Phi u = \Phi_1 P_1 u + \Phi_2 P_2 u$ and $\Phi_i : E_i \to E_i$ bounded and self-adjoint, $i = 1,2$;*
(I_2) *b' is compact, and*
(I_3) *there exists a subspace $\tilde{E} \subset E$ and sets $S \subset E$, $Q \subset \tilde{E}$ and constants $\alpha > \beta$ such that*
  (i) *$S \subset E_1$ and $\varphi|_S \geq \alpha$,*
  (ii) *Q is bounded and $\varphi|_{\partial Q} \leq \beta$,*
  (iii) *S and ∂Q link.*

Then φ possesses a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{h\in\Gamma} \sup_{u\in Q} \varphi\bigl(h(1,u)\bigr),$$

where

$$\Gamma \equiv \bigl\{h \in C\bigl([0,1] \times E, E\bigr) | h \text{ satisfies the following } (\Gamma_1)\text{-}(\Gamma_3)\bigr\},$$

(Γ_1) $h(0,u) = u$,
(Γ_2) $h(t,u) = u$ *for* $u \in \partial Q$, *and*
(Γ_3) $h(t,u) = e^{\theta(t,u)\Phi}u + K(t,u)$, *where* $\theta \in C([0,1] \times E, \mathbb{R})$ *and* K *is compact.*

Remark 2.2 As shown in [21], a deformation lemma can be proved with replacing the usual (PS)-condition with condition (C), and it turns out that Lemma 2.3 holds true under condition (C). We say φ satisfies condition (C), *i.e.*, for every sequence $\{u_n\} \subset H_T^1$, $\{u_n\}$ has a convergent subsequence if $\varphi(u_n)$ is bounded and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$.

3 Proofs of theorems

Proof of Theorem 1.1 It follows from Assumption (A)′ that the functional φ_k on H_{kT}^1 given by

$$\varphi_k(u) = \frac{1}{2}\int_0^{kT} \bigl|\dot{u}(t)\bigr|^2 \,dt - \frac{1}{2}\int_0^{kT} \bigl(Au(t), u(t)\bigr)\,dt - \int_0^{kT} F\bigl(t, u(t)\bigr)\,dt$$

is continuously differentiable. Moreover, one has

$$\bigl\langle\varphi_k'(u), v\bigr\rangle = \int_0^{kT} \bigl[\bigl(\dot{u}(t), \dot{v}(t)\bigr) - \bigl(Au(t), v(t)\bigr) - \bigl(\nabla F\bigl(t, u(t)\bigr), v(t)\bigr)\bigr]\,dt$$

for $u, v \in H_{kT}^1$ and the solutions of system (1.1) correspond to the critical points of φ_k (see [1]).

Obviously, there exists an orthogonal matrix Q such that

$$Q^\tau AQ = B = \begin{pmatrix} \lambda_1 &&&&&&&& \\ & \ddots &&&&&&& \\ && \lambda_r &&&&&& \\ &&& 0 &&&&& \\ &&&& \ddots &&&& \\ &&&&& 0 &&& \\ &&&&&& -\lambda_{r+s+1} && \\ &&&&&&& \ddots & \\ &&&&&&&& -\lambda_N \end{pmatrix} \tag{3.1}$$

Let $u = Qw$. Then by (1.1),

$$Q\ddot{w}(t) + AQw(t) + \nabla F\bigl(t, Qw(t)\bigr) = 0 \quad \text{a.e. } t \in \mathbb{R}.$$

Furthermore

$$\ddot{w}(t) + Q^{-1}AQw(t) + Q^{-1}\nabla F\big(t, Qw(t)\big) = 0 \quad \text{a.e. } t \in \mathbb{R},$$

that is,

$$\ddot{w}(t) + Bw(t) + Q^{-1}\nabla F\big(t, Qw(t)\big) = 0 \quad \text{a.e. } t \in \mathbb{R}. \tag{3.2}$$

Let $G(t,w) = F(t,Qw)$ and then $\nabla G(t,w) = Q^{-1}\nabla F(t,Qw(t))$. Let

$$\psi_k(w) = \frac{1}{2}\int_0^{kT} \big|\dot{w}(t)\big|^2\,dt - \frac{1}{2}\int_0^{kT} \big(Bw(t), w(t)\big)\,dt - \int_0^{kT} G\big(t, w(t)\big)\,dt.$$

Then the critical points of ψ_k correspond to solutions of system (3.2). It is easy to verify that $\varphi_k(u) = \psi_k(w)$ and G satisfies all the conditions of Theorem 1.1 and Theorem 1.2 if F satisfies them. Hence, w is the critical point of ψ_k if and only if $u = Qw$ is the critical point of φ_k. Therefore, we only need to consider the special case that $A = B$ is the diagonal matrix defined by (3.1). We divide the proof into six steps.

Step 1: Decompose the space H^1_{kT}. Let

$$I_N = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = (e_1, e_2, \ldots, e_N).$$

Note that

$$H^1_{kT} \subset \left\{ \sum_{i=0}^{\infty} \big(c_i \cos ik^{-1}\omega t + d_i \sin ik^{-1}\omega t\big) | c_i, d_i \in \mathbb{R}^N, i = 0, 1, 2 \cdots \right\}.$$

Define

$$H^-_{kT} = \left\{ u \in H^1_{kT} | u = u(t) = \sum_{i=1}^{r} e_i \sum_{j=0}^{kl_i} \big(c_{ij} \cos jk^{-1}\omega t + d_{ij} \sin jk^{-1}\omega t\big), c_{ij}, d_{ij} \in \mathbb{R} \right\},$$

$$H^0_{kT} = \left\{ u \in H^1_{kT} | u = u(t) = \sum_{i=r+1}^{r+s} e_i \sum_{j=0}^{\infty} \big(c_{ij} \cos jk^{-1}\omega t + d_{ij} \sin jk^{-1}\omega t\big), c_{ij}, d_{ij} \in \mathbb{R} \right\},$$

$$\begin{aligned} H^+_{kT} = \Bigg\{ u \in H^1_{kT} | u = u(t) &= \sum_{i=1}^{r} e_i \sum_{j=kl_i+1}^{\infty} \big(c_{ij} \cos jk^{-1}\omega t + d_{ij} \sin jk^{-1}\omega t\big) \\ &+ \sum_{i=r+s+1}^{N} e_i \sum_{j=0}^{\infty} \big(c_{ij} \cos jk^{-1}\omega t + d_{ij} \sin jk^{-1}\omega t\big), c_{ij}, d_{ij} \in \mathbb{R} \Bigg\}. \end{aligned}$$

