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Infinitely many sign-changing solutions for p-Laplacian equation involving the critical Sobolev exponent

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Abstract

In this paper, we study the following problem:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $1 < p < N$, $-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p^* = pN/(N-p)$ is the critical Sobolev exponent and $\lambda > 0$ is a parameter. By establishing a new deformation lemma, we show that if $N > p^2 + p$, then for each $\lambda > 0$, this problem has infinitely many sign-changing solutions, which extends the results obtained in (Cao *et al.* in J. Funct. Anal. 262: 2861-2902, 2012; Schechter and Zou in Arch. Ration. Mech. Anal. 197: 337-356, 2010).

1 Introduction

In this paper, we consider the following problem:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, $1 < p < N$, $-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p^* = pN/(N-p)$ is the critical Sobolev exponent and $\lambda > 0$ is a parameter.

The first existence result of Problem (1.1) with $p = 2$ was obtained by Brezis and Nirenberg in the celebrated paper [1]. In that paper, the authors proved that Problem (1.1) had a positive solution for $N \geq 4$ and $\lambda \in (0, \lambda_1^*)$ or $N = 3$ and $\lambda \in (\lambda_1^*/4, \lambda_1^*)$, where λ_1^* is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. After that, many existence results have appeared for (1.1); one can see, for example, [2–7] and the references therein for case of $p = 2$ and [8–11] and the references therein for case of $1 < p < N$. In particular, in the case of $p = 2$, the authors in [2] proved that the number of solutions of Problem (1.1) is bounded below by the number of eigenvalues of $(-\Delta, H_0^1(\Omega))$ lying in the open interval $(\lambda, \lambda + S|\Omega|^{-2/N})$, where S is the best Sobolev constant and $|\Omega|$ is the Lebesgue measure of Ω. In [5], the existence of infinitely many sign-changing solutions of (1.1) with $p = 2$ has been obtained when $N \geq 4$, $\lambda > 0$ and Ω is a ball, while it has been shown in [6] that (1.1) with $p = 2$ has infinitely many sign-changing radial solutions when $N \geq 7$, $\lambda > 0$ and Ω also is a ball. We remark that the methods used in [5, 6] are strongly dependent on the symmetry of the ball Ω. For a general bounded smooth domain Ω, by the method of invariant sets of the descending flow, the authors in [7] have shown that (1.1) with $p = 2$ has infinitely many sign-changing solutions when $N \geq 7$ and $\lambda > 0$, which extends the main result in [4].

The main purpose of this paper is to try to obtain the existence of infinitely many sign-changing solutions of Problem (1.1) for general $p \in (1, N)$. In a very recent work [9], the authors have proved that (1.1) has infinitely many solutions for $\lambda > 0$ and $N > p^2 + p$. However, by using the Picone identity (*cf.* [12, 13]), we see that every nonzero solution of Problem (1.1) is sign-changing for $\lambda \geq \lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ (see Lemma 2.1 for more details). Hence, to achieve our purpose, we mainly consider the situation of $\lambda \in (0, \lambda_1)$.

Our main result in this paper is the following.

Theorem 1.1 *Assume that* $N > p^2 + p$ *and* $\lambda > 0$. *Then Problem* (1.1) *has infinitely many sign-changing solutions*.

Since p^* is the critical Sobolev exponent, in order to overcome the lack of compactness of the embedding of $W_0^{1,p}(\Omega)$ in the Lebesgue space $L^{p^*}(\Omega)$, we follow the ideas of [4, 7, 9] to consider the following auxiliary problems:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + |u|^{p_n-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_n$}$$

where $p_n < p^*$ and p_n is increasing to p^*. It has been shown by [14, Theorem 1.2] that for every n, Problem $(\mathcal{P}_n)$ has infinitely many sign-changing solutions $\{u_{n,k}\}_{k\in\mathbb{N}}$. Hence, to prove Theorem 1.1, we will show that for every $k \in \mathbb{N}$, $\{u_{n,k}\}$ converges to some sign-changing solution u_k of (1.1) as $n \to \infty$, and that $\{u_k\}$ are different. The convergence of $\{u_{n,k}\}$ can be done with the help of [9, Theorem 1.2], which we show in Lemma 2.3. To distinguish $\{u_k\}$, we shall establish a new deformation lemma on special sets in $W_0^{1,p}(\Omega)$; see Lemma 2.5 for details.

Throughout this paper, we will always respectively denote $\|u\|$ and $\|u\|_r$ by the usual norm in spaces $W_0^{1,p}(\Omega)$ and $L^r(\Omega)$ $(r \geq 1)$. Let C be indiscriminately used to denote various positive constants.

2 Proof of Theorem 1.1

We first consider the case of $\lambda \geq \lambda_1$. Recall that λ_1, the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$, given by $\lambda_1 := \inf\{\int_\Omega |\nabla u|^p\,dx, \int_\Omega |u|^p\,dx = 1\}$, is simple and there exists a positive eigenfunction $e_1 \in W_0^{1,p}(\Omega)$ corresponding to λ_1 such that $\int_\Omega |\nabla e_1|^{p-2}\nabla e_1 \nabla\eta\,dx = \lambda_1 \int_\Omega e_1^{p-1}\eta\,dx$ for every $\eta \in W_0^{1,p}(\Omega)$ (*cf.* [15]). Moreover, by [16, Proposition 2.1], we know that $e_1 \in L^\infty(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega)$. On the other hand, we have the following proposition which is the so-called Picone identity.

