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# Existence of solutions for semilinear elliptic equations on $\mathbb{R}^N$

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**Abstract**
In this paper, the existence of at least one nontrivial solution for a class of semilinear elliptic equations on $\mathbb{R}^N$ is established by using the linking methods.

**Keywords:** Schrödinger equation; subcritical exponent; local linking

## 1 Introduction

In this paper we consider the question of the existence of solutions for a class of semilinear equations of the form

$$(P_\lambda) \quad -\triangle u + \lambda u = g(x,u), \quad x \in \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter and the nonlinearity $g \in C(\mathbb{R}^N \times \mathbb{R})$ is asymptotically linear, *i.e.*,

$$\lim_{|t|\to\infty} \frac{g(x,t)}{t} = V(x), \qquad \lim_{|x|\to\infty} V(x) = v_\infty \tag{1.1}$$

for some $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $v_\infty \in \mathbb{R}$. In case this equation is considered in a bounded domain $\Omega \subset \mathbb{R}^N$ (with, say, the Dirichlet boundary condition), there is a large amount of literature on existence and multiplicity results, with the case of resonance being of particular interest (see [1–3]). We recall that the problem is said to be at resonance if $-\lambda \in \sigma(S)$, where $\sigma(S)$ denotes the spectrum of $S$, the 'asymptotic linearization' of the problem. In other words, $S : D(S) \subset L^2(\Omega) \to L^2(\Omega)$ is the operator given by

$$Su(x) = -\triangle u(x) - V(x)u(x), \qquad D(S) = H_0^1(\Omega) \cap H^2(\Omega). \tag{1.2}$$

On the other hand, a systematic study of such asymptotically linear problems set in unbounded domains or the whole space $\mathbb{R}^N$ is more recent and presents a number of mathematical difficulties (see [4, 5]). As an example, we note that in the case of problem $(P_\lambda)$, the asymptotic linearization operator $S$ (now defined on $D(S) = H^2(\mathbb{R}^N)$) has a much more complicated spectrum (including an *essential* part $[-v_\infty, \infty)$), which in turn makes the study of this problem more challenging. In [4], motivated by the paper [5], Tehrani and Costa studied the existence of positive solutions to $(P_\lambda)$ by using the mountain pass theorem if $g(x,u)$ satisfies some strong asymptotically linear conditions. Comparing with previous paper [4], in [6], Tehrani obtained the existence of a (possibly sign-changing) solution for problem $(P_\lambda)$ under essentially condition (1.1) only. In fact, he proved the following.

**Theorem 1.0** [6] *Let $g_0(x,s) := g(x,s) - V(x)s$ and assume that*

(G) *for every $\epsilon > 0$, there exists $0 \le b_\epsilon(x) \in L^2(\mathbb{R}^N)$ such that*

$$\left|g_0(x,s)\right| \le b_\epsilon(x) + \epsilon|s| \quad \textit{a.e. } x \in \mathbb{R}^N, s \in \mathbb{R}.$$

*If $\Lambda \ge 0$ or $\max\{0, \nu_\infty\} < \lambda < -\Lambda$ and $-\lambda \notin \sigma_p(S)$, then $(P_\lambda)$ has a solution in $H^1(\mathbb{R}^N)$.*

Now, one naturally asks: Are there nontrivial solutions for problem $(P_\lambda)$ if $-\lambda \in \sigma(S)$ in the above theorem? Obviously, this case is resonance. But, this problem is not easy because we face the difficulties of verifying that the energy functional satisfies the (PS) condition if we still follow the idea of [6]. Here, there is still an interesting problem: Are there nontrivial solutions for problem $(P_\lambda)$ if $-\lambda \in \sigma(S)$ and $g_0(x,s)$ (in Theorem 1.0) is more generalized superlinear? We will answer the above problems affirmatively by using Li and Willem's local linking methods (see [7]).

Next, we recall a few basic facts in the theory of Schrödinger operators which are relevant to our discussion (see [6]).