Then H^-_{kT}, H^0_{kT} and H^+_{kT} are closed subsets of H^1_{kT} and

(1)

$$H^1_{kT} = H^-_{kT} \oplus H^0_{kT} \oplus H^+_{kT};$$

(2)

$$P_k(u,v)=0,\quad \forall u\in H_{kT}^-, v\in H_{kT}^0\oplus H_{kT}^+,\ \text{or}$$
$$P_k(u,v)=0,\quad \forall u\in H_{kT}^0, v\in H_{kT}^-\oplus H_{kT}^+,\ \text{or}$$
$$P_k(u,v)=0,\quad \forall u\in H_{kT}^+, v\in H_{kT}^-\oplus H_{kT}^0,$$

where

$$P_k(u,v)=\int_0^{kT}\bigl[\bigl(\dot{u}(t),\dot{v}(t)\bigr)-\bigl(Au(t),v(t)\bigr)\bigr]\,dt,\quad \forall u,v\in H_{kT}^1.$$

Let

$$H_{kT}^{01}=\left\{u\in H_{kT}^0 | u=\sum_{i=r+1}^{r+s}c_{i0}e_i, c_{i0}\in\mathbb{R}\right\},$$
$$H_{kT}^{02}=\left\{u\in H_{kT}^0 | u=u(t)=\sum_{i=r+1}^{r+s}e_i\sum_{j=1}^{\infty}\bigl(c_{ij}\cos jk^{-1}\omega t+d_{ij}\sin jk^{-1}\omega t\bigr), c_{ij},d_{ij}\in\mathbb{R}\right\},$$
$$H_{kT}^{+1}=\left\{u\in H_{kT}^+ | u=u(t)=\sum_{i=1}^{r}e_i\sum_{j=kl_i+1}^{kl_i+k-1}\bigl(c_{ij}\cos jk^{-1}\omega t+d_{ij}\sin jk^{-1}\omega t\bigr), c_{ij},d_{ij}\in\mathbb{R}\right\},$$
$$H_{kT}^{+2}=\left\{u\in H_{kT}^+ | u=u(t)=\sum_{i=1}^{r}e_i\sum_{j=kl_i+k}^{\infty}\bigl(c_{ij}\cos jk^{-1}\omega t+d_{ij}\sin jk^{-1}\omega t\bigr)\right.$$
$$\left.+\sum_{i=r+s+1}^{N}e_i\sum_{j=0}^{\infty}\bigl(c_{ij}\cos jk^{-1}\omega t+d_{ij}\sin jk^{-1}\omega t\bigr), c_{ij},d_{ij}\in\mathbb{R}\right\}.$$

Then

$$H_{kT}^0=H_{kT}^{01}\oplus H_{kT}^{02},\qquad H_{kT}^+=H_{kT}^{+1}\oplus H_{kT}^{+2},\qquad H_{kT}^1=H_{kT}^-\oplus H_{kT}^{01}\oplus H_{kT}^{02}\oplus H_{kT}^{+1}\oplus H_{kT}^{+2}$$

and

$$P_k(u,v)=0,\quad \forall u\in H_{kT}^{+1}, \forall v\in H_{kT}^{+2}.$$

Remark 3.1 When $k=1$, it is easy to see $H_T^{+1}=\{0\}$.

Step 2: Let

$$q_k(u)=\frac{1}{2}\int_0^{kT}\bigl[\bigl|\dot{u}(t)\bigr|^2-\bigl(Au(t),u(t)\bigr)\bigr]\,dt.$$

Next we consider the relationship between $q_k(u)$ and $\|u\|$ on those subspaces defined above. We only consider the case that (H3)(2) holds. For others, the conclusions are easy to be seen from the argument of this case.

(a) For $\forall u \in H_{kT}^{-}$, since

$$u = u(t) = \sum_{i=1}^{r} e_i \sum_{j=0}^{kl_i} \big(c_{ij} \cos jk^{-1}\omega t + d_{ij} \sin jk^{-1}\omega t\big),$$

then

$$\begin{aligned} q_k(u) &= \frac{1}{2}\int_0^{kT} \big[\big|\dot{u}(t)\big|^2 - \big(Au(t), u(t)\big)\big]\,dt \\ &= \frac{1}{2}\int_0^{kT} \Bigg[\Bigg(\sum_{i=1}^{r} e_i \sum_{j=0}^{kl_i} jk^{-1}\omega\big(d_{ij}\cos jk^{-1}\omega t - c_{ij}\sin jk^{-1}\omega t\big), \\ &\quad \sum_{i=1}^{r} e_i \sum_{j=0}^{kl_i} jk^{-1}\omega\big(d_{ij}\cos jk^{-1}\omega t - c_{ij}\sin jk^{-1}\omega t\big)\Bigg) \\ &\quad - \Bigg(\sum_{i=1}^{r} Ae_i \sum_{j=0}^{kl_i} \big(c_{ij}\cos jk^{-1}\omega t + d_{ij}\sin jk^{-1}\omega t\big), \\ &\quad \sum_{i=1}^{r} e_i \sum_{j=0}^{kl_i} \big(c_{ij}\cos jk^{-1}\omega t + d_{ij}\sin jk^{-1}\omega t\big)\Bigg)\Bigg]\,dt \\ &= \frac{1}{2}\sum_{i=1}^{r}\int_0^{kT} \Bigg\{\Bigg[\sum_{j=0}^{kl_i} jk^{-1}\omega\big(d_{ij}\cos jk^{-1}\omega t - c_{ij}\sin jk^{-1}\omega t\big)\Bigg]^2 \\ &\quad - \lambda_i\Bigg[\sum_{j=0}^{kl_i}\big(c_{ij}\cos jk^{-1}\omega t + d_{ij}\sin jk^{-1}\omega t\big)\Bigg]^2\Bigg\}\,dt \\ &= \frac{kT}{4}\sum_{i=1}^{r}\sum_{j=0}^{kl_i}\big[\big(jk^{-1}\omega\big)^2 - \lambda_i\big]\big(c_{ij}^2 + d_{ij}^2\big) \end{aligned}$$

and

$$\|u\|^2 = \int_0^{kT}\big(\big|\dot{u}(t)\big|^2 + \big|u(t)\big|^2\big)\,dt = \frac{kT}{2}\sum_{i=1}^{r}\sum_{j=0}^{kl_i}\big[\big(jk^{-1}\omega\big)^2 + 1\big]\big(c_{ij}^2 + d_{ij}^2\big).$$

Let

$$\delta = \min_{i\in\{1,\dots,r\}}\left\{\frac{\lambda_i - (l_i\omega)^2}{(l_i\omega)^2 + 1}\right\} > 0.$$

Then

$$q_k(u) \le -\frac{\delta}{2}\|u\|^2, \quad \forall u \in H_{kT}^{-}. \tag{3.3}$$

Remark 3.2 Obviously, if one of (H3)(5) and (H3)(6) holds, then $H_{kT}^{-} = \{0\}$. Hence,

$$q_k(u) = 0, \quad \forall u \in H_{kT}^{-}.$$

(b) For $\forall u \in H_{kT}^{+2} \oplus H_{kT}^{02}$, let

$$u = u(t) = u_1(t) + u_2(t) + u_3(t),$$

where

$$\begin{aligned}
u_1(t) &= \sum_{i=1}^{r} e_i \sum_{j=kl_i+k}^{\infty} \left(c_{ij}\cos jk^{-1}\omega t + d_{ij}\sin jk^{-1}\omega t\right),\\
u_2(t) &= \sum_{i=r+s+1}^{N} e_i \sum_{j=0}^{\infty} \left(c_{ij}\cos jk^{-1}\omega t + d_{ij}\sin jk^{-1}\omega t\right),\\
u_3(t) &= \sum_{i=r+1}^{r+s} e_i \sum_{j=1}^{\infty} \left(c_{ij}\cos jk^{-1}\omega t + d_{ij}\sin jk^{-1}\omega t\right).
\end{aligned}$$