Proposition 2.1 [13, Lemma A.6] *Let $u, v \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ be such that $u \geq 0$, $v > 0$ and $\frac{u}{v} \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then*

$$\begin{aligned}&\int_\Omega \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v\,dx\\ &\quad= \int_\Omega \left(p\left(\frac{u}{v}\right)^{p-1}|\nabla v|^{p-2}\nabla v\nabla u - (p-1)\left(\frac{u}{v}\right)^p|\nabla v|^p\right)dx.\end{aligned}$$

Moreover,

$$\int_\Omega \left(p\left(\frac{u}{v}\right)^{p-1}|\nabla v|^{p-2}\nabla v\nabla u - (p-1)\left(\frac{u}{v}\right)^p|\nabla v|^p\right)dx \leq \int_\Omega |\nabla u|^p\,dx,$$

and the equality holds if and only if $u = cv$ for some constant $c > 0$.

Lemma 2.1 *Assume that $u \in W^{1,p}_0(\Omega)$ is a nonzero solution of* (1.1) *for $\lambda \geq \lambda_1$. Then u is sign-changing.*

Proof By a contradiction, we may assume $u \geq 0$. By using a standard regularity argument and [17, Lemmas 3.2 and 3.3], we have $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. Thus, it follows from the strong maximum principle (*cf.* [18]) that $u > 0$. Now, for every $\varepsilon > 0$, by applying the above Picone identity (*i.e.*, Proposition 2.1) to $u + \varepsilon$ and e_1, we see

$$\int_\Omega |\nabla e_1|^p\,dx \geq \int_\Omega \nabla\left(\frac{e_1^p}{(u+\varepsilon)^{p-1}}\right)|\nabla u|^{p-2}\nabla u\,dx.$$

Noting that u is a solution of (1.1), we have

$$\int_\Omega |\nabla e_1|^p\,dx \geq \int_\Omega \left(\lambda\frac{u^{p-1}}{(u+\varepsilon)^{p-1}} + \frac{u^{p^*-1}}{(u+\varepsilon)^{p-1}}\right)e_1^p\,dx.$$

It follows from the Fatou lemma that

$$\begin{aligned}\int_\Omega |\nabla e_1|^p\,dx &\geq \liminf_{\varepsilon\to 0}\int_\Omega \left(\lambda\frac{u^{p-1}}{(u+\varepsilon)^{p-1}} + \frac{u^{p^*-1}}{(u+\varepsilon)^{p-1}}\right)e_1^p\,dx\\ &\geq \int_\Omega \left(\lambda + u^{p^*-p}\right)e_1^p\,dx,\end{aligned}$$

which is impossible since $\int_\Omega |\nabla e_1|^p\,dx = \lambda_1\int_\Omega e_1^p$, $u > 0$, $e_1 > 0$ and $\lambda \geq \lambda_1$. Therefore, we have proved Lemma 2.1. □

Next, we consider the case of $\lambda < \lambda_1$.

It is clear that the corresponding functional of $(\mathcal{P}_n)$ $I_n : W^{1,p}_0(\Omega) \to \mathbb{R}$, given by

$$I_n(u) = \frac{1}{p}\left(\|u\|^p - \lambda\|u\|_p^p\right) - \frac{1}{p_n}\|u\|_{p_n}^{p_n},$$

is C^1 Fréchet differentiable. Let $X_m = \operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$, where $\{\varphi_i\}$ is a linearly independent sequence of $W^{1,p}_0(\Omega)$. It is easy to show that there exists $R_m > 0$ such that $I_n(u) \leq -1$ for

$u \in X_m \backslash B_m$, where $B_m := \{u \in X_m : \|u\| \le R_m\}$ (*cf.* [14, Lemma 3.9]). We denote

$$P(-P) := \{u \in W_0^{1,p}(\Omega) : u \ge 0 (u \le 0) \text{ a.e.}\},$$
$$D_\mu^\pm := \{u \in W_0^{1,p}(\Omega) : \operatorname{dist}(u, \pm P) \le \mu\}, \qquad D_\mu := D_\mu^+ \cup D_\mu^-,$$
$$G_m := \{h \in C(B_m, W_0^{1,p}(\Omega)) : h \text{ is odd}, h(x) = x \text{ for } x \in \partial_{X_m} B_m\}.$$

Recall that the genus of a symmetric set A of $W_0^{1,p}(\Omega)$ is defined by

$$\operatorname{gen}(A) := \inf\{k \ge 0 : \exists f \in C(A, \mathbb{R}^k \backslash \{0\}), f \text{ is odd}\}.$$

Here, we say that A is symmetric if $x \in A$ implies $-x \in A$.

