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# Turing instability and stationary patterns in a predator-prey systems with nonlinear cross-diffusions

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**Abstract**

In this paper, we study a strongly coupled reaction-diffusion system which describes two interacting species in prey-predator ecosystem with nonlinear cross-diffusions and Holling type-II functional response. By a linear stability analysis, we establish some stability conditions of constant positive equilibrium for the ODE and PDE systems. In particular, it is shown that Turing instability can be induced by the presence of cross-diffusion. Furthermore, based on Leray-Schauder degree theory, the existence of non-constant positive steady state is investigated. Our results indicate that the model has no non-constant positive steady state with no cross-diffusion, while large cross-diffusion effect of the first species is helpful to the appearance of Turing instability as well as non-constant positive steady state (stationary patterns).

**Keywords:** cross-diffusion; Holling type-II functional response; Turing instability; non-constant positive steady state; stationary patterns

## 1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this paper, we are interested in a strongly coupled reaction-diffusion equations with Holling type-II functional response

$$\begin{cases} u_t - \Delta[(d_1 + \alpha_1 u + \frac{\beta_1}{1+v})u] = u(1 - \frac{u}{K} - \frac{mv}{1+u}) & \text{in } \Omega \times (0,\infty), \\ v_t - \Delta[(d_2 + \frac{\beta_2}{1+u} + \alpha_2 v)v] = v(\frac{mu}{1+u} - \theta) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\nu$ is the unit outward normal to $\partial\Omega$. The two unknown functions $u(x,t)$ and $v(x,t)$ represent the spatial distribution densities of the prey and predator, respectively. The constants $d_i$, $\alpha_i$, $\beta_i$ $(i = 1,2)$, $K$, $m$ and $\theta$ are all positive, and $u_0$, $v_0$ are nonnegative functions which are not identically zero. Moreover, $d_i$ is the diffusion rate of the two species, $\alpha_i$ expresses the self-diffusion effect, $\beta_i$ is called the cross-diffusion coefficient, $K$ accounts for the carrying capacity of the prey, $\theta$ is the death rate of the predator, and $m$ can be regarded as the measure of the interaction strength between the two species. In this model, the prey

$u$ and the predator $v$ diffuse with fluxes

$$J_1 = -\left(d_1 + 2\alpha_1 u + \frac{\beta_1}{1+v}\right)\nabla u + \frac{\beta_1 u}{(1+v)^2}\nabla v$$

and

$$J_2 = -\left(d_2 + \frac{\beta_2}{1+u} + 2\alpha_2 v\right)\nabla v + \frac{\beta_2 v}{(1+u)^2}\nabla u,$$

respectively. The cross-diffusion terms $\frac{\beta_1 u}{(1+v)^2}\nabla v$ and $\frac{\beta_2 v}{(1+u)^2}\nabla u$ can be explained that the prey keeps away from the predator while the predator moves away from a large group of prey. For more detailed biological meaning of the parameters, one can make some reference to [1–3].

The ODE system of (1.1)

$$\frac{du}{dt} = u\left(1 - \frac{u}{K} - \frac{mv}{1+u}\right), \qquad \frac{dv}{dt} = v\left(\frac{mu}{1+u} - \theta\right), \quad t > 0, \tag{1.2}$$

has been extensively studied in the existing literature; see, for example, [4–6]. The known results mainly focused on the existence and uniqueness of a limit cycle. In [6], Rosenzweig argued that enrichment of the environment (larger carrying capacity $K$) leads to destabilizing of the coexistence equilibrium, which is the so-called paradox of enrichment. Cheng [4] first proved the uniqueness of limit cycle. Hsu and Shi [5] discussed the relaxation oscillator profile of the unique limit cycle and found that (1.2) has a periodic orbit if $m$ is larger than a threshold value.

In mathematical biology, the classical prey-predator model (ODE system) reflects only population changes due to predation in a situation where predator and prey densities are not spatially dependent. It does not take into account either the fact that population is usually not homogeneously distributed, or the fact that predators and preys naturally develop strategies for survival. Both of these considerations involve diffusion processes which can be quite intricate as different concentration levels of predators and preys caused by different population movements. Such movements can be determined by the concentration of the same species (diffusion) and that of other species (cross-diffusion). In view of this, Shigesada, Kawasaki and Teramoto first proposed a strongly coupled reaction-diffusion model with Lotka-Volterra type reaction term (SKT model) to describe spatial segregation of interacting population species in one-dimensional space [3]. Since then the two-species SKT competing system and its overall behaviors continue to be of great interest in literature to both mathematical analysis and real-life modeling [7–10]. For the studies on biological models, since each model has rich and interesting properties and often describes complex biological process, it is very difficult to get some general conclusions for a class of mathematical models. So research in mathematical biology has often been performed by investigating a specific model, the focus of which is to discuss the influences of parameters on the behavior of species in the ecosystem. Thus, more and more attention has been recently focused on three or multi-species systems and the SKT model in any space dimension due to their more complicated patterns, and the SKT models with other types of reaction terms have also been proposed and investigated [11–19]. The obtained results mainly relate to the stability analysis of constant positive steady states and the existence of non-constant positive steady states (stationary patterns) [9, 10, 12–21], Turing instability [22, 23], and the global existence of non-negative time-dependent solutions [7, 8, 11, 24].

The role of diffusion in the modeling of many biological processes has been extensively studied. Starting with Turing's seminal work [25], diffusion and cross diffusion have been observed as causes of the spontaneous emergence of ordered structures, called patterns, in a variety of nonequilibrium situations. Diffusion-driven instability, also called Turing instability, has also been verified empirically in some chemical and biological models [26–28]. For the system with cross-diffusion, we can know that this kind of cross-diffusion may be helpful to create non-constant positive steady-state solutions for the predator-prey system, for example [9, 10, 16]. Recently, the authors of [22] discussed a two-species Holling-Tanner model with simple linear cross-diffusion

$$\begin{cases} u_t - \Delta(d_1 u + d_3 v) = u(a-u) - \frac{buv}{(1+\alpha u)(1+\beta v)}, \\ v_t - \Delta(d_2 u + d_4 v) = v(c - \frac{v}{\gamma u}) \end{cases}$$

and showed that under some parameters the positive equilibrium is stable for a diffusion system while unstable for a cross-diffusion system, which implies that cross-diffusion can induce the Turing instability of the uniform equilibrium. In [23], Xie investigated a class of strongly coupled prey-predator models with four Holling-type functional responses:

$$\begin{cases} u_t - \Delta[(d_1 + d_3 v)u] = g_1(u,v), \\ v_t - \Delta[(d_2 + \frac{d_4}{1+u})v] = g_2(u,v). \end{cases}$$

The results indicated that diffusion and cross-diffusion in these models cannot drive Turing instability. However, diffusion and cross-diffusion can still create non-constant positive solutions for the models.

As for reaction-diffusion system of (1.2), the diffusive predator-prey equations with no self- and cross-diffusion ($\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ in (1.1)) under Neumann boundary value conditions have also been investigated (see, for example, [29–33]). Ko and Ryu [29] obtained some results on the global stability of the constant steady state solutions and the existence of at least one non-constant equilibrium solution. Medvinsky *et al.* [30] used this model as a simple mathematical model to investigate the pattern formation of a phytoplankton-zooplankton system, and their numerical studies show a rich spectrum of spatiotemporal patterns. The discussion in [32] shows this system possesses complex spatiotemporal dynamics via a sequence of bifurcation of spatial nonhomogeneous periodic orbits and spatial nonhomogeneous steady state solutions. In [31], Peng and Shi proved the non-existence of non-constant positive steady state solutions. Recently, the existence, multiplicity and stability of positive solutions for the weakly coupled equations in (1.1) with Dirichlet boundary conditions were investigated in [33].