Then

$$\begin{aligned}
q_k(u) &= \frac{1}{2}\int_0^{kT} \left[\left|\dot u(t)\right|^2 - \left(Au(t), u(t)\right)\right] dt\\
&= \frac{1}{2}\int_0^{kT} \Big[\left(\dot u_1(t) + \dot u_2(t) + \dot u_3(t), \dot u_1(t) + \dot u_2(t) + \dot u_3(t)\right)\\
&\quad - \left(Au_1(t) + Au_2(t) + Au_3(t), u_1(t) + u_2(t) + u_3(t)\right)\Big] dt\\
&= \frac{1}{2}\int_0^{kT} \Big[\left(\dot u_1(t), \dot u_1(t)\right) + \left(\dot u_2(t), \dot u_2(t)\right) + \left(\dot u_3(t), \dot u_3(t)\right)\\
&\quad - \left(Au_1(t), u_1(t)\right) - \left(Au_2(t), u_2(t)\right) - \left(Au_3(t), u_3(t)\right)\Big]\\
&= \frac{kT}{4}\Bigg[\sum_{i=1}^{r}\sum_{j=kl_i+k}^{\infty}\left(jk^{-1}\omega\right)^2\left(c_{ij}^2 + d_{ij}^2\right) + \sum_{i=r+s+1}^{N}\sum_{j=0}^{\infty}\left(jk^{-1}\omega\right)^2\left(c_{ij}^2 + d_{ij}^2\right)\\
&\quad + \sum_{i=r+1}^{r+s}\sum_{j=1}^{\infty}\left(jk^{-1}\omega\right)^2\left(c_{ij}^2 + d_{ij}^2\right)\\
&\quad - \sum_{i=1}^{r}\lambda_i\sum_{j=kl_i+k}^{\infty}\left(c_{ij}^2 + d_{ij}^2\right) + \sum_{i=r+s+1}^{N}\lambda_i\sum_{j=0}^{\infty}\left(c_{ij}^2 + d_{ij}^2\right)\Bigg]\\
&= \frac{kT}{4}\Bigg\{\sum_{i=1}^{r}\sum_{j=kl_i+k}^{\infty}\left[\left(jk^{-1}\omega\right)^2 - \lambda_i\right]\left(c_{ij}^2 + d_{ij}^2\right) + \sum_{i=r+s+1}^{N}\sum_{j=0}^{\infty}\left[\left(jk^{-1}\omega\right)^2 + \lambda_i\right]\left(c_{ij}^2 + d_{ij}^2\right)\\
&\quad + \sum_{i=r+1}^{r+s}\sum_{j=1}^{\infty}\left(jk^{-1}\omega\right)^2\left(c_{ij}^2 + d_{ij}^2\right)\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
\|u\|^2 &= \int_0^{kT}\left(\left|\dot u(t)\right|^2 + \left|u(t)\right|^2\right) dt\\
&= \frac{kT}{2}\Bigg\{\sum_{i=1}^{r}\sum_{j=kl_i+k}^{\infty}\left[\left(jk^{-1}\omega\right)^2 + 1\right]\left(c_{ij}^2 + d_{ij}^2\right)
\end{aligned}$$

$$+\sum_{i=r+s+1}^{N}\sum_{j=0}^{\infty}\bigl[\bigl(jk^{-1}\omega\bigr)^2+1\bigr]\bigl(c_{ij}^2+d_{ij}^2\bigr)$$
$$\left.+\sum_{i=r+1}^{r+s}\sum_{j=1}^{\infty}\bigl[\bigl(jk^{-1}\omega\bigr)^2+1\bigr]\bigl(c_{ij}^2+d_{ij}^2\bigr)\right\}.$$

Since for fixed $i \in \{1,\dots,r\}$,

$$f(j)=\frac{(jk^{-1}\omega)^2-\lambda_i}{(jk^{-1}\omega)^2+1}\quad\text{and}\quad g(j)=\frac{(jk^{-1}\omega)^2}{(jk^{-1}\omega)^2+1}$$

are strictly increasing on $j \in \mathbb{N}$,

$$f(j)\geq f(kl_i+k)=\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}>0,\quad \forall j\geq kl_i+k$$

and

$$g(j)\geq g(1)=\frac{(k^{-1}\omega)^2}{(k^{-1}\omega)^2+1}=\frac{\omega^2}{\omega^2+k^2}>0.$$

Moreover, it is easy to verify that

$$\frac{(jk^{-1}\omega)^2+\lambda_i}{(jk^{-1}\omega)^2+1}\geq\frac{\lambda_{i_-}}{1+\lambda_{i_+}},\quad \forall j\in\mathbb{N}\cup\{0\}, i=r+s+1,\dots,N.$$

Let

$$\sigma_k=\min\left\{\min_{i\in\{1,\dots,r\}}\left\{\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}\right\},\frac{\omega^2}{\omega^2+k^2},\frac{\lambda_{i_-}}{1+\lambda_{i_+}}\right\}.$$

Then

$$q_k(u)\geq\frac{\sigma_k}{2}\|u\|^2,\quad \forall u\in H_{kT}^{+2}\oplus H_{kT}^{02}. \tag{3.4}$$

Remark 3.3 From the above discussion, it is easy to see the following conclusions:

(i) if (H3)(1) holds, then (3.4) holds with

$$\sigma_k=\min\left\{\min_{i\in\{1,\dots,r\}}\left\{\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}\right\},\frac{\omega^2}{\omega^2+k^2}\right\};$$

(ii) if (H3)(2) holds, then (3.4) holds with

$$\sigma_k=\min\left\{\min_{i\in\{1,\dots,r\}}\left\{\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}\right\},\frac{\omega^2}{\omega^2+k^2},\frac{\lambda_{i_-}}{1+\lambda_{i_+}}\right\};$$

(iii) if (H3)(3) holds, then (3.4) holds with

$$\sigma_k\equiv\sigma=\min\left\{\min_{i\in\{1,\dots,r\}}\left\{\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}\right\},\frac{\lambda_{i_-}}{1+\lambda_{i_+}}\right\};$$

(iv) if (H3)(4) holds, then (3.4) holds with

$$\sigma_k \equiv \sigma = \min_{i\in\{1,\dots,N\}} \left\{ \frac{(l_i+1)^2\omega^2 - \lambda_i}{(l_i+1)^2\omega^2 + 1} \right\};$$