By [14, Theorem 1.2], we know that, for every $n \in \mathbb{N}$, $I_n(u)$ has infinitely many critical points, denoted by $\{u_{n,k}\}_{k\in\mathbb{N}}$, in $X \backslash D_\mu$ for μ small enough. Moreover,

$$I_n(u_{n,k}) = d_{n,k} := \inf_{Z\in\Gamma_k} \sup_{u\in Z} I_n(u), \tag{2.1}$$

where $\Gamma_k := \{h(B_m \backslash B) \backslash D_\mu : h \in G_m \text{ for } m \ge n, B = -B \subset B_m \text{ open}, \operatorname{gen}(B) \le m - n\}$.

Lemma 2.2 *For every $k \in \mathbb{N}$, there exists $d_k^* > 0$ such that $\|u_{n,k}\| \le d_k^*$ for all $n \in \mathbb{N}$.*

Proof Consider the following auxiliary functional:

$$I_*(u) := \frac{1}{p}\left(\|u\|^p - \|u\|_p^p\right) - \frac{1}{p^*}\|u\|_\sigma^\sigma,$$

where $\sigma = (p + p^*)/2$. Since $p_n \to p^*$, we may assume $p_n > \sigma$ for all $n \in \mathbb{N}$. Then $\frac{1}{p^*}\|u\|_\sigma^\sigma \le \frac{\operatorname{meas}(\Omega)}{p^*} + \frac{1}{p_n}\|u\|_{p_n}^{p_n}$ for all $n \in \mathbb{N}$. This means

$$I_*(u) = I_n(u) + \left(\frac{1}{p_n}\|u\|_{p_n}^{p_n} - \frac{1}{p^*}\|u\|_\sigma^\sigma\right) \ge I_n(u) - \frac{\operatorname{meas}(\Omega)}{p^*}. \tag{2.2}$$

Note that $I_*(u)$ is the corresponding functional of the following equation:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \frac{\sigma}{p^*}|u|^{\sigma-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and the nonlinearity satisfies the assumptions of [14, Theorem 1.2]. Thus, this equation has a sequence of solutions $\{v_k\} \subset W_0^{1,p}(\Omega) \backslash (D_\mu^+ \cup D_\mu^-)$ such that

$$I_*(v_k) = \overline{d}_k := \inf_{Z\in\Gamma_k} \sup_{u\in Z} I_*(u)$$

for μ small enough. For every $k \in \mathbb{N}$, the definitions of $\overline{d}_k$ and $d_{k,n}$, together with (2.2), imply $\overline{d}_k + \frac{\operatorname{meas}(\Omega)}{p^*} \ge d_{k,n}$ for all $n \in \mathbb{N}$. On the other hand, since for every n, $\{u_{n,k}\}_{k\in\mathbb{N}}$ is a sequence of solutions for $(\mathcal{P}_n)$ whose energies satisfy (2.1), it follows that $d_{n,k} \ge (\frac{1}{p} - \frac{1}{p_1})(1 - \frac{\lambda}{\lambda_1})\|u_{n,k}\|^p$. We complete the proof by choosing $d_k^* = (\frac{(\overline{d}_k p^* + \operatorname{meas}(\Omega)) p_1 p \lambda_1}{p^*(p_1 - p)(\lambda_1 - \lambda)})^{1/p}$. □

By Lemma 2.2 and [9, Theorem 1.2], we know that for each $k \in \mathbb{N}$, there exists $u_k \in W_0^{1,p}(\Omega)$ such that $u_{n,k} \to u_k$ as $n \to \infty$ in $W_0^{1,p}(\Omega)$. The next lemma will give more information about u_k.

Lemma 2.3 *u_k is a sign-changing solution of Problem* (1.1) *for every $k \in \mathbb{N}$.*

Proof We first prove that u_k is a solution of Problem (1.1) for every $k \in \mathbb{N}$. Since $u_{n,k} \to u_k$ as $n \to \infty$ in $W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u_{n,k}|^{p-2}\nabla u_{n,k}\nabla\varphi\,dx \to \int_\Omega |\nabla u_k|^{p-2}\nabla u_k\nabla\varphi\,dx$$

and

$$\int_\Omega |u_{n,k}|^{p-2}u_{n,k}\varphi\,dx \to \int_\Omega |u_k|^{p-2}u_k\varphi\,dx$$

as $n \to \infty$ for every $\varphi \in W_0^{1,p}(\Omega)$. If we can prove

$$\int_\Omega |u_{n,k}|^{p_n-2}u_{n,k}\varphi\,dx \to \int_\Omega |u_k|^{2^*-2}u_k\varphi\,dx \tag{2.3}$$

as $n \to \infty$ for every $\varphi \in W_0^{1,p}(\Omega)$, then u_k is a solution of (1.1) for $u_{n,k}$ is a solution of $(\mathcal{P}_n)$. Indeed, $u_{n,k} \to u_k$ a.e. in Ω as $n \to \infty$ since $u_{n,k} \to u_k$ in $W_0^{1,p}(\Omega)$. By the Egoroff theorem, for every $\delta > 0$, there exists Ω_δ such that $u_{n,k} \to u_k$ uniformly in $\Omega\backslash\Omega_\delta$ and $|\Omega_\delta| < \delta$, where $|\Omega_\delta|$ is the Lebesgue measure of Ω_δ. This, together with the Lebesgue dominated convergence theorem, implies

$$\lim_{n\to\infty}\int_{\Omega\backslash\Omega_\delta} |u_{n,k}|^{p_n-2}u_{n,k}\varphi\,dx = \int_{\Omega\backslash\Omega_\delta} |u_k|^{p^*-2}u_k\varphi\,dx \quad \text{for every } \varphi \in W_0^{1,p}(\Omega). \tag{2.4}$$