From the above introductions, one can learn that few studies have been conducted into the occurrence of Turing instability for a strongly coupled reaction-diffusion system with nonlinear cross-diffusion terms in the literature. Motivated by a series of pioneering works such as [9, 10, 16], we are interested in the instability induced by cross-diffusion and the stationary patterns of strongly coupled model (1.1). The aim of this paper is to discuss Turing instability and establish the existence of non-constant positive steady states of system (1.1). The methods we employed are the classical linearization method and the Leray-Schauder degree theory. However, while performing *a priori* estimates and stability analysis, we must try a new method and techniques to solve difficulties caused by nonlinear cross-diffusion terms $\frac{\beta_1 u}{(1+v)^2}\nabla v$ and $\frac{\beta_2 v}{(1+u)^2}\nabla u$. Nonlinear cross-diffusion terms also add complexity of computation of characteristic equations. Moreover, this paper focuses on the influence of nonlinear cross-diffusion terms on the appearance of Turing instability, and the discussion shows that large cross-diffusion coefficient of the first species is helpful to the appearance of Turing instability as well as non-constant positive steady state.

The paper is organized as follows. In Section 2, we discuss the stability of a positive equilibrium point for ODE and PDE systems and then obtain sufficient conditions of the appearance of Turing pattern. The results imply that cross-diffusion $\beta_1$ has a destabilizing effect, which is helpful to the occurrence of Turing instability. In Section 3, we obtain *a priori* upper and lower bounds for the positive steady states problem of (1.1) in order to calculate the topological degree. In Section 4, the non-existence of non-constant positive steady state for (1.1) with vanished cross-diffusions is discussed. In Section 5, we establish the global existence of non-constant positive steady state of (1.1) for suitable values of cross-diffusion coefficient $\beta_1$ and then show that large cross-diffusion effect $\beta_1$ can create non-constant positive steady states.

## 2 Turing instability driven by cross-diffusion

Denote $\xi = \frac{\theta}{m-\theta}$. It is known from [31] that problem (1.1) has a unique positive equilibrium

$$\mathbf{w}^* = \left(u^*, v^*\right)^{\mathrm{T}} = \left(\xi, \frac{(K-\xi)(1+\xi)}{Km}\right)^{\mathrm{T}}$$

if and only if

$$m > \frac{(1+K)\theta}{K}. \tag{2.1}$$

Moreover, problem (1.1) has a trivial equilibrium $\mathbf{0}^* = (0,0)^{\mathrm{T}}$ and a semi-trivial equilibrium $\mathbf{u}^* = (K,0)^{\mathrm{T}}$.

We first investigate the stability of positive equilibrium for a reaction-diffusion system.

**Lemma 2.1** *Suppose that* $\theta < K(m-\theta) < m+\theta$, $\frac{\beta_1 m}{m-\theta} < \beta_2$. *Then the positive equilibrium* $\mathbf{w}^*$ *of* (1.1) *is uniformly asymptotically stable.*

*Proof* For simplicity, we denote $\mathbf{w} = (u,v)^{\mathrm{T}}$ and

$$\begin{aligned}
\boldsymbol{\Phi}(\mathbf{w}) &= \left(\phi_1(\mathbf{w}), \phi_2(\mathbf{w})\right)^{\mathrm{T}} = \left(\left(d_1 + \alpha_1 u + \frac{\beta_1}{1+v}\right)u, \left(d_2 + \frac{\beta_2}{1+u} + \alpha_2 v\right)v\right)^{\mathrm{T}}, \\
\mathbf{F}(\mathbf{w}) &= \left(f_1(\mathbf{w}), f_2(\mathbf{w})\right)^{\mathrm{T}} = \left(u\left(1 - \frac{u}{K} - \frac{mv}{1+u}\right), v\left(\frac{mu}{1+u} - \theta\right)\right)^{\mathrm{T}}.
\end{aligned}$$

Then problem (1.1) can be rewritten as

$$\begin{cases} \frac{\partial \mathbf{w}}{\partial t} - \Delta \mathbf{\Phi}(\mathbf{w}) = \mathbf{F}(\mathbf{w}) \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial \mathbf{w}}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ \mathbf{w}(x,0) = (u_0(x), v_0(x))^{\mathrm{T}} \quad \text{in } \Omega. \end{cases} \tag{2.2}$$

The linearization of problem (2.2) at the positive equilibrium $\mathbf{w}^*$ is

$$\begin{cases} \frac{\partial \mathbf{w}}{\partial t} - \mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)\Delta \mathbf{w} = \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\mathbf{w} \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial \mathbf{w}}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ \mathbf{w}(x,0) = (u_0(x), v_0(x))^{\mathrm{T}} \quad \text{in } \Omega, \end{cases} \tag{2.3}$$

where $\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*) = \begin{pmatrix} d_1+2\alpha_1 u^*+\frac{\beta_1}{1+v^*} & -\frac{\beta_1 u^*}{(1+v^*)^2} \\ -\frac{\beta_2 v^*}{(1+u^*)^2} & d_2+\frac{\beta_2}{1+u^*}+2\alpha_2 v^* \end{pmatrix}$, $\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & 0 \end{pmatrix}$. Here

$$b_{11} = 1 - \frac{2u^*}{K} - \frac{mv^*}{(1+u^*)^2} = \frac{u^*}{K(1+u^*)}\big(K - 1 - 2u^*\big),$$
$$b_{12} = -\frac{mu^*}{1+u^*} < 0, \qquad b_{21} = \frac{mv^*}{(1+u^*)^2} > 0.$$

It is easy to verify that $b_{11} < 0$ if $K(m-\theta) < m+\theta$.

Let $\{\lambda_i, \varphi_i\}_{i=1}^{\infty}$ be a set of eigenpairs for $-\Delta$ in $\Omega$ with no flux boundary condition, where $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, and let $E(\lambda_i)$ be the eigenspace corresponding to $\lambda_i$ in $C^1(\overline{\Omega})$, let $\varphi_{ij}$, $j = 1, \ldots, \dim E(\lambda_i)$, be an orthonormal basis of $E(\lambda_i)$. Let

$$\mathbf{X} = \big\{\mathbf{w} \in \big[C^2(\Omega) \cap C^1(\overline{\Omega})\big]^2 | \partial\mathbf{w}/\partial\nu = 0 \text{ on } \partial\Omega\big\}, \qquad \mathbf{X}_{ij} = \big\{\mathbf{c}\varphi_{ij} | \mathbf{c} \in \mathbb{R}^2\big\}.$$

Then we can do the following decomposition:

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i, \quad \text{where } \mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\lambda_i)} \mathbf{X}_{ij}. \tag{2.4}$$

For each $i \ge 1$, $\mathbf{X}_i$ is invariant under the operator $\mathfrak{L} = \mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)\Delta + \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)$. Then problem (2.3) has a non-trivial solution of the form $\mathbf{w} = \mathbf{c}\varphi \exp\{\mu t\}$ ($\mathbf{c} \in \mathbb{R}^2$ is a constant vector) if and only if $(\mu, \mathbf{c})$ is an eigenpair for the matrix $-\lambda_i \mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*) + \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)$.