(v) if (H3)(5) holds, then (3.4) holds with

$$\sigma_k = \min\left\{ \frac{\omega^2}{\omega^2+k^2}, \frac{\lambda_{i_-}}{1+\lambda_{i_+}} \right\};$$

(vi) if (H3)(6) holds, then (3.4) holds with

$$\sigma_k \equiv \sigma = \frac{\lambda_{i_-}}{1+\lambda_{i_+}}.$$

(c) For $\forall u \in H_{kT}^{+1}$, since

$$u = \sum_{i=1}^{r} e_i \sum_{j=kl_i+1}^{kl_i+k-1} \bigl(c_{ij}\cos jk^{-1}\omega t + d_{ij}\sin jk^{-1}\omega t\bigr),$$

$$q_k(u) = \frac{kT}{4} \sum_{i=1}^{r} \sum_{j=kl_i+1}^{kl_i+k-1} \bigl[\bigl(jk^{-1}\omega\bigr)^2 - \lambda_i\bigr]\bigl(c_{ij}^2 + d_{ij}^2\bigr)$$

and

$$\|u\|^2 = \frac{kT}{2} \sum_{i=1}^{r} \sum_{j=kl_i+1}^{kl_i+k-1} \bigl[\bigl(jk^{-1}\omega\bigr)^2 + 1\bigr]\bigl(c_{ij}^2 + d_{ij}^2\bigr).$$

Obviously, when $k = 1$, $u = 0$. So $q_1(u) = 0$. When $k > 1$, it follows from

$$\left(l_i + 1 - \frac{1}{k}\right)^2 \omega^2 \le \lambda_i < (l_i+1)^2\omega^2, \quad \forall i \in \{1,\dots,r\}$$

that

$$q_k(u) \le 0, \quad \forall u \in H_{kT}^{+1}. \tag{3.5}$$

(d) Obviously, for $\forall u \in H_{kT}^{01}$, we have

$$q_k(u) = 0, \quad \forall u \in H_{kT}^{01}. \tag{3.6}$$

Step 3: Assume that (H3)(2) holds. We prove that there exist $\rho_k > 0$ and $b_k > 0$ such that

$$\varphi_k(u) \ge b_k > 0, \quad \forall u \in \bigl(H_{kT}^{+2} \oplus H_{kT}^{02}\bigr) \cap \partial B_{\rho_k}.$$

Let

$$C_k = \sqrt{\frac{12 + k^2T^2}{12kT}}.$$

Choosing $\rho_k = \min\{1, l_k/C_k\} > 0$ and $b_k = (\frac{\sigma_k}{2} - \alpha_k)\rho_k^2 > 0$, by Lemma 2.1, (H4) and (3.4), we have, for all $u \in (H_{kT}^{+2} \oplus H_{kT}^{02}) \cap \partial B_{\rho_k}$,

$$\begin{aligned}
\varphi_k(u) &\geq \frac{1}{2}\int_0^{kT} \left|\dot{u}(t)\right|^2 dt - \frac{1}{2}\int_0^{kT} \big(Au(t), u(t)\big)\, dt - \int_0^{kT} F\big(t, u(t)\big)\, dt \\
&\geq \frac{\sigma_k}{2}\|u\|^2 - \alpha_k \int_0^{kT} \left|u(t)\right|^2 dt \\
&\geq \left(\frac{\sigma_k}{2} - \alpha_k\right)\|u\|^2 \\
&= \left(\frac{\min\{\min_{i\in\{1,\ldots,r\}}\{\frac{(l_i+1)^2\omega^2-\lambda_i}{(l_i+1)^2\omega^2+1}\}, \frac{\omega^2}{\omega^2+k^2}, \frac{\lambda_{i_-}}{1+\lambda_{i_+}}\}}{2} - \alpha_k\right)\rho_k^2.
\end{aligned}$$

For cases (H3)(1) and (H3)(3)-(H3)(6), correspondingly, by (H4) and Remark 3.3, similar to the above argument, we can also obtain that

$$\varphi_k(u) \geq \left(\frac{\sigma_k}{2} - \alpha_k\right)\rho_k^2 > 0, \quad \forall u \in \big(H_{kT}^{+2} \oplus H_{kT}^{02}\big) \cap \partial B_{\rho_k}.$$

Step 4: Let

$$Q_k = \big\{sh_k | s \in [0, s_1]\big\} \oplus \big(B_{s_2} \cap \big(H_{kT}^{-} \oplus H_{kT}^{01} \oplus H_{kT}^{+1}\big)\big),$$

where $h_k \in H_{kT}^{+2} \oplus H_{kT}^{02}$, s_1 and s_2 will be determined later. In this step, we prove $\varphi_k|_{\partial Q_k} \leq 0$. We only consider the case that F satisfies (H3)(2). For other cases, the results can be seen easily from the argument of case (H3)(2).

Assume that F satisfies (H3)(2). Let

$$d_k = \min\left\{\frac{(l_{i_0}+1)^2\omega^2 - \lambda_{i_0}}{2}, \frac{\omega^2}{2k^2}, \frac{\lambda_{i_-}}{2}\right\}.$$

Case (i): if

$$d_k := d = \frac{(l_{i_0}+1)^2\omega^2 - \lambda_{i_0}}{2},$$

then we choose

$$h_k(t) = \sin(l_{i_0}+1)\omega t \cdot e_{i_0}, \quad \forall t \in \mathbb{R}.$$

Obviously, $h_k \in H_{kT}^{+2}$ and $\dot{h}_k(t) = (l_{i_0}+1)\omega \cos(l_{i_0}+1)\omega t \cdot e_{i_0}$, $\forall t \in \mathbb{R}$. Then

$$\|h_k\|_{L^2}^2 = \frac{kT}{2}, \qquad \|\dot{h}_k\|_{L^2}^2 = \frac{kT(l_{i_0}+1)^2\omega^2}{2}.$$

By (H3)(2), (1.5) and the periodicity of F, we have

$$F(t,x) \geq \beta_k|x|^2 - \beta_k\hat{L}_k^2 = (d + \varepsilon_{0k})|x|^2 - \beta_k\hat{L}_k^2, \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0, kT], \tag{3.7}$$

where $\varepsilon_{0k} = \beta_k - d > 0$ and $\hat{L}_k > \max\{1, L_k\}$. Since $H_{kT}^{-} \oplus H_{kT}^{01} \oplus H_{kT}^{+1}$ is the finite dimensional space, there exists a constant $K_{1k} > 0$ such that

$$K_{1k}\|u\|^2 \le \|u\|_{L^2}^2 \le \|u\|^2, \quad \forall u \in H_{kT}^{-} \oplus H_{kT}^{01} \oplus H_{kT}^{+1}. \tag{3.8}$$