On the other hand, for every $\varphi \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\int_{\Omega_\delta} \big||u_{n,k}|^{p_n-2}u_{n,k} - |u_k|^{p^*-2}u_k\big||\varphi|\,dx \\
&\quad\le \int_{\Omega_\delta} \big||u_{n,k}|^{p_n-2}u_{n,k} - |u_{n,k}|^{p-1} - |u_{n,k}|^{p^*-1}\big||\varphi|\,dx \\
&\qquad + \int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_k|^{p^*-2}u_k\big||\varphi|\,dx \\
&\qquad + \int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_{n,k}|^{p-1} - |u_{n,k}|^{p^*-1}\big||\varphi|\,dx \\
&\quad\le 2\int_{\Omega_\delta} \big||u_{n,k}|^{p-1} + |u_{n,k}|^{p^*-1}\big||\varphi|\,dx + \int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_k|^{p^*-2}u_k\big||\varphi|\,dx \\
&\qquad + \int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_{n,k}|^{p-1} - |u_{n,k}|^{p^*-1}\big||\varphi|\,dx \\
&\quad\le 2\int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1}\big||\varphi|\,dx + \int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_k|^{p^*-2}u_k\big||\varphi|\,dx \\
&\qquad + 3\int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_{n,k}|^{p-1} - |u_{n,k}|^{p^*-1}\big||\varphi|\,dx.
\end{aligned}$$

For every $\varepsilon > 0$, by the above inequality and the absolute continuity of the integral, we can take δ small enough such that

$$2\int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1}\big||\varphi|\,dx + \int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_k|^{p^*-2}u_k\big||\varphi|\,dx < \varepsilon/3.$$

For this δ, since $u_{n,k} \to u_k$ in $W_0^{1,p}(\Omega)$,

$$3\int_{\Omega_\delta} \big||u_k|^{p-1} + |u_k|^{p^*-1} - |u_{n,k}|^{p-1} - |u_{n,k}|^{p^*-1}\big||\varphi|\,dx < \varepsilon/3$$

for n large enough. By (2.4), for this δ, we have

$$\int_{\Omega\setminus\Omega_\delta} \big||u_{n,k}|^{p_n-2}u_{n,k}\varphi - |u_k|^{p^*-2}u_k\varphi\big|\,dx < \varepsilon/3$$

for n large enough. So (2.3) holds. Moreover, by a similar proof, we can show $d_k := \lim_{n\to\infty} d_{n,k} = I_n(u_{n,k}) = I(u_k)$.

Next, we will show u_k is sign-changing for all $k \in \mathbb{N}$. Since for each $n \in \mathbb{N}$, $u_{n,k}$ is a sign-changing solution of $(\mathcal{P}_n)$, multiplying $(\mathcal{P}_n)$ by $u_{n,k}^{\pm}$, we obtain $\|u_{n,k}^{\pm}\|^p = \lambda\|u_{n,k}^{\pm}\|_p^p + \|u_{n,k}^{\pm}\|_{p_n}^{p_n}$, where $u^{\pm} = \max\{\pm u, 0\}$. Note that $\lambda < \lambda_1$, by the Sobolev imbedding theorem, we have $0 < (1 - \frac{\lambda}{\lambda_1})C \le \|u_{n,k}^{\pm}\|_{p^*}^{p_n-p}$. It follows that $u_{n,k} \to u_k$ in $L^{p^*}(\Omega)$ as $n \to \infty$ for $u_{n,k} \to u_k$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. This gives $0 < (1 - \frac{\lambda}{\lambda_1})C \le \|u_k^{\pm}\|_{p^*}^{p^*-p}$, *i.e.*, $u_k^{\pm} \not\equiv 0$ for all $k \in \mathbb{N}$. □

Let $\varepsilon > 0$ and $c \in \mathbb{R}$, we denote

$$\begin{aligned} &K := \{u \in W_0^{1,p}(\Omega) : I'(u) = 0\}, \qquad K_c := \{u \in W_0^{1,p}(\Omega) : I(u) = c, I'(u) = 0\},\\ &K_\mu^* := K\setminus(\operatorname{int}(D_\mu^+) \cup \operatorname{int}(D_\mu^-)), \qquad K_{c,\mu}^* := K_c\setminus(\operatorname{int}(D_\mu^+) \cup \operatorname{int}(D_\mu^-)),\\ &\mathcal{N}_{c,\mu,\varepsilon} := \{u \in W_0^{1,p}(\Omega) : \operatorname{dist}(u, K_c^*) < \varepsilon\}. \end{aligned}$$