The characteristic equation of the matrix $-\lambda_i \mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*) + \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)$ is given by

$$p_i(\mu) = \mu^2 - \operatorname{trace}\big[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}\big(\mathbf{w}^*\big) + \mathbf{F}_{\mathbf{w}}\big(\mathbf{w}^*\big)\big]\mu + \det\big[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}\big(\mathbf{w}^*\big) + \mathbf{F}_{\mathbf{w}}\big(\mathbf{w}^*\big)\big] = 0.$$

Notice that

$$\begin{aligned} &\operatorname{trace}\big[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}\big(\mathbf{w}^*\big) + \mathbf{F}_{\mathbf{w}}\big(\mathbf{w}^*\big)\big] \\ &\quad = -\left(d_1 + 2\alpha_1 u^* + \frac{\beta_1}{1+v^*} + d_2 + \frac{\beta_2}{1+u^*} + 2\alpha_2 v^*\right)\lambda_i + b_{11} < 0, \\ &\det\big[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}\big(\mathbf{w}^*\big) + \mathbf{F}_{\mathbf{w}}\big(\mathbf{w}^*\big)\big] = A\lambda_i^2 + B\lambda_i + C, \end{aligned}$$

where

$$A = \left(d_1 + 2\alpha_1 u^* + \frac{\beta_1}{1+v^*}\right)\left(d_2 + \frac{\beta_2}{1+u^*} + 2\alpha_2 v^*\right) - \frac{\beta_1\beta_2 u^* v^*}{(1+u^*)^2(1+v^*)^2} > 0,$$
$$B = -\left[b_{11}\left(d_2 + \frac{\beta_2}{1+u^*} + 2\alpha_2 v^*\right) + \frac{\beta_1 u^*}{(1+v^*)^2} b_{21} + \frac{\beta_2 v^*}{(1+u^*)^2} b_{12}\right],$$
$$C = -b_{12}b_{21} > 0.$$

Obviously, if $\frac{\beta_1 m}{m-\theta} < \beta_2$, then $\frac{\beta_1 u^*}{(1+v^*)^2} b_{21} + \frac{\beta_2 v^*}{(1+u^*)^2} b_{12} < 0$ and so $B > 0$. Thus, $\det[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*) + \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)] > 0$. It follows from Routh-Hurwitz criterion that the two roots $\mu_{1,i}$, $\mu_{2,i}$ of $p_i(\mu) = 0$ have both negative real parts for all $i \geq 1$.

In order to obtain the local stability of $\mathbf{u}^*$, we need to prove that there exists a positive constant $\delta$ such that

$$\operatorname{Re}\{\mu_{1,i}\}, \operatorname{Re}\{\mu_{2,i}\} \leq -\delta \quad \text{for all } i \geq 1. \tag{2.5}$$

Let $\mu = \lambda_i \zeta$, then

$$p_i(\mu) = \lambda_i^2 \zeta^2 - \operatorname{trace}\left[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}\left(\mathbf{w}^*\right) + \mathbf{F}_{\mathbf{w}}\left(\mathbf{w}^*\right)\right]\lambda_i \zeta + \det\left[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}\left(\mathbf{w}^*\right) + \mathbf{F}_{\mathbf{w}}\left(\mathbf{w}^*\right)\right] \triangleq \tilde{p}_i(\zeta).$$

Notice that $\lambda_i \to \infty$ as $i \to \infty$. We can calculate that

$$\lim_{i\to\infty} \frac{\tilde{p}_i(\zeta)}{\lambda_i^2} = \zeta^2 + \left(d_1 + \frac{\beta_1}{1+v^*} + d_2 + \frac{\beta_2}{1+u^*}\right)\zeta + A \triangleq \tilde{p}(\zeta).$$

By Routh-Hurwitz criterion, the two roots $\zeta_1$, $\zeta_2$ of $\tilde{p}(\zeta) = 0$ have both negative real parts. Then we can conclude that there exists a positive constant $\tilde{\delta}$ such that $\operatorname{Re}\{\zeta_1\}, \operatorname{Re}\{\zeta_2\} \leq -\tilde{\delta}$. By continuity, we see that there exists $i_0$ such that the two roots of $\tilde{p}_i(\zeta) = 0$ satisfy $\operatorname{Re}\{\zeta_{1,i}\}, \operatorname{Re}\{\zeta_{2,i}\} \leq -\frac{\tilde{\delta}}{2}$ for all $i \geq i_0$. Then $\operatorname{Re}\{\mu_{1,i}\}, \operatorname{Re}\{\mu_{2,i}\} \leq -\frac{\lambda_i \tilde{\delta}}{2} \leq -\frac{\tilde{\delta}}{2}$ for all $i \geq i_0$. Let

$$\delta = \min\left\{\frac{\tilde{\delta}}{2}, \max_{1\leq i\leq i_0}\left\{\operatorname{Re}\{\mu_{1,i}\}, \operatorname{Re}\{\mu_{2,i}\}\right\}\right\}.$$

Then (2.5) holds true. The theorem is thus proved. □

Similarly, we can also learn, by the proof of Lemma 2.1, a series of stability results about the positive equilibrium for problem (1.1) with different cross-diffusion cases.

**Lemma 2.2** *Suppose that* $\theta < K(m-\theta) < m+\theta$, $B < 0$, $\beta_1, \beta_2 \neq 0$. *The positive equilibrium* $\mathbf{w}^*$ *of* (1.1) *is unstable if* $B^2 - 4AC > 0$ *and*

$$\frac{-B - \sqrt{B^2 - 4AC}}{2A} < \lambda_i < \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

*for some* $i \geq 1$, *whereas it is uniformly asymptotically stable if* $B^2 - 4AC < 0$.

**Lemma 2.3** *Suppose that* $\theta < K(m-\theta) < m+\theta$, $\beta_1 = \beta_2 = 0$. *Then the positive equilibrium* $\mathbf{w}^*$ *of* (1.1) *is uniformly asymptotically stable for disappeared cross-diffusion.*

**Lemma 2.4** *Suppose that $\theta < K(m-\theta) < m+\theta$, $\beta_1 = 0$, $\beta_2 \neq 0$. Then the positive equilibrium $\mathbf{w}^*$ of* (1.1) *is uniformly asymptotically stable.*

Now we consider the case when $\beta_1 \neq 0$, $\beta_2 = 0$. For simplicity, denote

$$A_0 = \left(d_1 + 2\alpha_1 u^* + \frac{\beta_1}{1+v^*}\right)\left(d_2 + 2\alpha_2 v^*\right), \qquad B_0 = -\left[b_{11}\left(d_2 + 2\alpha_2 v^*\right) + \frac{\beta_1 u^*}{(1+v^*)^2} b_{21}\right].$$

Then

$$\det\left[-\lambda_i \mathbf{\Phi}_{\mathbf{w}}\left(\mathbf{w}^*\right) + \mathbf{F}_{\mathbf{w}}\left(\mathbf{w}^*\right)\right] = A_0\lambda_i^2 + B_0\lambda_i + C.$$

We thus have the following result.

**Lemma 2.5** *Suppose that $\theta < K(m-\theta) < m+\theta$, $\beta_1 \neq 0$, $\beta_2 = 0$. The positive equilibrium $\mathbf{w}^*$ of* (1.1) *is unstable if $B_0 < 0$, $B_0^2 - 4A_0C > 0$ and*

$$\frac{-B_0 - \sqrt{B_0^2 - 4A_0C}}{2A_0} < \lambda_i < \frac{-B_0 + \sqrt{B_0^2 - 4A_0C}}{2A_0}$$

*for some $i \geq 1$, whereas it is uniformly asymptotically stable if $B_0 > 0$, or $B_0 < 0$ and $B_0^2 - 4A_0C < 0$.*

Now we consider the corresponding ODE system. Let $\mathbf{w} = (u(t), v(t))^{\mathrm{T}}$ be a positive solution of (1.2). It is easy to show that $u(t)$ and $v(t)$ are both well posed. Similar to the proof of Lemma 2.1, we can get the following stability result.

**Lemma 2.6** *Assume that $\theta < K(m-\theta) < m+\theta$. The positive equilibrium point $\mathbf{w}^*$ of* (1.2) *is locally asymptotically stable. In particular, $\mathbf{w}^*$ is globally asymptotically stable if $\theta < K(m-\theta) < m$.*

*Proof* According to the proof of Lemma 2.1, we can easily obtain local asymptotical stability of $\mathbf{w}^*$ for ODE system (1.2).