1. Since $\lim_{|x|\to\infty} V(x) = \nu_\infty$, one has $\sigma_{\mathrm{ess}}(S) = [-\nu_\infty, \infty)$.
2. The bottom of the spectrum $\sigma(S)$ of the operator $S$ is given by

$$\Lambda = \lambda_0 = \inf_{0\ne u\in H^2(\mathbb{R}^N)} \frac{\int |\nabla u|^2 - V(x)u^2}{\int u^2}.$$

Therefore we clearly have $\Lambda \le -\nu_\infty$. If $\Lambda < -\nu_\infty$, then by using the concentration compactness principle of Lions, one shows that $\Lambda$ is the principle eigenvalue of $S$ with a positive eigenfunction $\Phi_0$:

$$S\Phi_0 = \lambda_0\Phi_0, \quad \Phi_0 \in H^2\left(\mathbb{R}^N\right), \Phi_0 > 0.$$

3. The spectrum of $S$ in $(-\infty, -\nu_\infty)$, namely $\sigma(S) \cap (-\infty, -\nu_\infty)$, is at most a countable set, which we denote by

$$\Lambda = \lambda_0 < \lambda_1 < \lambda_2 < \cdots,$$

where each $\lambda_k$ is an isolated eigenvalue of $S$ of the finite multiplicity. Let $E_{\lambda_j}$ denote the eigenspace of $S$ corresponding to the eigenvalue $\lambda_j$.

Now, we state our main results. In this paper, we always assume that $\lim_{|x|\to\infty} V(x) = \nu_\infty$ and $\nu_\infty < 0$. The conditions imposed on $g_0(x,t)$ (see Theorem 1.0) are as follows:

($H_1$) $g_0 \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there are constants $C_1, C_2 \ge 0$ such that

$$\left|g_0(x,t)\right| \le C_1 + C_2|t|^{s-1}, \quad \forall x \in \mathbb{R}^N, \forall t \in R, s \in \left(2, p^*\right) (N \ge 3),$$

where $p^* = \frac{2N}{N-2}$;

($H_2$) $g_0(x,t) = \circ(|t|)$, $|t| \to 0$, uniformly on $\mathbb{R}^N$;

($H_3$) $\lim_{|t|\to\infty} \frac{g_0(x,t)}{t} = +\infty$ uniformly on $\mathbb{R}^N$;

($H_4$) There is a constant $\theta \ge 1$ such that for all $(x,t) \in \mathbb{R}^N \times R$ and $s \in [0,1]$,

$$\theta\left(g_0(x,t)t - 2G_0(x,t)\right) \ge \left(sg_0(x,st)t - 2G_0(x,st)\right),$$

where $G_0(x,t) = \int_0^t g_0(x,s)\,ds$;

($H_5$) For some $\delta > 0$, either

$$G_0(x,t) \geq 0 \quad \text{for } |t| \leq \delta, x \in \mathbb{R}^N$$

or

$$G_0(x,t) \leq 0 \quad \text{for } |t| \leq \delta, x \in \mathbb{R}^N;$$

($H_6$) $\lim_{|x|\to\infty} \sup_{|t|\leq r} \frac{g_0(x,t)}{|t|} = 0$ for every $r > 0$.

**Theorem 1.1** *Assume that conditions* ($H_1$)-($H_4$) *hold. If* $-\lambda$ *is an eigenvalue of* $S(-\lambda < -\nu_\infty)$, *assume also that* ($H_5$) *and* ($H_6$) *hold. Then the problem* $(P_\lambda)$ *has at least one nontrivial solution.*

**Remark 1.1** It follows from the condition ($H_3$) that our nonlinearity $g_0(x,t)$ does not satisfy the classical condition of Ambrosetti and Rabinowitz:

(AR) There is $\mu > 2$ such that $0 < \mu G_0(x,u) \leq u g_0(x,u)$ for all $x \in \mathbb{R}^N$ and $u \neq 0$.