Thanks to Lemma 2.3, $u_k \in K_{\mu_k}^*$ for some $\mu_k > 0$. We claim that $\{u_k\} \subset K_\mu^*$ for some $\mu > 0$. Indeed, if not, then $\operatorname{dist}(u_k, P) \to 0$ as $k \to \infty$ without loss of generality. On the one hand, since u_k is a solution of (1.1), $\langle I'(u_k), u_k - S_\lambda(u_k)\rangle_{W_0^{1,p}(\Omega), W_0^{-1,p}(\Omega)} = 0$, where $S_\lambda(u_k) : (-\Delta_p)^{-1}(\lambda|u_k|^{p-2}u_k + |u_k|^{p^*-2}u_k)$. On the other hand, by [17, Lemma 3.7], we have

$$\langle I'(u_k), u_k - S_\lambda(u_k)\rangle_{W_0^{1,p}(\Omega), W_0^{-1,p}(\Omega)} \ge C\|u_k - S_\lambda(u_k)\|^2(\|u_k\| + \|S_\lambda(u_k)\|)^{p-2}$$

for $1 < p < 2$ and

$$\langle I'(u_k), u_k - S_\lambda(u_k)\rangle_{W_0^{1,p}(\Omega), W_0^{-1,p}(\Omega)} \ge C\|u_k - S_\lambda(u_k)\|^p$$

for $p \ge 2$. Note that by a similar proof of [14, Lemma 3.3], we can see that $S_\lambda(D_\mu^{\pm}) \subset \operatorname{int}(D_\mu^{\pm})$ for μ small enough. Thus, $\|u_k - S_\lambda(u_k)\| > 0$ for k large enough. This implies

$$\langle I'(u_k), u_k - S_\lambda(u_k)\rangle_{W_0^{1,p}(\Omega), W_0^{-1,p}(\Omega)} \ge C_k > 0$$

for k large enough, which contradicts $\langle I'(u_k), u_k - S_\lambda(u_k)\rangle_{W_0^{1,p}(\Omega), W_0^{-1,p}(\Omega)} = 0$. For the sake of convenience, we denote K_μ^*, $K_{c,\mu}^*$, $\mathcal{N}_{c,\mu,\varepsilon}$ by K^*, K_c^*, $\mathcal{N}_{c,\varepsilon}$. Note that for every $c \in \mathbb{R}$, K_c is compact in $W_0^{1,p}(\Omega)$ (*cf.* [9, Theorem 1.2]). It follows from [19, Proposition 7.5] that there exists $\varepsilon > 0$ such that

$$\operatorname{gen}(\mathcal{N}_{c,2\varepsilon}) = \operatorname{gen}(K_c^*) < +\infty. \tag{2.5}$$

Let $J_n^c := \{u \in W_0^{1,p}(\Omega) : I_n(u) \le c\}$ and $\mathcal{Q}_n^c := D_\mu \cup J_n^c$. Let $J^c := \{u \in W_0^{1,p}(\Omega) : I(u) \le c\}$. For $\delta > 0$ small enough, we define $\mathcal{A}_{n,\varepsilon}^{c,\delta} := (\mathcal{Q}_n^{c+\delta} \setminus \mathcal{Q}_n^{c-\delta}) \setminus \mathcal{N}_{c,\varepsilon}$, then we have the following.

Lemma 2.4 *Assume that there exists $\delta > 0$ such that $K^* \cap J^{c+\delta} \setminus \operatorname{int}(J^{c-\delta}) = K_c^*$ for n large. Then there exists $\gamma > 0$ such that $\|I_n'(u)\| \ge \gamma$ for $u \in \mathcal{A}_{n,\varepsilon}^{c,\delta}$ and large n.*

Proof Assume a contradiction. Then, for every $n \in \mathbb{N}$, there exists $\{v_{n,k}\} \subset \mathcal{A}_{n,\varepsilon}^{c,\delta}$ such that $\lim_{k\to\infty} I_n'(v_{n,k}) = 0$. It is clear that I_n satisfies the (PS) condition for every $n \in \mathbb{N}$. Hence there exists $v_n \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $v_{n,k} \to v_n$ in $W_0^{1,p}(\Omega)$ as $k \to \infty$ with $I_n'(v_n) = 0$ and $I_n(v_n) \in [c-\delta, c+\delta]$. This implies

$$c + \delta \ge I_n(v_n) = \left(\frac{1}{p} - \frac{1}{p_n}\right)\left(1 - \frac{\lambda}{\lambda_1}\right)\|v_n\|^p \ge \left(\frac{1}{p} - \frac{1}{p_1}\right)\left(1 - \frac{\lambda}{\lambda_1}\right)\|v_n\|^p.$$