Define the following Lyapunov function:

$$E(t) = E(\mathbf{w})(t) = \left(u - u^* - u^* \ln\frac{u}{u^*}\right) + \rho\left(v - v^* - v^* \ln\frac{v}{v^*}\right),$$

where $\rho$ is a positive constant to be determined. Obviously, $E(\mathbf{w}^*) = 0$, and $E(\mathbf{w}) > 0$ if $\mathbf{w} \neq \mathbf{w}^*$. We compute the derivative of $E(t)$ for system (1.2):

$$\begin{aligned}\frac{dE}{dt} &= \frac{u-u^*}{u}\frac{du}{dt} + \rho\frac{v-v^*}{v}\frac{dv}{dt} \\ &= -\left[\left(\frac{1}{K} - \frac{mv^*}{(1+u)(1+u^*)}\right)\left(u-u^*\right)^2 + \frac{m}{1+u}\left(1 - \frac{\rho}{1+u^*}\right)\left(u-u^*\right)\left(v-v^*\right)\right].\end{aligned}$$

It is easy to demonstrate that $\frac{1}{K} - \frac{mv^*}{(1+u)(1+u^*)} > 0$ if $K(m-\theta) < m$. On the other hand, we can choose $\rho = u^*$ and then $1 - \frac{\rho}{1+u^*} = \frac{1}{1+u^*} > 0$. Then we get

$$\frac{dE}{dt} < 0 \quad \text{if } \mathbf{w} \neq \mathbf{w}^*.$$

By the Lyapunov-LaSalle invariance principle [34], $\mathbf{w}^*$ is globally asymptotically stable. So the proof of Lemma 2.6 is completed. □

Based on the above discussion, we now can establish some sufficient conditions for the occurrence of Turing instability induced by cross-diffusion. Our main result in this section is the following theorem.

**Theorem 2.7** *Assume that* $\theta < K(m-\theta) < m+\theta$. *The stability of the constant equilibrium* $\mathbf{w}^*$ *is stable for the ODE dynamics* (1.2) *while unstable for the PDE dynamics* (1.1) *if one of the following two conditions is fulfilled*:

(C1) $B < 0$, $B^2 - 4AC > 0$, *and* $\frac{-B-\sqrt{B^2-4AC}}{2A} < \lambda_i < \frac{-B+\sqrt{B^2-4AC}}{2A}$ *for some* $i \geq 1$,

(C2) $\beta_2 = 0$, $B_0 < 0$, $B_0^2 - 4A_0C > 0$, *and* $\frac{-B_0-\sqrt{B_0^2-4A_0C}}{2A_0} < \lambda_i < \frac{-B_0+\sqrt{B_0^2-4A_0C}}{2A_0}$ *for some* $i \geq 1$.

**Remark 2.8** The Turing instability refers to 'diffusion driven instability', *i.e.*, the stability of the constant equilibrium changing from stable for the ODE dynamics, to unstable for the PDE dynamics. Lemma 2.4 and Theorem 2.7 imply that cross-diffusion $\beta_1$ has a destabilizing effect, which is helpful to the occurrence of Turing instability. Moreover, we can see that sufficiently large cross-diffusion $\beta_1$ can guarantee $B < 0$ and $B_0 < 0$, even $B^2 - 4AC > 0$ and $B_0^2 - 4A_0C > 0$ under a proper parameter condition. So large cross-diffusion effect $\beta_1$ can induce Turing instability.

## 3 Prior bounds for the positive steady states of the PDE system

The corresponding steady state problem of (1.1) is

$$\begin{cases} -\Delta \mathbf{\Phi}(\mathbf{w}) = \mathbf{F}(\mathbf{w}) & \text{in } \Omega, \\ \frac{\partial \mathbf{w}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

In this section, we give *a priori* positive upper and lower bounds for positive solutions to the elliptic system (3.1). For this, we need to make use of the following two results.

**Lemma 3.1** (Maximum principle [9]) *Let* $g(x,w) \in C(\Omega \times \mathbb{R}^1)$ *and* $b_j(x) \in C(\overline{\Omega})$, $j = 1,\dots,N$.

(1) *If* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies*

$$\begin{cases} -\Delta w(x) \leq \sum_{j=1}^N b_j(x) w_{x_j} + g(x, w(x)) & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} \leq 0 & \text{on } \partial\Omega, \end{cases}$$

*and* $w(x_0) = \max_{\overline{\Omega}} w$, *then* $g(x_0, w(x_0)) \geq 0$.

(2) *If* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies*

$$\begin{cases} -\Delta w(x) \geq \sum_{j=1}^N b_j(x) w_{x_j} + g(x, w(x)) & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} \geq 0 & \text{on } \partial\Omega, \end{cases}$$

*and* $w(x_0) = \min_{\overline{\Omega}} w$, *then* $g(x_0, w(x_0)) \leq 0$.

**Lemma 3.2** (Harnack inequality [35]) *Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to $-\Delta w(x) = c(x)w(x)$ with $c \in C(\overline{\Omega})$ subject to the homogeneous Neumann boundary condition. Then there exists a positive constant $C = C(N, \Omega, \|c\|_\infty)$ such that*

$$\max_{\overline{\Omega}} w \le C \min_{\overline{\Omega}} w.$$

In this paper, we assume that the classical solution is in $[C^2(\Omega) \cap C^1(\overline{\Omega})]^2$. The results of upper and lower bounds can be stated as follows.

**Theorem 3.3** (Upper bound) *For any positive classical solution* $\mathbf{w}$ *of* (3.1), *there exist two positive constants $C_i = C_i(d_j, \alpha_j, \beta_j, j = 1, 2, K, \theta)$, $i = 1, 2$, such that*

$$\max_{\overline{\Omega}} u \le C_1, \qquad \max_{\overline{\Omega}} v \le C_2.$$

*Proof* Problem (3.1) can be rewritten as

$$\begin{cases} -\Delta\phi_1 = u(1 - \frac{u}{K} - \frac{mv}{1+u}) & \text{in } \Omega, \\ -\Delta\phi_2 = v(\frac{mu}{1+u} - \theta) & \text{in } \Omega, \\ \frac{\partial\phi_1}{\partial\nu} = \frac{\partial\phi_2}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $x_1 \in \overline{\Omega}$ be a point such that $\phi_1(x_1) = \max_{\overline{\Omega}} \phi_1$. Applying Lemma 3.1 to the first equation in (3.2) yields $u(x_1) \le K$ and

$$\max_{\overline{\Omega}} u \le \frac{1}{d_1} \max_{\overline{\Omega}} \phi_1 = \frac{1}{d_1}\left(d_1 + \alpha_1 u(x_1) + \frac{\beta_1}{1 + v(x_1)}\right)u(x_1) \le \left(1 + \frac{\alpha_1 K + \beta_1}{d_1}\right)K \triangleq C_1.$$

Denote $\phi = \phi_1 + \phi_2$. Let $x_2 \in \overline{\Omega}$ be a point such that $\phi(x_2) = \max_{\overline{\Omega}} \phi$. Since

$$-\Delta\phi = \left(1 - \frac{u}{K}\right)u - \theta v,$$

from Lemma 3.1, we can obtain $v(x_2) \le \frac{u(x_2)}{\theta} \le \frac{C_1}{\theta}$ and

$$\begin{aligned} \max_{\overline{\Omega}} v &\le \frac{1}{d_2} \max_{\overline{\Omega}} \phi = \frac{1}{d_2}\phi(x_2) \\ &= \frac{1}{d_2}\left[\left(d_1 + \alpha_1 u(x_2) + \frac{\beta_1}{1 + v(x_2)}\right)u(x_2) + \left(d_2 + \frac{\beta_2}{1 + u(x_2)} + \alpha_2 v(x_2)\right)v(x_2)\right] \\ &\le \frac{1}{d_2}\left[(d_1 + \alpha_1 C_1 + \beta_1)C_1 + \left(d_2 + \beta_2 + \frac{\alpha_2 C_1}{\theta}\right)\frac{C_1}{\theta}\right] \triangleq C_2. \end{aligned}$$