By (3.3), (3.5), (3.6), (3.7) and (3.8), we know that for all $s > 0$ and $u = u^- + u^{01} + u^{+1} \in H_{kT}^{-} \oplus H_{kT}^{01} \oplus H_{kT}^{+1}$,

$$\begin{aligned}
\varphi_k(sh_k + u) &\le -\frac{\delta}{2}\|u^-\|^2 + \frac{s^2}{2}\int_0^{kT}|\dot{h}_k(t)|^2\,dt - \frac{\lambda_{i_0}s^2}{2}\int_0^{kT}|h_k(t)|^2\,dt \\
&\quad - \int_0^{kT} F\big(t, sh_k(t) + u(t)\big)\,dt \\
&\le -\frac{\delta}{2}\|u^-\|^2 + \frac{s^2}{2}\cdot\frac{kT(l_{i_0}+1)^2\omega^2}{2} - \frac{\lambda_{i_0}s^2}{2}\cdot\frac{kT}{2} \\
&\quad - (d + \varepsilon_{0k})\int_0^{kT}|sh_k(t) + u(t)|^2\,dt + \beta_k\hat{L}_k^2 kT \\
&= -\frac{\delta}{2}\|u^-\|^2 + \frac{s^2}{2}\cdot\frac{kT(l_{i_0}+1)^2\omega^2}{2} - \frac{\lambda_{i_0}s^2}{2}\cdot\frac{kT}{2} \\
&\quad - (d + \varepsilon_{0k})\big(s^2\|h_k\|_{L^2}^2 + \|u\|_{L^2}^2\big) + \beta_k\hat{L}_k^2 kT \\
&\le -\frac{\delta}{2}\|u^-\|^2 + \left(\frac{kT(l_{i_0}+1)^2\omega^2}{4} - \frac{\lambda_{i_0}kT}{4} - \frac{dkT}{2} - \frac{kT\varepsilon_{0k}}{2}\right)s^2 \\
&\quad - (d + \varepsilon_{0k})\|u\|_{L^2}^2 + \beta_k\hat{L}_k^2 kT \\
&\le -\frac{kT\varepsilon_{0k}}{2}s^2 - \varepsilon_{0k}\|u\|_{L^2}^2 + \beta_k\hat{L}_k^2 kT \\
&\le -\frac{kT\varepsilon_{0k}}{2}s^2 - \varepsilon_{0k}K_{1k}\|u\|^2 + \beta_k\hat{L}_k^2 kT.
\end{aligned} \tag{3.9}$$

Hence,

$$\varphi_k(sh_k + u) \le 0, \quad \text{either } s \ge s_1 \text{ or } \|u\| \ge s_2,$$

where

$$s_1 = \sqrt{\frac{2\beta_k\hat{L}_k^2}{\varepsilon_{0k}}}, \qquad s_2 = \sqrt{\frac{\beta_k\hat{L}_k^2 kT}{\varepsilon_{0k}K_{1k}}}.$$

Case (ii): if $d_k = \omega^2/(2k^2)$, then we choose

$$h_k(t) = \sin k^{-1}\omega t \cdot e_{r+1} \in H_{kT}^{02}, \quad \forall t \in \mathbb{R}.$$

Then

$$\dot{h}_k(t) = \frac{\omega}{k}\cos k^{-1}\omega t \cdot e_{r+1}, \quad \forall t \in \mathbb{R},$$

and

$$(Ah_k, h_k) = 0, \qquad \|h_k\|_{L^2}^2 = \frac{kT}{2}, \qquad \|\dot{h}_k\|_{L^2}^2 = \frac{T\omega^2}{2k}. \tag{3.10}$$

By (H3)(2), (1.5) and the periodicity of F, we have

$$F(t,x) \geq \beta_k|x|^2 - \beta_k\hat{L}_k^2 = \left(\frac{\omega^2}{2k^2} + \varepsilon'_{0k}\right)|x|^2 - \beta_k\hat{L}_k^2, \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T], \tag{3.11}$$

where $\hat{L}_k > \max\{1, L_k\}$ and $\varepsilon'_{0k} = \beta_k - \frac{\omega^2}{2k^2}$. By (3.3), (3.5), (3.6), (3.8) and (3.11), we know that for all $s > 0$ and $u = u^- + u^{01} + u^{+1} \in H_{kT}^- \oplus H_{kT}^{01} \oplus H_{kT}^{+1}$,

$$\begin{aligned}
\varphi_k(sh_k + u) &\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{s^2}{2}\int_0^{kT}|\dot{h}_k(t)|^2\,dt - \int_0^{kT} F(t, sh_k(t) + u)\,dt \\
&\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{s^2}{2}\cdot\frac{T\omega^2}{2k} - \left(\frac{\omega^2}{2k^2} + \varepsilon'_{0k}\right)\int_0^{kT}|sh_k(t) + u(t)|^2\,dt \\
&\quad + \beta_k\hat{L}_k^2 kT \\
&= -\frac{\delta}{2}\|u^-\|^2 + \frac{s^2}{2}\cdot\frac{T\omega^2}{2k} - \left(\frac{\omega^2}{2k^2} + \varepsilon'_{0k}\right)\left(s^2\|h_k\|_{L^2}^2 + \|u\|_{L^2}^2\right) \\
&\quad + \beta_k\hat{L}_k^2 kT \\
&= -\frac{\delta}{2}\|u^-\|^2 + \left(\frac{T\omega^2}{4k} - \frac{T\omega^2}{4k} - \frac{kT\varepsilon'_{0k}}{2}\right)s^2 - \left(\frac{\omega^2}{2k^2} + \varepsilon'_{0k}\right)\|u\|_{L^2}^2 \\
&\quad + \beta_k\hat{L}_k^2 kT \\
&\leq -\frac{kT\varepsilon'_{0k}}{2}s^2 - \varepsilon'_{0k}\|u\|_{L^2}^2 + \beta_k\hat{L}_k^2 kT \\
&\leq -\frac{kT\varepsilon'_{0k}}{2}s^2 - \varepsilon'_{0k}K_{1k}\|u\|^2 + \beta_k\hat{L}_k^2 kT.
\end{aligned}$$

Hence,

$$\varphi_k(sh_k + u) \leq 0, \quad \text{either } s \geq s_1 \text{ or } \|u\| \geq s_2,$$

where

$$s_1 = \sqrt{\frac{2\beta_k\hat{L}_k^2}{\varepsilon'_{0k}}}, \qquad s_2 = \sqrt{\frac{\beta_k\hat{L}_k^2 kT}{\varepsilon'_{0k}K_{1k}}}.$$

Case (iii): if $d_k = \lambda_{i_-}/2$, then we choose

$$h_k = \frac{1}{\sqrt{kT}}\cdot e_{i_-} \in H_{kT}^{+2}.$$

Then

$$\dot{h}_k = 0, \qquad (Ah_k, h_k) = -\lambda_{i_-}(h_k, h_k), \qquad \|h_k\|_{L^2}^2 = 1.$$