In recent years, there have been some papers devoted to replacing (AR) with more natural conditions (see [8–10]). But our methods are different from the references therein.

We also consider asymptotically quadratic functions. We assume that:

($H_7$) For every $\epsilon > 0$, there exists $0 \leq b_\epsilon(x) \in L^2(\mathbb{R}^N)$ such that

$$\left|g_0(x,s)\right| \leq b_\epsilon(x) + \epsilon|s| \quad \text{a.e. } x \in \mathbb{R}^N, s \in R,$$

and $\lambda_k < -\lambda < \lambda_{k+1}$.

**Theorem 1.2** *Assume that conditions* ($H_2$), ($H_6$), ($H_7$) *and one of the following conditions hold*:

($A_1$) $\lambda_j < 0 < \lambda_{j+1}$, $j \neq k$;

($A_2$) $\lambda_j = 0 < \lambda_{j+1}$, $j \neq k$ *for some* $\delta > 0$,

$$G_0(x,u) \geq 0 \quad \textit{for } |u| > \delta, x \in \mathbb{R}^N;$$

($A_3$) $\lambda_j < 0 = \lambda_{j+1}$, $j \neq k$ *for some* $\delta > 0$,

$$G_0(x,u) \geq 0 \quad \textit{for } |u| \leq \delta, x \in \mathbb{R}^N.$$

*Then problem* $(P_\lambda)$ *has at least one nontrivial solution.*

## 2 Preliminaries

Let $X$ be a Banach space with a direct sum decomposition

$$X = X^1 \oplus X^2.$$

Consider two sequences of subspaces

$$X_0^1 \subset X_1^1 \subset \cdots \subset X^1, \qquad X_0^2 \subset X_1^2 \subset \cdots \subset X^2$$

such that

$$X^j = \bigcup_{n\in N} X_n^j, \quad j = 1, 2.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, let $X_\alpha = X_{\alpha_1} \oplus X_{\alpha_2}$. We know that

$$\alpha \le \beta \quad \Leftrightarrow \quad \alpha_1 \le \beta_1, \qquad \alpha_2 \le \beta_2.$$

A sequence $(\alpha_n) \subset N^2$ is admissible if, for every $\alpha \in N^2$, there is $m \in N$ such that $n \ge m \Rightarrow \alpha_n \ge \alpha$. For every $I : X \to R$, we denote by $I_\alpha$ the function $I$ restricted $X_\alpha$.

**Definition 2.1** Let $I$ be locally Lipschitz on $X$ and $c \in R$. The functional $I$ satisfies the $(C)_c^*$ condition if every sequence $(u_{\alpha_n})$ such that $(\alpha_n)$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad I(u_{\alpha_n}) \to c, \qquad \bigl(1 + \|u_{\alpha_n}\|\bigr) I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of $I$.

**Definition 2.2** Let $I$ be locally Lipschitz on $X$ and $c \in R$. The functional $I$ satisfies the $(C)^*$ condition if every sequence $(u_{\alpha_n})$ such that $(\alpha_n)$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup_n I(u_{\alpha_n}) \le c, \qquad \bigl(1 + \|u_{\alpha_n}\|\bigr) I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of $I$.

**Remark 2.1** 1. The $(C)^*$ condition implies the $(C)_c^*$ condition for every $c \in R$.

2. When the $(C)_c^*$ sequence is bounded, then the sequence is a $(\mathrm{PS})_c^*$ sequence (see [11]).

3. Without loss of generality, we assume that the norm in $X$ satisfies

$$\|u_1 + u_2\|^2 = \|u_1\|^2 + \|u_2\|^2, \quad u_j \in X_j, j = 1, 2.$$

**Definition 2.3** Let $X$ be a Banach space with a direct sum decomposition

$$X = X_1 \oplus X_2.$$

The function $I \in C^1(X, R)$ has a local linking at 0, with respect to $(X^1, X^2)$ if, for some $r > 0$,

$$I(u) \ge 0, \quad u \in X^1, \|u\| \le r,$$
$$I(u) \le 0, \quad u \in X^2, \|u\| \le r.$$