This completes the proof. □

**Theorem 3.4** (Lower bound) *Suppose that $\frac{mK}{1+K} \neq \theta$. For any positive classical solution* $\mathbf{w}$ *of* (3.1), *there exists a positive constant $c_i = c_i(N, \Omega, d_j, \alpha_j, \beta_j, j = 1, 2, K, m, \theta)$, $i = 1, 2$, such that*

$$\min_{\overline{\Omega}} u \ge c_1, \qquad \min_{\overline{\Omega}} v \ge c_2.$$

*Proof* Since the inequalities

$$\left\| \frac{1-\frac{u}{K}-\frac{mv}{1+u}}{d_1+\alpha_1 u+\frac{\beta_1}{1+v}} \right\|_\infty, \left\| \frac{\frac{mu}{1+u}-\theta}{d_2+\frac{\beta_2}{1+u}+\alpha_2 v} \right\|_\infty \le \bar{C} = \bar{C}(d_j,\alpha_j,\beta_j,j=1,2,K,m,\theta),$$

Harnack inequality in Lemma 3.2 shows that there exist two positive constants $\bar{M}_i = \bar{M}_i(N,\Omega,d_j,\alpha_j,\beta_j,j=1,2,K,m,\theta)$, $i=1,2$, such that

$$\max_{\overline{\Omega}} \phi_i \le \bar{M}_i \min_{\overline{\Omega}} \phi_i, \quad i=1,2.$$

Thus,

$$\frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} \le \frac{\max_{\overline{\Omega}} \phi_1}{\min_{\overline{\Omega}} \phi_1} \frac{d_1+\alpha_1 \max_{\overline{\Omega}} u+\frac{\beta_1}{1+\min_{\overline{\Omega}} v}}{d_1+\alpha_1 \min_{\overline{\Omega}} u+\frac{\beta_1}{1+\max_{\overline{\Omega}} v}} \le \bar{M}_1 \frac{d_1+\alpha_1 C_1+\beta_1}{d_1+\frac{\beta_1}{1+C_2}} \triangleq M_1'.$$

By the same way, we have

$$\frac{\max_{\overline{\Omega}} v}{\min_{\overline{\Omega}} v} \le \frac{\max_{\overline{\Omega}} \phi_2}{\min_{\overline{\Omega}} \phi_2} \frac{d_2+\frac{\beta_2}{1+\min_{\overline{\Omega}} u}+\alpha_2 \max_{\overline{\Omega}} v}{d_2+\frac{\beta_2}{1+\max_{\overline{\Omega}} u}+\alpha_2 \min_{\overline{\Omega}} v} \le \bar{M}_2 \frac{d_2+\beta_2+\alpha_2 C_2}{d_2+\frac{\beta_2}{1+C_1}} \triangleq M_2'.$$

On the other hand, by integrating the second equation in (3.1), we have $\int_\Omega v(\frac{mu}{1+u}-\theta)\,dx = 0$, which implies that there exists a point $y_1 \in \Omega$ such that $\frac{mu(y_1)}{1+u(y_1)}-\theta=0$, *i.e.*,

$$mu(y_1) = \theta\bigl(1+u(y_1)\bigr).$$

So $u(y_1) \ge \frac{\theta}{m}$ and

$$\min_{\overline{\Omega}} u \ge \frac{\max_{\overline{\Omega}} u}{M_1'} \ge \frac{u(y_1)}{M_1'} \ge \frac{\theta}{mM_1'} \triangleq c_1.$$

Now we need to prove $v$ has a positive lower bound. Suppose on the contrary that $\min_{\overline{\Omega}} v \ge c_2 > 0$ does not hold. Then there exists a sequence $\{d_{1,n},d_{2,n},\alpha_{1,n},\alpha_{2,n},\beta_{1,n},\beta_{2,n}\}_{n=1}^\infty$ with $(d_{1,n},d_{2,n},\alpha_{1,n},\alpha_{2,n},\beta_{1,n},\beta_{2,n}) \in [\underline{d_1},\infty) \times [\underline{d_2},\infty) \times [\underline{\alpha_1},\infty) \times [\underline{\alpha_2},\infty) \times [\underline{\beta_1},\infty) \times [\underline{\beta_2},\infty)$ such that the corresponding nonnegative solution $(u_n,v_n)$ of (3.1) with $(d_1,d_2,\alpha_1,\alpha_2,\beta_1,\beta_2) = (d_{1,n},d_{2,n},\alpha_{1,n},\alpha_{2,n},\beta_{1,n},\beta_{2,n})$ satisfies

$$\min_{\overline{\Omega}} u_n \ge c_1, \qquad \min_{\overline{\Omega}} v_n \to 0 \quad \text{as } n \to \infty,$$

and then

$$\max_{\overline{\Omega}} v_n \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

We may assume, by passing to a subsequence if necessary, that as $n \to \infty$,

$$(d_{1,n},d_{2,n},\alpha_{1,n},\alpha_{2,n},\beta_{1,n},\beta_{2,n}) \to (d_1,d_2,\alpha_1,\alpha_2,\beta_1,\beta_2),$$

$$(u_n,v_n) \to (\tilde{u},\tilde{v}).$$

By (3.1) and the $L^p$ regularity theory of elliptic equations, we can conclude that $u_n, v_n \in W^{2,p}(\Omega)$ for any $p > 1$. Then, for $p > N$, by Sobolev embedding theorem, we have $u_n, v_n \in C^{2,\alpha}(\overline{\Omega})$. It follows, by passing to a subsequence if necessary, that $(u_n, v_n)$ converges uniformly to the nonnegative function $(\tilde{u}, \tilde{v})$ in $C^2(\overline{\Omega})$ as $n \to \infty$. Then

$$0 < c_1 \le \min_{\overline{\Omega}} \tilde{u} \le \max_{\overline{\Omega}} \tilde{u} \le C_1, \qquad 0 \le \max_{\overline{\Omega}} \tilde{v} \le C_2.$$

By (3.3), we note that $\tilde{v} \equiv 0$. Moreover, since

$$-\Delta\left[\left(d_{1,n} + \alpha_{1,n}u_n + \frac{\beta_{1,n}}{1+v_n}\right)u_n\right] = u_n\left(1 - \frac{u_n}{K} - \frac{mv_n}{1+u_n}\right),$$

we have

$$-\Delta\big[(d_1 + \beta_1 + \alpha_1\tilde{u})\tilde{u}\big] = \tilde{u}\left(1 - \frac{\tilde{u}}{K}\right).$$

Multiplying the above equation by $\frac{1-\frac{\tilde{u}}{K}}{\tilde{u}}$ and then integrating the resulting equation over $\Omega$, we can obtain

$$0 \ge -\int_\Omega \frac{d_1 + \beta_1 + 2\alpha_1\tilde{u}}{\tilde{u}^2}|\nabla\tilde{u}|^2\,dx = \int_\Omega \left(1 - \frac{\tilde{u}}{K}\right)^2 dx \ge 0.$$

Thus, $\tilde{u} \equiv K$ and then $(u_n, v_n) \to (K, 0)$ as $n \to \infty$. At the same time, we consider the integral equation

$$\int_\Omega v_n\left(\frac{mu_n}{1+u_n} - \theta\right) dx = 0, \quad \forall n \ge 1.$$

However, since $\frac{mu_n}{1+u_n} - \theta \to \frac{mK}{1+k} - \theta \ne 0$ as $n \to \infty$, we can conclude that $v_n(\frac{mu_n}{1+u_n} - \theta)$ is positive or negative as $n$ is large enough. It is a contradiction. □

## 4 Non-existence of non-constant positive steady states

The aim of this section is to investigate the non-existence of non-constant positive steady states of problem (1.1) with no cross-diffusion.