By (H3)(2), (1.5) and the periodicity of F, we have

$$F(t,x) \geq \beta_k|x|^2 - \beta_k\hat{L}_k^2 = \left(\frac{\lambda_{i_-}}{2} + \varepsilon''_{0k}\right)|x|^2 - \beta_k\hat{L}_k^2, \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0,kT], \tag{3.12}$$

where $\hat{L}_k > \max\{\sqrt{1+\frac{1}{T}}, L_k\}$ and $\varepsilon''_{0k} = \beta_k - \lambda_{i\_}/2$. By (3.3), (3.5), (3.6), (3.8) and (3.12), for all $s > 0$ and $u = u^- + u^{01} + u^{+1} \in H_{kT}^- \oplus H_{kT}^{01} \oplus H_{kT}^{+1}$, we have

$$\begin{aligned}
\varphi_k(sh_k + u) &\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{s^2}{2}\int_0^{kT}|\dot{h}_k(t)|^2\,dt + \frac{\lambda_{i_}s^2}{2}\int_0^{kT}|h_k(t)|^2\,dt \\
&\quad - \int_0^{kT} F(t, sh_k(t) + u(t))\,dt \\
&\leq -\frac{\delta}{2}\|u^-\|^2 + \frac{\lambda_{i_}s^2}{2} - \left(\frac{\lambda_{i_}}{2} + \varepsilon''_{0k}\right)\int_0^{kT}|sh_k(t) + u(t)|^2\,dt + \beta_k\hat{L}_k^2 kT \\
&= -\frac{\delta}{2}\|u^-\|^2 + \frac{\lambda_{i_}s^2}{2} - \left(\frac{\lambda_{i_}}{2} + \varepsilon''_{0k}\right)\left(s^2\|h_k\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \beta_k\hat{L}_k^2 kT \\
&= -\frac{\delta}{2}\|u^-\|^2 + \left(\frac{\lambda_{i_}}{2} - \frac{\lambda_{i_}}{2} - \varepsilon''_{0k}\right)s^2 - \left(\frac{\lambda_{i_}}{2} + \varepsilon''_{0k}\right)\|u\|_{L^2}^2 + \beta_k\hat{L}_k^2 kT \\
&\leq -\varepsilon''_{0k}s^2 - \varepsilon''_{0k}K_{1k}\|u\|^2 + \beta_k\hat{L}_k^2 kT.
\end{aligned}$$

Hence,

$$\varphi_k(se_k + u) \leq 0, \quad \text{either } s \geq s_1 \text{ or } \|u\| \geq s_2,$$

where

$$s_1 = \sqrt{\frac{\beta_k\hat{L}_k^2 kT}{\varepsilon''_{0k}}}, \qquad s_2 = \sqrt{\frac{\beta_k\hat{L}_k^2 kT}{\varepsilon''_{0k}K_{1k}}}.$$

Combining cases (i), (ii) and (iii), if we let

$$\begin{aligned}
s_1 &= \max\left\{\sqrt{\frac{2\beta_k\hat{L}_k^2}{\varepsilon_{0k}}}, \sqrt{\frac{2\beta_k\hat{L}_k^2}{\varepsilon'_{0k}}}, \sqrt{\frac{\beta_k\hat{L}_k^2 kT}{\varepsilon''_{0k}}}\right\}, \\
s_2 &= \max\left\{\sqrt{\frac{\beta_k\hat{L}_k^2 kT}{\varepsilon_{0k}K_{1k}}}, \sqrt{\frac{\beta_k\hat{L}_k^2 kT}{\varepsilon'_{0k}K_{1k}}}, \sqrt{\frac{\beta_k\hat{L}_k^2}{\varepsilon''_{0k}K_{1k}}}\right\},
\end{aligned}$$

then

$$\varphi_k(sh_k + u) \leq 0, \quad \text{either } s \geq s_1 \text{ or } \|u\| \geq s_2. \tag{3.13}$$

By (1.5), (3.3), (3.5) and (3.6), for all $u \in H_{kT}^- \oplus H_{kT}^{01} \oplus H_{kT}^{+1}$, we have

$$\begin{aligned}
\varphi_k(u) &= \frac{1}{2}\int_0^{kT}|\dot{u}(t)|^2\,dt - \frac{1}{2}\int_0^{kT}(Au(t), u(t))\,dt - \int_0^{kT}F(t, u(t))\,dt \\
&\leq -\frac{\delta}{2}\|u^-\|^2 \\
&\leq 0.
\end{aligned} \tag{3.14}$$

Thus, it follows from (3.13) and (3.14) that $\varphi|_{\partial Q_k} \leq 0 < b_k$.

Step 5: We prove that φ_k satisfies (C)-condition in H^1_{kT}. The proof is similar to that in Theorem 1.1 in [17]. We omit it.

Step 6: We claim that φ_k has a nontrivial critical point $u_k \in H^1_{kT}$ such that $\varphi_k(u_k) \geq b_k > 0$. Especially, we claim that, for cases (H3)(1) and (H3)(4), since A is a positive semidefinite matrix, (1.5) implies that u_k is nonconstant.

In fact, it is easy to see that

$$\begin{aligned} q_k(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_0^{kT} \big((A+I)u(t), u(t)\big)\,dt \\ &= \frac{1}{2}\big\langle (I-K)u, u\big\rangle, \end{aligned}$$

where $K : H^1_{kT} \to H^1_{kT}$ is the linear self-adjoint operator defined, using the Riesz representation theorem, by

$$\int_0^{kT} \big((A+I)u(t), v(t)\big)\,dt = \big\langle (Ku, v)\big\rangle, \quad \forall u, v \in H^1_T.$$

The compact imbedding of H^1_{kT} into $C([0,kT];\mathbb{R}^N)$ implies that K is compact. In order to use Lemma 2.3, we let $\Phi = I - K$ and define $\Phi_i : E_i \to E_i$, $i = 1, 2$ by

$$\langle \Phi_i u, v\rangle = \big\langle (I-K)u, v\big\rangle, \quad u, v \in E_i,$$

where $E_1 = H^{+2}_{kT} \oplus H^{02}_{kT}$ and $E_2 = H^{-}_{kT} \oplus H^{01}_{kT} \oplus H^{+1}_{kT}$. Since K is a self-adjoint compact operator, it is easy to see that Φ_i $(i = 1, 2)$ are bounded and self-adjoint. Let

$$b(u) = -\int_0^{kT} F\big(t, u(t)\big)\,dt.$$

Assumption (A)$'$ and Lemma 2.2 imply that b is weakly continuous and is uniformly differentiable on bounded subsets of $E = H^1_{kT}$. Furthermore, by standard theorems in [22], we conclude that b' is compact. Let $S_k = (H^{+2}_{kT} \oplus H^{02}_{kT}) \cap \partial B_{\rho_k}$. Then S_k and ∂Q_k link. Hence, by Step 1-Step 5, Lemma 2.3 and Remark 2.2, there exists a critical point $u_k \in H^1_{kT}$ such that $\varphi_k(u_k) \geq b_k > 0$, which implies that u_k is nonzero. For cases (H3)(1) and (H3)(4), since A is a positive semidefinite matrix, it follows from (1.5) that u_k is nonconstant. The proof is complete. □

Proof of Theorem 1.2 Obviously, when $r = 0$, $s > 0$ and $s < N$, (H1) holds for any $k \in \mathbb{N}$. Moreover, since (H3)$'$ implies that (H3)(5) and (H4)$'$ implies that (H4), system (1.1) has kT-periodic solution for every $k \in \mathbb{N}$.