**Lemma 2.1** (see [7]) *Suppose that* $I \in C^1(X, R)$ *satisfies the following assumptions*:

$(\mathrm{B}_1)$ *I has a local linking at* 0 *and* $X^1 \ne \{0\}$;

($B_2$) *I satisfies* (PS)$^*$;

($B_3$) *I maps bounded sets into bounded sets*;

($B_4$) *For every* $m \in N$, $I(u) \to -\infty$, $\|u\| \to \infty$, $u \in X = X_m^1 \oplus X^2$. *Then I has at least two critical points*.

**Remark 2.2** Assume that $I$ satisfies the $(C)_c^*$ condition. Then this theorem still holds.

Let $X$ be a real Hilbert space and let $I \in C^1(X, R)$. The gradient of $I$ has the form

$$\nabla I(u) = Au + B(u),$$

where $A$ is a bounded self-adjoint operator, 0 is not the essential spectrum of $A$, and $B$ is a nonlinear compact mapping.

We assume that there exist an orthogonal decomposition,

$$X = X_1 + X_2,$$

and two sequences of finite-dimensional subspaces,

$$X_0^1 \subset X_1^1 \subset X_1^1 \subset \cdots \subset X^1, \qquad X_0^2 \subset X_1^2 \subset \cdots \subset X^2,$$

such that

$$X^j = \overline{\bigcup_{n \in N} X_n^j}, \quad j = 1, 2,$$

$$AX_n^j \subset X_n^j, \quad j = 1, 2, n \in N.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, we denote by $X_\alpha$ the space

$$X_\alpha^1 \oplus X_\alpha^2,$$

by $p_\alpha : X \to X_\alpha$ the orthogonal projector onto $X_\alpha$, and by $M^-(L)$ the Morse index of a self-adjoint operator $L$.

**Lemma 2.2** (see [7]) *I satisfies the following assumptions*:

(i) *I has a local linking at* 0 *with respect to* $(X^1, X^2)$;

(ii) *There exists a compact self-adjoint operator* $B_\infty$ *such that*

$$B(u) = B_\infty(u) + \circ(\|u\|), \quad \|u\| \to \infty;$$

(iii) $A + B_\infty$ *is invertible*;

(vi) *For infinitely many multiple-indices* $\alpha := (n, n)$,

$$M^-\big((A + P_\alpha B_\infty)|_{X_\alpha}\big) \neq \dim X_n^2.$$

*Then I has at least two critical points*.

## 3 The proof of main results

*Proof of Theorem* 1.1 (1) We shall apply Lemma 2.1 to the functional

$$I(u) = \frac{1}{2}\int \left(|\nabla u|^2 - V(x)|u|^2\right) + \frac{1}{2}\lambda \int u^2 - \int_\Omega G_0(x,u)$$

defined on $X = H^1(\mathbb{R}^N)$. We consider only the case $-\lambda \in \sigma(S)$, and

$$G_0(x,u) \le 0 \quad \text{for } |u| \le \delta, x \in \mathbb{R}^N. \tag{3.1}$$

Then other case is similar and simple.

Let $X^2$ be a finite dimensional space spanned by the eigenfunctions corresponding to negative eigenvalues of $S + \lambda$ and let $X^1$ be its orthogonal complement in $X$. Choose a Hilbertian basis $e_n$ $(n \ge 0)$ for $X$ and define

$$X_n^1 = \operatorname{span}(e_0, e_1, \ldots, e_n), \quad n \in N;$$
$$X_n^2 = X^2, \quad n \in N;$$
$$X^1 = \overline{\bigcup_{n\in N} X_n^1}.$$

By the condition $(\mathrm{H}_1)$ and Sobolev inequalities, it is easy to see that the functional $I$ belongs to $C^1(X,R)$ and maps bounded sets to bounded sets.