**Theorem 4.1** *Let $\beta_1 = \beta_2 = 0$, $\theta > mK^2(1 + \frac{\alpha_1 K}{d_1})$. Then there exists a positive constant $D_2 = D_2(N, \Omega, d_1, \alpha_1, \alpha_2, K, m, \theta)$ such that problem* (1.1) *has no non-constant positive steady state provided that $d_2 \ge D_2$.*

*Proof* For any $U \in L^1(\Omega)$, denote $\bar{U} = \frac{1}{|\Omega|}\int_\Omega U\,dx$. Assume that $\mathbf{w} = (u, v)^{\mathrm{T}}$ is a positive solution of (3.1) with $\beta_1 = \beta_2 = 0$. Multiplying the two equations in (3.1) by $\frac{u-\bar{u}}{u}$ and $\frac{v-\bar{v}}{v}$, respectively, and integrating the results over $\Omega$ by parts, we can obtain

$$\begin{aligned}
&\int_\Omega \left(\frac{(d_1 + 2\alpha_1 u)\bar{u}}{u^2}|\nabla u|^2 + \frac{(d_2 + 2\alpha_2 v)\bar{v}}{v^2}|\nabla v|^2\right) dx \\
&\quad = \int_\Omega \big[\big(f_1(u,v) - f_1(\bar{u},\bar{v})\big)(u - \bar{u}) + \big(f_2(u,v) - f_2(\bar{u},\bar{v})\big)(v - \bar{v})\big]\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \left[ \left( \frac{mv}{(1+u)(1+\bar{u})} - \frac{1}{K} \right)(u-\bar{u})^2 + \left( \frac{m}{(1+u)(1+\bar{u})} - \frac{m}{1+\bar{u}} \right)(u-\bar{u})(v-\bar{v}) \right] dx \\
&< \int_\Omega \left[ \left( mv - \frac{1}{K} + \epsilon \right)(u-\bar{u})^2 + C(\epsilon)(v-\bar{v})^2 \right] dx,
\end{aligned}
$$

where $\epsilon$ is the arbitrary small positive constant arising from Young's inequality.

Similar to the proof of Lemma 3.3 and Lemma 3.4, we can conclude that

$$
\begin{aligned}
&0 < \tilde{c}_1 \le u \le \left(1 + \frac{\alpha_1 K}{d_1}\right) K \triangleq \tilde{C}_1, \\
&0 < \tilde{c}_2 \le v \le \frac{K}{d_2}\left(1 + \frac{\alpha_1 K}{d_1}\right)\left[ d_1 + \frac{d_2}{\theta} + K\left(1 + \frac{\alpha_1 K}{d_1}\right)\left(\alpha_1 + \frac{\alpha_2}{\theta^2}\right)\right] \triangleq \tilde{C}_2.
\end{aligned}
$$

It follows from the Poincaré inequality that

$$
\begin{aligned}
&\lambda_2 \int_\Omega \left[ \frac{d_1 \tilde{c}_1}{\tilde{C}_1^{\,2}} (u-\bar{u})^2 + \frac{d_2 \tilde{c}_2}{\tilde{C}_2^{\,2}} (v-\bar{v})^2 \right] dx \\
&\quad \le \int_\Omega \left[ \left( m\tilde{C}_2 - \frac{1}{K} + \epsilon \right)(u-\bar{u})^2 + C(\epsilon)(v-\bar{v})^2 \right] dx.
\end{aligned}
$$

Since $m\tilde{C}_2 - \frac{1}{K} < 0$ if

$$
d_2 > \frac{mK^2(d_1 + \alpha_1 K)}{d_1\theta - mK^2(d_1 + \alpha_1 K)} \left[ d_1 + K\left(\alpha_1 + \frac{\alpha_2}{\theta^2}\right)\left(1 + \frac{\alpha_1 K}{d_1}\right)\right],
$$

we may choose $\epsilon$ sufficiently small and $d_2$ sufficiently large such that $m\tilde{C}_2 - \frac{1}{K} + \epsilon < 0$, $\lambda_2 \frac{d_2 \tilde{c}_2}{\tilde{C}_2^{\,2}} > C(\epsilon)$. Thus, we can conclude that $u \equiv \bar{u}$, $v \equiv \bar{v}$. Then the proof is completed. □

## 5 Existence of non-constant positive steady states

In this section, we shall use the Leray-Schauder degree theory to develop a general setting to establish the existence of stationary patterns for system (1.1). Denote

$$
\begin{aligned}
&\mathbf{X}^+ = \{\mathbf{w} \in \mathbf{X} | \mathbf{w} > \mathbf{0} \text{ on } \overline{\Omega}\}, \\
&B(C) = \left\{ \mathbf{w} = (u,v)^{\mathrm{T}} \in \mathbf{X}^+ | C^{-1} < u, v < C \text{ on } \overline{\Omega} \right\},
\end{aligned}
$$

where $C$ is a positive constant whose existence is guaranteed by Theorems 3.3 and 3.4.

Since the determinant $\det[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w})]$ is positive for all non-negative $\mathbf{w}$, $[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1}$ exists and $\det\{[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1}\}$ is positive, thus $\mathbf{w}$ is a positive solution of system (3.1) if and only if

$$
\Psi(\mathbf{w}) \triangleq \mathbf{w} - (\mathbf{I} - \Delta)^{-1}\left\{ \left[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w})\right]^{-1} \left[ \mathbf{F}(\mathbf{w}) + \nabla \mathbf{w} \mathbf{\Phi}_{\mathbf{ww}}(\mathbf{w}) \nabla \mathbf{w} \right] + \mathbf{w} \right\} = 0 \quad \text{in } \mathbf{X}^+, \tag{5.1}
$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ in $\mathbf{X}$, subject to the homogeneous Neumann boundary condition. Since $\Psi(\cdot)$ is a compact perturbation of the identity operator, the Leray-Schauder degree $\deg(\Psi(\cdot), 0, B(C))$ is well defined if $\Psi(\mathbf{w}) \ne 0$ for any $\mathbf{w} \in \partial B(C)$. Further, we calculate

$$
\mathrm{D}_{\mathbf{w}} \Psi\left(\mathbf{w}^*\right) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\left\{ \left[\mathbf{\Phi}_{\mathbf{w}}\left(\mathbf{w}^*\right)\right]^{-1} \mathbf{F}_{\mathbf{w}}\left(\mathbf{w}^*\right) + \mathbf{I} \right\} \quad \text{in } \mathcal{L}(\mathbf{X}, \mathbf{X}).
$$

We recall that if $D_{\mathbf{w}}\Psi$ does not have any pure imaginary or zero eigenvalue, the index of the operator $\Psi$ at the fixed point $\mathbf{w}^*$ is defined as $\mathrm{index}(\Psi(\cdot),\mathbf{w}^*) = (-1)^r$, where $r$ is the total number of eigenvalues of $D_{\mathbf{w}}\Psi$ with negative real parts (counting multiplicities). Then the degree $\deg(\Psi(\cdot),0,B(C))$ is equal to the sum of the indexes over all solutions to equation $\Psi = 0$ in $B(C)$, provided that $\Psi \neq 0$ on $\partial B(C)$.