Let $d = \frac{\omega^2}{2}$. Like the argument of case (ii) in the proof of Theorem 1.1, choose

$$e_k(t) = \sin k^{-1}\omega t e_{r+1} \in H^{02}_{kT}, \quad \forall t \in \mathbb{R}.$$

By (H3)$'$, (1.5) and the T-periodicity of F, we have

$$F(t,x) \geq \beta|x|^2 - \beta L^2 = \left(\frac{\omega^2}{2} + \varepsilon_1\right)|x|^2 - \beta L^2, \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0,kT], \tag{3.15}$$

where $\varepsilon_1 = \beta - \frac{\omega^2}{2}$. In the proof of Theorem 1.1, if we replace (3.15) with (3.11), then we obtain

$$\varphi_k(se_k + u) \leq 0, \quad \text{either } s \geq s_1 \text{ or } \|u\| \geq s_2,$$

where

$$s_1 = \sqrt{\frac{2\beta L^2}{\varepsilon_1}} = \sqrt{\frac{2\beta L^2}{\beta - \frac{\omega^2}{2}}}, \qquad s_2 = \sqrt{\frac{\beta L^2 kT}{\varepsilon_1 K_{1k}}}.$$

Note that s_1 is independent of k. Hence, if u_k is the critical point of φ_k, then it follows from (3.3), (3.5), (3.6), the definitions of critical value c in Lemma 2.3 and Q_k that

$$\begin{aligned}
\varphi_k(u_k) &\leq \sup_{u\in Q_k} \varphi_k(u) \\
&\leq \sup_{s\in[0,s_1]} \left\{ \frac{s^2}{2} \int_0^{kT} \left|\dot{e}_k(t)\right|^2 dt - \frac{s^2}{2} \int_0^{kT} \big(Ae_k(t), e_k(t)\big)\, dt \right\} \\
&\leq \frac{s_1^2}{2} \int_0^{kT} \left|\dot{e}_k(t)\right|^2 dt \\
&= \frac{\beta L^2 T\omega^2}{2k(\beta - \frac{\omega^2}{2})} \\
&\leq \frac{\beta L^2 T\omega^2}{2(\beta - \frac{\omega^2}{2})} := M.
\end{aligned} \tag{3.16}$$

Hence, $\varphi_k(u_k)$ is bounded for any $k \in \mathbb{N}$.

Obviously, we can find $k_1 \in \mathbb{N}/\{1\}$ such that $k_1 > \frac{M}{b_1}$, then we claim that u_k is distinct from u_1 for all $k \geq k_1$. In fact, if $u_k = u_1$ for some $k \geq k_1$, it is easy to check that

$$\varphi_k(u_k) = k\varphi_1(u_1) \geq kb_1.$$

Then by (3.16), we have $k_1 \leq k \leq \frac{M}{b_1}$, a contradiction. We also can find $k_2 > \max\{k_1, \frac{k_1 M}{b_{k_1}}\}$ such that $u_{k_1 k} \neq u_{k_1}$ for all $k \geq \frac{k_2}{k_1}$. Otherwise, if $u_{k_1 k} = u_{k_1}$ for some $k \geq k_1$, we have $\varphi_{k_1 k}(u_{k_1 k}) = k\varphi_{k_1}(u_{k_1}) \geq kb_{k_1}$. Then by (3.16), we have $\frac{k_2}{k_1} \leq k \leq \frac{M}{b_{k_1}}$, a contradiction. In the same way, we can obtain that system (1.1) has a sequence of distinct periodic solutions with period $k_j T$ satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$. The proof is complete. □

Proof of Theorem 1.3 Except for verifying (C) condition, the proof is the same as in Theorem B (that is Theorem 1 in [15]). To verify (C) condition, we only need to prove the sequence $\{u_n\}$ is bounded if $\varphi(u_n)$ is bounded and $\|\varphi'(u_n)\|(\|1 + \|u_n\|) \to 0$ as $n \to +\infty$. Other proofs are the same as in [15]. The proof of boundedness of $\{u_n\}$ is essentially the same as in Theorem 1.1 in [17] except that 2 is replaced by p, H^1_{kT} by

$$W^{1,p}_{kT} = \big\{u : \mathbb{R} \to \mathbb{R}^N | u \text{ is absolutely continuous}, u(t) = u(t+T) \text{ and } \dot{u} \in L^p\big([0,T]\big)\big\}$$

equipped with the norm

$$\|u\| = \left(\int_0^{kT} |u(t)|^p \, dt + \int_0^{kT} |\dot{u}(t)|^p \, dt\right)^{1/p},$$

and

$$F(t,x) \geq \beta_k |x|^2, \quad \forall x \in \mathbb{R}^N, |x| > L$$

by

$$F(t,x) \geq \varepsilon |x|^p, \quad \forall x \in \mathbb{R}^N, |x| > L$$

for some $\varepsilon > 0$. So, we omit the details. □

4 Examples

Example 4.1 Let $T = 2\pi$ and

$$A = \begin{pmatrix} 7.5 & 0 & 0 & 0 & 0 \\ 0 & 7.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

Then $\omega = 1$, $r = 2$, $\lambda_1 = 7.5$, $\lambda_2 = 7.4$, $\lambda_3 = 0$, $\lambda_4 = -3$, $\lambda_5 = -4$, $\lambda_{i_+} = 4$ and $\lambda_{i_-} = 3$. Obviously, the matrix A satisfies Assumption (A0) and $l_1 = l_2 = 2$ such that

$$l_i^2 \omega^2 < \lambda_i < (l_i + 1)^2 \omega^2, \quad i = 1, 2.$$

It is easy to verify that (H1) holds with $k = 1, 2, 3$. Let

$$F(t,x) \equiv \frac{4}{63k^2} |x|^2 \left(11^{\frac{|x|^{3/2}}{1+|x|^{3/2}}} - \frac{1}{2}\right) \quad \text{a.e. } t \in [0,T].$$

Then $F(t,x) \geq 0$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$ and

$$\lim_{|x| \to 0} \frac{F(t,x)}{|x|^2} = \frac{2}{63k^2} \quad \text{uniformly for a.e. } t \in [0,T], \tag{4.1}$$

$$\lim_{|x| \to \infty} \frac{F(t,x)}{|x|^2} = \frac{2}{3k^2} \quad \text{uniformly for a.e. } t \in [0,T]. \tag{4.2}$$

It is easy to verify that

$$\big(\nabla F(t,x), x\big) - 2F(t,x) = \frac{6 \ln 11}{63k^2} |x|^2 \cdot 11^{\frac{|x|^{3/2}}{1+|x|^{3/2}}} \cdot \frac{|x|^{3/2}}{(1 + |x|^{3/2})^2}.$$

Choose $\xi = 1$, $\eta = 1$ and $\nu = 3/2$. Moreover, obviously, there exists $m > 0$ such that $\frac{|x|^{3/2}}{1+|x|^{3/2}} > \frac{2}{3}$. Then

$$\frac{(\nabla F(t,x),x) - 2F(t,x)}{\frac{F(t,x)}{\xi+\eta|x|^{\nu}}} = \frac{\frac{3}{2}\ln 11 \cdot 11^{\frac{|x|^{3/2}}{1+|x|^{3/2}}} \cdot \frac{|x|^{3/2}}{1+|x|^{3/2}}}{11^{\frac{|x|^{3/2}}{1+|x|^{3/2}}} - \frac{1}{2}} > \ln 11 > 1.$$

Hence, (H2) holds.