(2) We claim that $I$ has a local linking at 0 with respect to $(X^1, X^2)$. Decompose $X^1$ into $V + W$ when $V = E_{-\lambda}$, $W = (X^2 + V)^\perp$. Also, set $u = v + w$, $u \in X^1$, $v \in V$, $w \in W$.

For the convenience of our proof, we state some facts for the norm of the whole space $X$. It is well known that there is an equivalent norm $\|\cdot\|$ on $X = H^1(\mathbb{R}^N)$ such that

$$\int \left(|\nabla u|^2 - V(x)|u|^2\right) = -\|u\|^2, \quad u \in X^2$$

and

$$\int \left(|\nabla u|^2 - V(x)|u|^2\right) = \|u\|^2, \quad u \in W.$$

By the equivalence of norm in the finite-dimensional space, there exists $C > 0$ such that

$$\|v\|_\infty \le C\|v\|, \quad \forall v \in V. \tag{3.2}$$

It follows from $(\mathrm{H}_1)$ and $(\mathrm{H}_2)$ that for any $\epsilon > 0$, there exists $C_\epsilon$ such that

$$\left|G_0(x,u)\right| \le \epsilon u^2 + C_\epsilon |u|^s. \tag{3.3}$$

Hence, we obtain

$$I(u) \le -m\|u\|^2 + c^*\|u\|^{s+1},$$

where $m > 0$, $c^*$ is a constant and hence, for $r > 0$ small enough,

$$I(u) \le 0, \quad u \in X^2, \|u\|_X \le r.$$

Let $u = v + w \in X^1$ be such that $\|u\|_X \le r_1 = \frac{\delta}{2C}$ and let

$$\mathbb{A}_1 = \left\{ x \in \mathbb{R}^N : \left|w(x)\right| \le \frac{\delta}{2} \right\},$$
$$\mathbb{A}_2 = \mathbb{R}^N \setminus \mathbb{A}_1.$$

From (3.2), we have

$$\left|v(x)\right| \le \|v\|_\infty \le C\|v\| \le \frac{\delta}{2}$$

for all $\|u\| \le r_1$ and $x \in \mathbb{R}^N$. On the one hand, one has $|u(x)| \le |v(x)| + |w(x)| \le \|v\|_\infty + \frac{\delta}{2} \le \delta$ for all $x \in \mathbb{A}_1$. Hence, from $(H_5)$, we obtain

$$\int_{\mathbb{A}_1} G_0(x,u)\,dx \le 0.$$

On the other hand, we have

$$\left|u(x)\right| \le \left|v(x)\right| + \left|w(x)\right| \le \frac{\delta}{2} + \left|w(x)\right| \le 2\left|w(x)\right|$$

for all $x \in \mathbb{A}_2$. It follows from (3.3) that

$$G_0(x,u) \le \epsilon u^2 + C_\epsilon |u|^{s+1} \le 4\epsilon w^2 + 2^{s+1} C_\epsilon |w|^{s+1}$$

for all $x \in \mathbb{A}_2$ and all $u \in X_1$ with $\|u\| \le r_1$, which implies that

$$\begin{aligned}\int G_0(x,u) &\le 4\epsilon \int_{\mathbb{A}_2} w^2\,dx + \int_{\mathbb{A}_2} 2^{s+1} C_\epsilon |w|^{s+1}\,dx \\ &\le 4(C_3)^2 \epsilon \|w\|^2 + (2C_3)^{\lambda+1} C_\epsilon \|w\|^{s+1},\end{aligned}$$

where $C_3$ is a constant. Hence, there exist positive constants $C^{**}$, $C_4$ and $C_5$ such that

$$\begin{aligned}I(u) &= \frac{1}{2}\|w\|^2 - \int_{\mathbb{A}_2} G_0(x,u)\,dx - \int_{\mathbb{A}_1} G_0(x,u)\,dx \\ &\ge C^{**}\|w\|^2 - 4(C_3)^2\epsilon\|w\|^2 - (2C_3)^{\lambda+1} C_\epsilon \|w\|^{s+1} - \int_{\mathbb{A}_1} G(x,u)\,dx \\ &\ge C_4\|w\|^2 - C_5\|w\|^{s+1}\end{aligned}$$

for all $u \in X^1$ with $\|u\| \le r_1$, which implies that

$$I(u) \ge 0, \quad \forall u \in X^1 \text{ with } \|u\| \le r$$

for $0 < r$ small enough.