In order to calculate $r$, we employ the eigenspaces of $-\Delta$. Using the decomposition (2.4) we investigate the eigenvalues of matrix $D_{\mathbf{w}}\Psi(\mathbf{w}^*)$. First, we know $\mathbf{X}_{ij}$ is invariant under $D_{\mathbf{w}}\Psi(\mathbf{w}^*)$ for each $i \in \mathbb{N}$ and each $j \in [1, \dim E(\lambda_i)] \cap \mathbb{N}$, *i.e.*, $D_{\mathbf{w}}\Psi(\mathbf{w}^*)$, $\mathbf{w} \in \mathbf{X}_{ij}$ for any $\mathbf{w} \in \mathbf{X}_{ij}$. Hence, $\mu$ is an eigenvalue of $D_{\mathbf{w}}\Psi(\mathbf{w}^*)$ on $\mathbf{X}_{ij}$ if and only if it is an eigenvalue of the matrix

$$\mathbf{I} - \frac{1}{1+\lambda_i}\bigl\{\bigl[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)\bigr]^{-1}\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*) + \mathbf{I}\bigr\} = \frac{1}{1+\lambda_i}\bigl\{\lambda_i\mathbf{I} - \bigl[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)\bigr]^{-1}\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\bigr\}.$$

So $D_{\mathbf{w}}\Psi(\mathbf{w}^*)$ is invertible if and only if, for any $i \geq 1$, the matrix $\frac{1}{1+\lambda_i}\{\lambda_i\mathbf{I} - [\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1} \times \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\}$ is non-singular. Denote

$$H(\lambda) \triangleq H(\mathbf{w}^*,\lambda) = \det\bigl\{\lambda\mathbf{I} - \bigl[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)\bigr]^{-1}\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\bigr\}.$$

We notice that if $H(\lambda_i) \neq 0$, then for each $j \in [1, \dim E(\lambda_i)]$, the number of negative eigenvalues of $D_{\mathbf{w}}\Psi(\mathbf{w}^*)$ on $\mathbf{X}_{ij}$ is odd if and only if $H(\lambda_i) < 0$. In conclusion, we have the following result.

**Lemma 5.1** *Assume that, for each $i \geq 1$, the matrix $\lambda_i\mathbf{I} - [\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1}\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)$ is non-singular. Then*

$$\mathrm{index}\bigl(\Psi(\cdot),\mathbf{w}^*\bigr) = (-1)^\sigma, \quad \text{where } \sigma = \sum_{i\geq 1, H(\lambda_i)<0} \dim E(\lambda_i).$$

According to the above lemma, we should consider the sign of $H(\lambda_i)$ in order to calculate $\mathrm{index}(\Psi(\cdot),\mathbf{w}^*)$. Since

$$H(\lambda) = \det\bigl\{\bigl[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)\bigr]^{-1}\bigr\}\det\bigl\{\lambda\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*) - \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\bigr\}$$

and $\det\{[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1}\} > 0$, we only need to consider the sign of $\det\{\lambda\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*) - \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\}$. A direct calculation shows

$$\det\bigl\{\lambda\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*) - \mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\bigr\} = A_2\lambda^2 + A_1\lambda + A_0 \triangleq q(\lambda),$$

where

$$\begin{aligned}
A_2 &= \left(d_1 + 2\alpha_1 u^* + \frac{\beta_1}{1+v^*}\right)\left(d_2 + \frac{\beta_2}{1+u^*} + 2\alpha_2 v^*\right) - \frac{\beta_1\beta_2 u^* v^*}{(1+u^*)^2(1+v^*)^2} > 0,\\
A_1 &= -\left[\left(d_2 + \frac{\beta_2}{1+u^*} + 2\alpha_2 v^*\right)b_{11} + \frac{\beta_1 u^*}{(1+v^*)^2}b_{21} + \frac{\beta_2 v^*}{(1+u^*)^2}b_{12}\right],\\
A_0 &= -b_{12}b_{21} > 0.
\end{aligned}$$

Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be the two roots of $q(\lambda) = 0$ with $\mathrm{Re}\{\bar{\lambda}_1\} \leq \mathrm{Re}\{\bar{\lambda}_2\}$. Then

$$\bar{\lambda}_1\bar{\lambda}_2 = \det\{\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)\} > 0.$$

So the signs of $\mathrm{Re}\,\bar{\lambda}_1$ and $\mathrm{Re}\,\bar{\lambda}_2$ are identical. Perform the following limits:

$$\lim_{\beta_1\to\infty} \frac{q(\lambda)}{\beta_1} = \lambda(\Lambda_2\lambda^2 + \Lambda_1),$$

where

$$\Lambda_2 = \frac{1}{1+v^*}\left(d_2 + \frac{\beta_2}{1+u^*} + 2\alpha_2 v^*\right) - \frac{\beta_2 u^* v^*}{(1+u^*)^2(1+v^*)^2} > 0,$$
$$\Lambda_1 = -\frac{u^*}{(1+v^*)^2} b_{21} < 0.$$

Then we have the following result.

**Lemma 5.2** *Assume that $\theta < K(m-\theta) < m+\theta$. Then there exists a positive constant $\beta_1^*$ such that for any $\beta_1 \geq \beta_1^*$, the two roots $\bar{\lambda}_1$, $\bar{\lambda}_2$ of $q(\lambda) = 0$ are all real and satisfy*

$$\lim_{\beta_1\to\infty} \bar{\lambda}_1 = 0, \qquad \lim_{\beta_1\to\infty} \bar{\lambda}_2 = -\frac{\Lambda_1}{\Lambda_2} \triangleq \bar{\lambda} > 0. \tag{5.2}$$

*Moreover, we can conclude that*

$$\begin{cases} 0 < \bar{\lambda}_1 < \bar{\lambda}_2, \\ q(\lambda) < 0 \quad \textit{when } \lambda \in (\bar{\lambda}_1, \bar{\lambda}_2), \\ q(\lambda) > 0 \quad \textit{when } \lambda \in (-\infty, \bar{\lambda}_1) \cup (\bar{\lambda}_2, +\infty). \end{cases} \tag{5.3}$$

Now we establish the global existence of non-constant positive solution to (3.1) with respect to the cross-diffusion coefficients $\beta_1$, as the other parameters are all fixed positive constants.

**Theorem 5.3** *Assume that the parameters $d_1, d_2, \alpha_1, \alpha_2, \beta_2, K, M$ and $\theta$ are all fixed and satisfy $d_2 \geq D_2$, $\frac{mK}{1+K} \neq \theta$ and*

$$mK^2\left(1 + \frac{\alpha_1 K}{d_1}\right) < \theta < K(m-\theta) < m+\theta. \tag{5.4}$$

*Let $\bar{\lambda}$ be given by the limit in* (5.2). *If $\bar{\lambda} \in (\lambda_n, \lambda_{n+1})$ for some $n \geq 1$ and the sum $\sigma_n = \sum_{i=2}^{n} \dim E(\lambda_i)$ is odd, then there exists a positive constant $\beta_1^*$ such that, if $\beta_1 > \beta_1^*$, problem* (1.1) *has at least one non-constant positive steady state.*

*Proof* By Lemma 5.2, there exists a positive constant $\beta_1^*$ such that, if $\beta_1 > \beta_1^*$, (5.3) holds and

$$0 = \lambda_1 < \bar{\lambda}_1 < \bar{\lambda}_2, \qquad \bar{\lambda}_2 \in (\lambda_n, \lambda_{n+1}). \tag{5.5}$$

We will prove that for any $\beta_1 > \beta_1^*$, (1.1) has at least one non-constant positive steady state. The proof will be fulfilled by contradiction. Suppose on the contrary that the assertion is not true for some $\beta_1 = \bar{\beta}_1 \geq \beta_1^*$. Let $\beta_1$ be fixed as $\bar{\beta}_1$.