Thus, by [9, Theorem 1.2], up to a subsequence, we see that there exists $v_0 \in W_0^{1,p}(\Omega)$ such that $v_n \to v_0$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Moreover, by using the arguments in the proof of Lemma 2.3, we have $I'(v_0) = 0$ and $I(v_0) \in [c-\delta, c+\delta]$. On the other hand, for large n, $v_n \notin (\operatorname{int}(D_\mu^+) \cup \operatorname{int}(D_\mu^-)) \cup \mathcal{N}_{c,\varepsilon}$ since $v_{n,k} \in \mathcal{A}_{n,\varepsilon}^{c,\delta}$. It follows that $v_0 \notin (\operatorname{int}(D_\mu^+) \cup \operatorname{int}(D_\mu^-)) \cup \mathcal{N}_{c,\varepsilon}$. This contradicts the fact that $K^* \cap J_n^{c+\delta} \setminus \operatorname{int}(J_n^{c-\delta}) = K_c^*$. □

Lemma 2.5 *Assume that there exists $\gamma > 0$ such that $\|I_n'(u)\| \ge \gamma$ for every $u \in \mathcal{A}_{n,\varepsilon}^{c,\delta}$ and large n. Then there exist $\delta > 0$ and an odd continuous map η_n such that $\eta_n : \mathcal{A}_{n,2\varepsilon}^{c,\delta} \cup \mathcal{Q}_n^{c-\delta} \to \mathcal{Q}_n^{c-\delta}$ and $\eta|_{\mathcal{Q}_n^{c-\delta}} = Id$ for large n.*

Proof We first assume $1 < p < 2$. It is clear that there exists $L > 0$ such that

$$\|u\| + \left\|S_{n,\lambda}(u)\right\| \le L \quad \text{for all } u \in \mathcal{N}_{c,2\varepsilon}, \tag{2.6}$$

where

$$\left\langle S_{n,\lambda}(u), \varphi\right\rangle := \int_\Omega \left(\lambda|u|^{p-2}u + |u|^{p_n-2}u\right)\varphi\,dx \quad \text{for } u \in W_0^{1,p}(\Omega) \text{ and } \varphi \in W_0^{-1,p}(\Omega).$$

Let $T_{n,\lambda} : W_0^{1,p}(\Omega)\setminus K \to W_0^{1,p}(\Omega)$ be the local Lipschitz continuous operator obtained in [14, Lemma 2.1] and let $\phi_u(t)$ be the solution of the following O.D.E.

$$\begin{cases} \frac{d\phi}{dt} = -\phi + T_{n,\lambda}(\phi), \\ \phi = u \in W_0^{1,p}(\Omega)\setminus K. \end{cases}$$

Denote $\tau(u)$ to be the maximal interval of existence of $\phi_u(t)$.

Claim 1: $\phi_u(t)$ cannot enter $\mathcal{N}_{c,\varepsilon}$ before it enters $\mathcal{Q}_n^{c-\delta}$ for small δ, large n and $u \in \mathcal{A}_{n,2\varepsilon}^{c,\delta}$.

Indeed, if the claim fails, then for every $\delta > 0$, $\phi_u(t)$ will enter $\mathcal{N}_{c,\varepsilon}$ before it enters $\mathcal{Q}_n^{c-\delta}$. Since $u \in \mathcal{A}_{n,2\varepsilon}^{c,\delta} \subset W_0^{1,p}(\Omega)\backslash\mathcal{N}_{c,2\varepsilon}$, there exist $0 \leq t_1 < t_2 < \tau(u)$ such that $\phi_u(t) \in \mathcal{N}_{c,2\varepsilon}\backslash\mathcal{N}_{c,\varepsilon}$ for $t \in (t_1, t_2]$ and

$$\operatorname{dist}\bigl(\phi_u(t_1), K_c^*\bigr) = 2\varepsilon, \qquad \operatorname{dist}\bigl(\phi_u(t_2), K_c^*\bigr) = \varepsilon.$$

By [14, Lemma 2.1], $C\|u - S_{n,\lambda}(u)\|^2(\|u\| + \|S_{n,\lambda}(u)\|)^{p-2} \leq \langle I_n(u), u - T_{n,\lambda}(u)\rangle$. On the other hand, by the choice of t_1 and t_2, we know that $\phi_u(t) \in \mathcal{A}_{n,\varepsilon}^{c,\delta}$ for $t \in (t_1, t_2]$. Thanks to [17, Lemma 3.8], $\|u - S_{n,\lambda}(u)\| \geq (\frac{\gamma}{C})^{1/(p-1)}$ for large n. This, together with (2.6) and [14, Lemma 2.1], implies

$$\begin{aligned}
\varepsilon &\leq \bigl\|\phi_u(t_2) - \phi_u(t_1)\bigr\| \leq \int_{t_1}^{t_2} \bigl\|\phi_u(t) - T_{n,\lambda}\bigl(\phi_u(t)\bigr)\bigr\|\,dt \\
&\leq C\int_{t_1}^{t_2} \bigl\|\phi_u(t) - S_{n,\lambda}\bigl(\phi_u(t)\bigr)\bigr\|\,dt \\
&\leq C\int_{t_1}^{t_2} \bigl\|\phi_u(t) - S_{n,\lambda}\bigl(\phi_u(t)\bigr)\bigr\|^2\bigl(\bigl\|\phi_u(t)\bigr\| + \bigl\|S_{n,\lambda}\bigl(\phi_u(t)\bigr)\bigr\|\bigr)^{p-2}\,dt \\
&\leq C\int_{t_1}^{t_2} \bigl\langle I_n\bigl(\phi_u(t)\bigr), \phi_u(t) - T_{n,\lambda}\bigl(\phi_u(t)\bigr)\bigr\rangle\,dt \\
&= C\bigl(I_n(t_1) - I_n(t_2)\bigr) \leq 4C\delta.
\end{aligned}$$

A contradiction with $\delta < 4C/\varepsilon$.