(3) We claim that $I$ satisfies $(C)_c^*$. Consider a sequence $(u_{\alpha_n})$ such that $(u_{\alpha_n})$ is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad I(u_{\alpha_n}) \to c, \qquad \big(1 + \|u_{\alpha_n}\|\big) I'(u_{\alpha_n}) \to 0 \tag{3.4}$$

and

$$\lim_{n\to\infty} \int \left( \frac{1}{2} g_0(x, u_{\alpha_n}) u_{\alpha_n} - G_0(x, u_{\alpha_n}) \right) = c. \tag{3.5}$$

Let $w_{\alpha_n} = \|u_{\alpha_n}\|^{-1} u_{\alpha_n}$. Up to a subsequence, we have

$$w_{\alpha_n} \rightharpoonup w \quad \text{in } X, \qquad w_{\alpha_n} \to w \quad \text{in } L^2_{\text{loc}}, \quad w_{\alpha_n}(x) \to w(x) \quad \text{a.e. } x \in \mathbb{R}^N.$$

If $w = 0$, we choose a sequence $\{t_n\} \subset [0,1]$ such that

$$I(t_n u_{\alpha_n}) = \max_{t\in[0,1]} I(t u_{\alpha_n}).$$

For any $m > 0$, let $v_{\alpha_n} = 2\sqrt{m} w_{\alpha_n}$. Now, we claim that

$$\lim_{n\to\infty} \int G_0(x, v_{\alpha_n}) = 0.$$

Let $\epsilon > 0$; for $r \geq 1$, then,

$$\begin{aligned} \int_{|v_{\alpha_n}|\geq r} G_0(x, v_{\alpha_n})\,dx &\leq C_6 r^{p-2^*} \int_{|v_{\alpha_n}|\geq r} |v_{\alpha_n}|^{2^*}\,dx \\ &\leq C_7 r^{p-2^*} |v_{\alpha_n}|_{2^*}^{2^*}. \end{aligned}$$

Since $p < 2^*$, we may fix $r$ large enough such that

$$\left| \int_{|v_{\alpha_n}|\geq r} G_0(x, v_{\alpha_n})\,dx \right| \leq \frac{\epsilon}{3}$$

for all $n$. Moreover, by $(\mathrm{H}_6)$, there exists $R > 0$ such that

$$\left| \int_{|v_{\alpha_n}|\geq r} G_0(x, v_{\alpha_n})\,dx \right| \leq |v_{\alpha_n}|_2^2 \sup_{|t|\leq r, |x|\geq R} \frac{|G_0(x,t)|}{t^2} \leq \frac{\epsilon}{3}$$

for all $n$. Finally, since $v_{\alpha_n} \to 0$ in $L^s(B_R(0))$ for $s \in [2, 2^*)$, we can use $(\mathrm{H}_1)$ again to derive

$$\left| \int_{|v_{\alpha_n}|\leq r \cap |x|\leq R} G_0(x, v_{\alpha_n})\,dx \right| \leq \frac{\epsilon}{3}$$

for $n$ large enough. Combining the above three formulas, our claim holds.

So, for $n$ large enough, $2\sqrt{m}\|u_{\alpha_n}\|^{-1} \in (0,1)$, we have

$$I(t_n u_{\alpha_n}) \geq I(v_{\alpha_n}) \geq m - \epsilon \geq \frac{m}{2}, \tag{3.6}$$

where $\epsilon$ is a small enough constant.