For $t \in [0,1]$, define

$$d_1(t) \equiv d_1, \qquad d_2(t) = 2D_2 + t(d_2 - 2D_2),$$
$$\alpha_i(t) \equiv \alpha_i, \qquad \beta_1(t) \equiv t\bar{\beta}_1, \qquad \beta_2(t) = t\beta_2$$

and

$$\begin{aligned}\mathbf{\Phi}(t;\mathbf{w}) &= \bigl(\phi_1(t;\mathbf{w}),\phi_2(t;\mathbf{w})\bigr)^{\mathrm{T}} \\ &= \left(\left(d_1(t) + \alpha_1(t)u + \frac{\beta_1(t)}{1+v}\right)u, \left(d_2(t) + \frac{\beta_2(t)}{1+u}\right)v + \alpha_2(t)v\right)^{\mathrm{T}},\end{aligned}$$

and then consider the problem

$$\begin{cases} -\Delta\mathbf{\Phi}(t;\mathbf{w}) = \mathbf{F}(\mathbf{w}) & \text{in } \Omega, \\ \frac{\partial \mathbf{w}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

Then $\mathbf{w}$ is a non-constant positive steady state of (1.1) if and only if it is a non-constant positive solution of problem (5.6) for $t = 1$. It is obvious that $\mathbf{w}^*$ is the unique constant positive solution of (5.6) for any $t \in [0,1]$. From (5.1), we know that for any $t \in [0,1]$, $\mathbf{w}$ is a positive solution of problem (5.6) if and only if

$$\Psi(t;\mathbf{w}) \triangleq \mathbf{w} - (\mathbf{I}-\Delta)^{-1}\bigl\{\bigl[\mathbf{\Phi}_{\mathbf{w}}(t;\mathbf{w})\bigr]^{-1}\bigl[\mathbf{F}(\mathbf{w}) + \nabla\mathbf{w}\mathbf{\Phi}_{\mathbf{ww}}(t;\mathbf{w})\nabla\mathbf{w}\bigr] + \mathbf{w}\bigr\} = 0 \quad \text{in } \mathbf{X}^+.$$

It is obvious that $\Psi(1;\mathbf{w}) = \Psi(\mathbf{w})$. Theorem 4.1 indicates that $\Psi(0;\mathbf{w}) = 0$ only has the constant positive solution $\mathbf{w}^*$ in $\mathbf{X}^+$. A direct calculation shows that

$$\mathrm{D}_{\mathbf{w}}\Psi\bigl(t;\mathbf{w}^*\bigr) = \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\bigl\{\bigl[\mathbf{\Phi}_{\mathbf{w}}\bigl(t;\mathbf{w}^*\bigr)\bigr]^{-1}\mathbf{F}_{\mathbf{w}}\bigl(\mathbf{w}^*\bigr) + \mathbf{I}\bigr\}.$$

In particular,

$$\begin{aligned}\mathrm{D}_{\mathbf{w}}\Psi\bigl(0;\mathbf{w}^*\bigr) &= \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\bigl\{\bigl[\hat{\mathbf{\Phi}}_{\mathbf{w}}\bigl(\mathbf{w}^*\bigr)\bigr]^{-1}\mathbf{F}_{\mathbf{w}}\bigl(\mathbf{w}^*\bigr) + \mathbf{I}\bigr\}, \\ \mathrm{D}_{\mathbf{w}}\Psi\bigl(1;\mathbf{w}^*\bigr) &= \mathbf{I} - (\mathbf{I}-\Delta)^{-1}\bigl\{\bigl[\mathbf{\Phi}_{\mathbf{w}}\bigl(\mathbf{w}^*\bigr)\bigr]^{-1}\mathbf{F}_{\mathbf{w}}\bigl(\mathbf{w}^*\bigr) + \mathbf{I}\bigr\} = \mathrm{D}_{\mathbf{w}}\Psi\bigl(\mathbf{w}^*\bigr).\end{aligned}$$

Here $\hat{\mathbf{\Phi}}_{\mathbf{w}}(\mathbf{w}^*) = \mathrm{diag}(d_1 + 2\alpha_1 u^*, 2D_2 + 2\alpha_2 v^*)$. Moreover, we already know that

$$H(\lambda) = \det\bigl\{\bigl[\mathbf{\Phi}_{\mathbf{w}}\bigl(\mathbf{w}^*\bigr)\bigr]^{-1}\bigr\}q(\lambda) \tag{5.7}$$

and $\det\{[\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1}\} > 0$.

For $t = 1$, by (5.3), (5.5) and (5.7), we have

$$\begin{cases} H(\lambda_1) = H(0) > 0, \\ H(\lambda_i) < 0 & \text{when } 2 \leq i \leq n, \\ H(\lambda_i) > 0 & \text{when } i > n. \end{cases}$$

Thus, 0 is not an eigenvalue of the matrix $\lambda_i \mathbf{I} - [\mathbf{\Phi}_{\mathbf{w}}(\mathbf{w}^*)]^{-1}\mathbf{F}_{\mathbf{w}}(\mathbf{w}^*)$ for all $i \geq 0$, and

$$\sum_{i\geq 1, H(\lambda_i)<0} \dim E(\lambda_i) = \sum_{i=2}^{n} \dim E(\lambda_i) = \sigma_n$$

is odd. It follows from Lemma 5.1 that

$$\operatorname{index}\bigl(\Psi(1;\cdot), \mathbf{w}^*\bigr) = (-1)^r = (-1)^{\sigma_n} = -1. \tag{5.8}$$

For $t = 0$, we have

$$\operatorname{index}\bigl(\Psi(0;\cdot), \mathbf{w}^*\bigr) = (-1)^0 = 1 \tag{5.9}$$

from Theorem 4.1.

On the other hand, by Theorems 3.3 and 3.4, there exists a positive constant $M$ such that for all $t \in [0,1]$, the positive solution of (5.6) satisfies $M^{-1} < u, v < M$ and $\Psi(t;\mathbf{w}) \neq 0$ on $\partial B(M)$. By the homotopy invariance of the topological degree, we can obtain

$$\deg\bigl(\Psi(1;\cdot), 0, B(M)\bigr) = \deg\bigl(\Psi(0;\cdot), 0, B(M)\bigr). \tag{5.10}$$

Now, by our supposition, both equations $\Psi(1;\mathbf{w}) = 0$ and $\Psi(0;\mathbf{w}) = 0$ have only the constant positive solution $\mathbf{w}^*$ in $B(M)$. Thus, by (5.8) and (5.9),

$$\begin{aligned}
\deg\bigl(\Psi(1;\cdot), 0, B(M)\bigr) &= \operatorname{index}\bigl(\Psi(1;\cdot), \mathbf{w}^*\bigr) = -1, \\
\deg\bigl(\Psi(0;\cdot), 0, B(M)\bigr) &= \operatorname{index}\bigl(\Psi(0;\cdot), \mathbf{w}^*\bigr) = 1,
\end{aligned}$$

which contradicts (5.10). The proof is completed. □

**Remark 5.4** Condition (5.4) may be fulfilled if $m$ is much larger than $K$, and $K$ is rather small in comparison with $m$ and $\theta$. Moreover, the conclusion in Theorem 5.3 coincides with the discussion in Section 2. So we know that large cross-diffusion effect $\beta_1$ is helpful to the formation of stationary patterns.

**Remark 5.5** The results of Theorems 2.7, 4.1 and 5.3 show that large cross-diffusion effect of the first species can create not only Turing patterns but also stationary patterns (non-constant positive steady states).

**Competing interests**
The author declares that she has no competing interests.

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