When $k = 1$,

$$\min\left\{\frac{(l_{i_0}+1)^2\omega^2 - \lambda_{i_0}}{2}, \frac{\omega^2}{2k^2}, \frac{\lambda_{i_-}}{2}\right\} = \frac{1}{2} \quad \text{and} \quad \sigma_1 = 0.15.$$

By (4.2), we can find $L_1 > 0$ such that

$$F(t,x) \geq \left(\frac{2}{3} - \frac{1}{10}\right)|x|^2 = \frac{17}{30}|x|^2, \quad \forall |x| > L_1, \text{ and a.e. } t \in [0,T].$$

Let $\beta_1 = \frac{17}{30}$. Then (H3)(2) holds with $k = 1$. Moreover, by (4.1), we can find $l_1 > 0$ such that

$$F(t,x) \leq \left(\frac{2}{63} + \frac{23}{2520}\right)|x|^2 \approx 0.0409|x|^2, \quad \forall |x| \leq l_1 \text{ and a.e. } t \in [0,T].$$

Let $\alpha_1 = 0.0409$. Then (H4) holds. By Theorem 1.1, we obtain that system (1.1) has a T-periodic solution.

When $k = 2$,

$$\min\left\{\frac{(l_{i_0}+1)^2\omega^2 - \lambda_{i_0}}{2}, \frac{\omega^2}{2k^2}, \frac{\lambda_{i_-}}{2}\right\} = \frac{1}{8} \quad \text{and} \quad \sigma_2 = 0.15.$$

By (4.2), we can find $L_2 > 0$ such that

$$F(t,x) \geq \left(\frac{1}{6} - \frac{1}{100}\right)|x|^2 \approx 0.1567|x|^2, \quad \forall |x| > L_2 \text{ and a.e. } t \in [0,T].$$

Let $\beta_2 = 0.1567$. Then (H3)(2) holds with $k = 2$. Moreover, by (4.1), we can find $l_2 > 0$ such that

$$F(t,x) \leq \left(\frac{1}{126} + \frac{1}{1000}\right)|x|^2 \approx 0.00894|x|^2, \quad \forall |x| \leq l_2 \text{ and a.e. } t \in [0,T].$$

Let $\alpha_2 = 0.00894$. Then (H4) holds. Note that $\frac{1}{6} < \frac{1}{2} = \min\{\frac{(l_{i_0}+1)^2\omega^2-\lambda_{i_0}}{2}, \frac{\omega^2}{2}, \frac{\lambda_{i_-}}{2}\}$. So, when $k = 2$, by Theorem 1.1, we cannot judge that system (1.1) has a T-periodic solution. However, we can obtain that system (1.1) has a $2T$-periodic solution.

When $k = 3$,

$$\min\left\{\frac{(l_{i_0}+1)^2\omega^2 - \lambda_{i_0}}{2}, \frac{\omega^2}{2k^2}, \frac{\lambda_{i_-}}{2}\right\} = \frac{1}{18} \quad \text{and} \quad \sigma_3 = 0.1.$$

By (4.2), we can find $L_3 > 0$ such that

$$F(t,x) \geq \left(\frac{2}{27} - \frac{1}{100}\right)|x|^2 \approx 0.0641|x|^2, \quad \forall |x| > L_3 \text{ and a.e. } t \in [0,T].$$

Let $\beta_3 = 0.0641$. Then (H3)(2) holds with $k = 3$. Moreover, by (4.1), we can find $l_3 > 0$ such that

$$F(t,x) \leq \left(\frac{2}{567} + \frac{1}{1000}\right)|x|^2 \approx 0.00453|x|^2, \quad \forall |x| \leq l_3 \text{ and a.e. } t \in [0,T].$$

Let $\alpha_3 = 0.00453$. Then (H4) holds. Note that $\frac{2}{27} < \frac{1}{8} = \min\{\frac{(l_{i_0}+1)^2\omega^2-\lambda_{i_0}}{2}, \frac{\omega^2}{2\times 2^2}, \frac{\lambda_{i_-}}{2}\} < \frac{1}{2} = \min\{\frac{(l_{i_0}+1)^2\omega^2-\lambda_{i_0}}{2}, \frac{\omega^2}{2}, \frac{\lambda_{i_-}}{2}\}$. So, when $k = 3$, by Theorem 1.1, we cannot judge that system (1.1) has T-periodic solution and $2T$-periodic solution. However, we can obtain that system (1.1) has a $3T$-periodic solution. It is easy to verify that Example 4.1 does not satisfy the theorem in [19] even if $k = 1$.

Example 4.2 Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and

$$F(t,x) \equiv \frac{2\pi^2}{T^2}|x|^2\left(e^{\frac{|x|^{3/2}}{1+|x|^{3/2}}} - 1\right) \quad \text{a.e. } t \in [0,T].$$

Then

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0 \quad \text{uniformly for a.e. } t \in [0,T],$$
$$\lim_{|x|\to \infty} \frac{F(t,x)}{|x|^2} = \frac{2\pi^2}{T^2}(e-1) \quad \text{uniformly for a.e. } t \in [0,T].$$

Obviously, (A0), (A)′, (1.5), (H3)′ and (H4)′ hold. Let $\xi = 1$, $\eta = 1$ and $\nu = \frac{3}{2}$. Similar to the argument in Example 4.1, we obtain (H2) also holds. Then by Theorem 1.2, system (1.1) has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.

Example 4.3 Let $p = 4$ and

$$F(t,x) \equiv |x|^p\left(e^{|x|^p} - 1\right) = |x|^4\left(e^{|x|^4} - 1\right) \quad \text{a.e. } t \in [0,T].$$

Then (1.5) holds and

$$\lim_{|x|\to 0} \frac{F(t,x)}{|x|^4} = 0, \qquad \lim_{|x|\to \infty} \frac{F(t,x)}{|x|^4} = +\infty \quad \text{uniformly for a.e. } t \in [0,T].$$

Let $\xi = 1$, $\eta = 1$ and $\nu = 1/2$. Then it is easy to obtain that there exists $m > 1$ such that (H5) holds. By Theorem 1.3, system (1.8) has a sequence of distinct periodic solutions with period $k_j T$ satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$. It is easy to see that Example 4.3 does not satisfy (1.3). Hence, Theorem 1.3 improved Theorem B.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XZ proposed the idea of the paper and finished the main proofs. XT provided some important techniques in the process of proofs.

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