That is, $I(t_n u_{\alpha_n}) \to \infty$. Now, $I(0) = 0$, $I(u_{\alpha_n}) \to c$, we know that $t_n \in [0,1]$ and

$$\int \left(\left|\nabla(t_n u_{\alpha_n})\right|^2 - V(x)t_n^2|u_{\alpha_n}|^2\right) \\ + \lambda \int t_n^2|u_{\alpha_n}|^2 - \int g_0(x, t_n u_{\alpha_n})t_n u_{\alpha_n} = t_n \frac{d}{dt}\bigg|_{t=t_n} I(tu_{\alpha_n}) = 0. \tag{3.7}$$

Therefore, using $(H_4)$, we have

$$\int \frac{1}{2}g_0(x, u_{\alpha_n})u_{\alpha_n} - G_0(x, u_{\alpha_n}) \geq \frac{1}{\theta}\int \left(\frac{1}{2}g_0(x, t_n u_{\alpha_n})t_n u_{\alpha_n} - G_0(x, t_n u_{\alpha_n})\right) \to +\infty.$$

This contradicts (3.5).

If $w \neq 0$, then the set $\ominus = \{x \in \mathbb{R}^N : w(x) \neq 0\}$ has a positive Lebesgue measure. For $x \in \ominus$, we have $|u_{\alpha_n}(x)| \to \infty$. Hence, by $(H_3)$, we have

$$\frac{g_0(x, u_{\alpha_n}(x))u_{\alpha_n}(x)}{|u_{\alpha_n}(x)|^2}\left|w_{\alpha_n}(x)\right|^2 \to \infty. \tag{3.8}$$

From (3.4), we obtain

$$1 - o(1) \geq \left(\int_{w\neq 0} + \int_{w=0}\right)\frac{g_0(x, u_{\alpha_n}(x))u_{\alpha_n}(x)}{|u_{\alpha_n}(x)|^2}\left|w_{\alpha_n}(x)\right|^2 dx. \tag{3.9}$$

By (3.8), the right-hand side of (3.9) $\to +\infty$. This is a contradiction.

In any case, we obtain a contradiction. Therefore, $\{u_{\alpha_n}\}$ is bounded.

Next, we denote $\{u_{\alpha_n}\}$ as $\{u_n\}$ and prove $\{u_n\}$ contains a convergent subsequence.

In fact, we know that $\{u_n\}$ is bounded in $X$. Passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $X$. In order to establish strong convergence, it suffices to show that

$$\|u_n\| \to \|u\|.$$

By the condition $(H_6)$ and $\langle I'(u_n), u_n - u\rangle \to 0$, we can similarly conclude it according to the above proof of our claim.

Finally, we claim that for every $m \in N$,

$$I(u) \to -\infty \quad \text{as } \|u\| \to \infty, u \in X_m^1 \oplus X^2.$$

By $(H_2)$ and $(H_3)$, there exist large enough $M$ and some positive constant $T$ such that

$$G_0(x,t) \geq Mt^2, \quad x \in \mathbb{R}^N, t \geq T.$$

So, for any $u \in X_m^1 \oplus X^2$, we have

$$\begin{aligned} I(tu) &= \frac{1}{2}t^2\int \left(|\nabla u|^2 - V(x)|u|^2\right) + \frac{t^2}{2}\lambda\int u^2 - \int G_0(x, tu) \\ &\leq \frac{1}{2}t^2\int \left(|\nabla u|^2 - V(x)|u|^2\right) + \frac{t^2}{2}\lambda\int u^2 - Mt^2\int u^2 \to -\infty \quad \text{as } t \to +\infty. \end{aligned}$$

Hence, our claim holds. □

*Proof of Theorem* 1.2 We omit the proof which depends on Lemma 2.2 and is similar to the preceding one since our result is a variant of Ding Yanheng's Theorem 1.2 (see [12]). □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors read and approved the final manuscript.

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