Claim 2: There exists $\tau_1(t) < \tau(u)$ such that $\phi_u(\tau_1(u)) \in \mathcal{Q}_n^{c-\delta}$ for large n and $u \in \mathcal{A}_{n,2\varepsilon}^{c,\delta}$.

If the claim is not true, then $\phi_u(t) \in \mathcal{Q}_n^{c+\delta}\backslash\mathcal{Q}_n^{c-\delta}$ for all $t \in (0, \tau(u))$. We first consider the case of $\tau(u) < +\infty$. In fact, by Claim 1, $\phi_u(t) \notin \mathcal{N}_{c,\varepsilon}$, *i.e.*, $\phi_u(t) \in \mathcal{A}_{n,\varepsilon}^{c,\delta}$ for all $t \in (0, \tau(u))$. Since $\|I_n'(u)\| \geq \gamma > 0$ for $u \in \mathcal{A}_{n,\varepsilon}^{c,\delta}$ and large n, we must have

$$\bigl\|\phi_u(t)\bigr\| \to \infty \quad \text{as } t \to \tau(u). \tag{2.7}$$

On the other hand, by [14, Lemma 2.1] and [17, Lemma 5.2], we have

$$\begin{aligned}
\bigl\|\phi_u(t) - \phi_u(0)\bigr\| &\leq \int_0^t \bigl\|\phi_u(s) - T_{\lambda,n}\bigl(\phi_u(s)\bigr)\bigr\|\,ds \\
&\leq C\int_0^t \bigl\|\phi_u(s) - S_{\lambda,n}\bigl(\phi_u(s)\bigr)\bigr\|\,ds \\
&\leq C\int_0^t \bigl(1 + \bigl\|\phi_u(s) - S_{\lambda,n}\bigl(\phi_u(s)\bigr)\bigr\|\bigr)^p\,ds \\
&\leq C\int_0^t \bigl(1 + \bigl\|\phi_u(s) - S_{\lambda,n}\bigl(\phi_u(s)\bigr)\bigr\|\bigr)^2\bigl(\bigl\|\phi_u(s)\bigr\| + \bigl\|S_{\lambda,n}\bigl(\phi_u(s)\bigr)\bigr\|\bigr)^{p-2}\,ds \\
&\leq C\int_0^t \bigl\|\phi_u(s) - S_{\lambda,n}\bigl(\phi_u(s)\bigr)\bigr\|^2\bigl(\bigl\|\phi_u(s)\bigr\| + \bigl\|S_{\lambda,n}\bigl(\phi_u(s)\bigr)\bigr\|\bigr)^{p-2}\,ds \\
&\leq C\bigl(I_n\bigl(\phi_u(0)\bigr) - I_n\bigl(\phi_u(t)\bigr)\bigr) \leq C.
\end{aligned}$$

This means $\|\phi_u(t)\| \leq \|u\| + C$ for all $t \in (0, \tau(u))$, which contradicts with (2.7). It follows that there must exist $\tau_1(u) < \tau(u)$ such that $\phi_u(\tau_1(u)) \in \mathcal{Q}_n^{c-\delta}$ for $u \in \mathcal{A}_{n,2\varepsilon}^{c,\delta}$, large n and

$\tau(u) < +\infty$. Next, we consider the case of $\tau(u) = +\infty$. Since $\|u - S_{n,\lambda}(u)\| \geq (\frac{\gamma}{C})^{1/(p-1)}$ for all $u \in \mathcal{A}_{n,\varepsilon}^{c,\delta}$ and large n, it follows from [14, Lemma 2.1] and [17, Lemma 5.2] that

$$\begin{aligned}\frac{dI_n(\phi_u(t))}{dt} &= \langle I_n(\phi_u(t)), -\phi_u(t) + T_{n,\lambda}(\phi_u(t))\rangle \\ &\leq -C\|\phi_u(t) - S_{n,\lambda}(\phi_u(t))\|^2(\|\phi_u(t)\| + \|S_{n,\lambda}(\phi_u(t))\|)^{p-2} \\ &\leq -C\|\phi_u(t) - S_{n,\lambda}(\phi_u(t))\|^2(1 + \|\phi_u(t) - S_{n,\lambda}(\phi_u(t))\|)^{p-2} \\ &\leq -C < 0.\end{aligned}$$

Thus, there also exists $\tau_1(u) \in (0, +\infty)$ such that $\phi_u(\tau_1(u)) \in \mathcal{Q}_n^{c-\delta}$ for $u \in \mathcal{A}_{n,2\varepsilon}^{c,\delta}$ and $\tau(u) = +\infty$. Moreover, we must have $\phi_u(t) \in \mathcal{Q}_n^{c-\delta}$ for $t \in (\tau_1(u), \tau(u))$ since $\frac{dI_n(\phi_u(t))}{dt} \leq 0$ for all $u \in W_0^{1,p}(\Omega)\backslash K$.

Let

$$\eta_n(u) = \begin{cases} \phi_u(\tau_1(u)), & u \in \mathcal{A}_{n,2\varepsilon}^{c}, \\ u, & u \in \mathcal{Q}_n^{c-\delta}. \end{cases}$$

Then, by the continuity of $\phi_u(t)$, $\eta_n(u)$ is continuous. Note that $\phi_u(t)$ is odd and $\tau_1(u)$ is even, we see that $\eta_n(u)$ is odd and it is the desired map. The situation of $p \geq 2$ can be proved in a similar way. Therefore, we complete the proof of this lemma. □

Proof of Theorem 1.1 We first consider the case $\lambda \geq \lambda_1$. Thanks to Lemma 2.1 and [9, Theorem 1.1], (1.1) has infinitely many sign-changing solutions. Next, we consider the case of $\lambda \in (0, \lambda_1)$. Since for every $n \in \mathbb{N}$, $0 \leq d_{n,k} \leq d_{n,k+1}$ for all $k \in \mathbb{N}$, $d_k \leq d_{k+1}$ for all $k \in \mathbb{N}$. It follows that two cases may occur:

Case 1: There are $1 < k_1 < k_2 < \cdots$ such that $d_{k_1} < d_{k_2} < \cdots$.

In this case, Problem (1.1) has infinitely many sign-changing solutions.

Case 2: There exists $k_0 > 0$ such that $d_* = d_k$ for all $k \geq k_0$.

In this case, if $(K^* \cap J^{d_*+\delta} \backslash J^{d_*-\delta}) \backslash K_{d_*}^* \neq \emptyset$ for every $\delta > 0$ small enough, then Problem (1.1) also has infinitely many sign-changing solutions. Otherwise, there exists $\delta_0 > 0$ such that $(K^* \cap J^{d_*+\delta} \backslash J^{d_*-\delta}) = K_{d_*}^*$ for $\delta < \delta_0$. Thanks to Lemmas 2.4 and 2.5, there exists η_n such that $\eta_n(\mathcal{A}_{n,2\varepsilon}^{d_*} \cup \mathcal{Q}_n^{d_*-\delta}) \subset \mathcal{Q}_n^{d_*-\delta}$ for small δ and large n. Fix $l \in \mathbb{N}$ and $k \geq k_0$, the definitions of d_k and d_{k+l} give that there exists a large n such that $d_{n,k} > d_* - \delta$ and $d_{n,k+l} < d_* + \delta$ for small $\delta \in (0,1)$. By the definition of $d_{n,k+l}$, there exists $Z \in \Gamma_{k+l}$ such that $\sup_Z I_n(u) < d_* + \delta$, where $Z = h(B_m \backslash B) \backslash D_\mu$, $h \in G_m$ and $\operatorname{gen}(B) \leq m - k - l$. It follows that $h(B_m \backslash B) \backslash \mathcal{N}_{d_*,2\varepsilon} \subset \mathcal{A}_{n,2\varepsilon}^{c,\delta} \cup \mathcal{Q}_n^{c-\delta}$. Thus, $\eta_n(h(B_m \backslash B) \backslash \mathcal{N}_{d_*,2\varepsilon}) \subset \mathcal{Q}_n^{d_*-\delta}$. By the choice of δ and B_m, we have $\eta_n \circ h \in G_m$. If $\operatorname{gen}(B \cup h^{-1}(\mathcal{N}_{d_*,2\varepsilon})) \leq m - k$, then we have

$$d_* - \delta < d_{n,k} \leq \sup_{\eta_n \circ h(B_m \backslash (B \cup h^{-1}(\mathcal{N}_{d_*,2\varepsilon})))} I_n(u) \leq d_* - \delta.$$

A contradiction. By the properties of gen, we have

$$m - k + 1 \leq \operatorname{gen}(B \cup h^{-1}(\mathcal{N}_{d_*,2\varepsilon})) \leq \operatorname{gen}(B) + \operatorname{gen}(\mathcal{N}_{d_*,2\varepsilon}) \leq m - k - l + \operatorname{gen}(\mathcal{N}_{d_*,2\varepsilon}).$$

This implies $\operatorname{gen}(\mathcal{N}_{d_*,2\varepsilon}) \geq l + 1$. Since $l \in \mathbb{N}$ is arbitrary, we have $\operatorname{gen}(\mathcal{N}_{d_*,2\varepsilon}) = +\infty$, which contradicts with (2.5). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors typed, read and approved the final manuscript.

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