Zhuang and Zhang *Boundary Value Problems* 2013, **2013**:168
http://www.boundaryvalueproblems.com/content/2013/1/168

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Global existence and asymptotic behavior of solutions to a class of fourth-order wave equations

Zhitao Zhuang and Yuanzhang Zhang*

*Correspondence:
yzzhangmath@126.com
School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou, 450011, China

Abstract
This paper is concerned with the Cauchy problem for a class of fourth-order wave equations in an *n*-dimensional space. Based on the decay estimate of solutions to the corresponding linear equation, a solution space is defined. We prove the global existence and optimal decay estimate of the solution in the corresponding Sobolev spaces by the contraction mapping principle provided that the initial value is suitably small.
MSC: 35L30; 35L75

Keywords: fourth-order wave equation; global existence; decay estimate

1 Introduction

We investigate the Cauchy problem for a class of fourth-order wave equations

$$au_{tt} + \Delta^2 u + u_t = \Delta f(u) \tag{1.1}$$

with the initial value

$$t = 0 : u = u_0(x), \qquad u_t = u_1(x). \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $t > 0$, and $a > 0$ is a constant. The nonlinear term $f(u)$ is a smooth function with $f(u) = O(u^2)$ for $u \to 0$.

Equation (1.1) is reduced to the classical Cahn-Hilliard equation if $a = 0$ (see [1]), which has been widely studied by many authors. Galenko *et al.* [2–5] proposed to add inertial term au_{tt} to the classical Cahn-Hilliard equation in order to model non-equilibrium decompositions caused by deep supercooling in certain glasses. For more background, we refer to [4–6] and references therein. It is obvious that (1.1) is a fourth-order wave equation. For global existence and asymptotic behavior of solutions to more higher order wave equations, we refer to [7–14] and references therein.

Very recently, global existence and asymptotic behavior of solutions to the problem (1.1), (1.2) were established by Wang and Wei [7] under smallness condition on the initial data. When $u_0 \in H^{s+2} \cap L^1$, $u_1 \in H^s \cap L^1$, they obtained the following decay estimate:

$$\|\partial_x^k u(t)\|_{H^{s+2-k}} \le C\big(\|u_0\|_{L^1 \cap H^{s+2}} + \|u_1\|_{L^1 \cap H^s}\big)(1+t)^{-\frac{n}{8}-\frac{k}{4}} \tag{1.3}$$

© 2013 Zhuang and Zhang; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

for $0 \le k \le s+2$ and $s \ge [n/2]+5$. The main purpose of this paper is to refine the result in [7] and prove the following decay estimate for the solution to the problem (1.1), (1.2) for $n \ge 1$ with L^1 data,

$$\left\| \partial_x^k u(t) \right\|_{H^{s+2-k}} \le CE_0(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}, \tag{1.4}$$

for $0 \le k \le s+2$ and $s \ge \max\{0, [n/2]-1\}$. Here $E_0 := \|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}} + \|u_1\|_{\dot{W}^{-2,1}\cap H^s}$ is assumed to be small. We also establish the decay estimate for the solution to the problem (1.1), (1.2) for $n \ge 1$ with L^2 data,

$$\left\| \partial_x^k u(t) \right\|_{H^{s+2-k}} \le CE_1(1+t)^{-\frac{k}{4}-\frac{1}{2}}, \tag{1.5}$$

for $0 \le k \le s+2$ and $s \ge \max\{0, [n/2]-1\}$. Here $E_1 := \|u_0\|_{\dot{H}^{-2}\cap H^{s+2}} + \|u_1\|_{\dot{H}^{-2}\cap H^s}$ is assumed to be small. Compared to the result in [7], we obtain a better decay estimate of solutions for small initial data.

The paper is organized as follows. In Section 2, we study the decay property of the solution operators appearing in the solution formula. We prove global existence and asymptotic behavior of solutions for the Cauchy problem (1.1), (1.2) in Sections 3 and 4, respectively.

Notations We introduce some notations which are used in this paper. Let $\mathcal{F}[u]$ denote the Fourier transform of u defined by

$$\hat{u}(\xi) = \mathcal{F}[u](\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx.$$

We denote its inverse transform by $\mathcal{F}^{-1}$. For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. The usual Sobolev space of order s is defined by $W^{s,p} = (I-\Delta)^{-\frac{s}{2}}L^p$ with the norm $\|f\|_{W^{s,p}} = \|(I-\Delta)^{\frac{s}{2}}f\|_{L^p}$. The corresponding homogeneous Sobolev space of order s is defined by $\dot{W}^{s,p} = (-\Delta)^{-\frac{s}{2}}L^p$ with the norm $\|f\|_{\dot{W}^{s,p}} = \|(-\Delta)^{\frac{s}{2}}f\|_{L^p}$; when $p=2$, we write $H^s = W^{s,2}$ and $\dot{H}^s = \dot{W}^{s,2}$. We note that $W^{s,p} = L^p \cap \dot{W}^{s,p}$ for $s \ge 0$.

For a nonnegative integer k, ∂_x^k denotes the totality of all the kth order derivatives with respect to $x \in \mathbb{R}^n$. Also, for an interval I and a Banach space X, $C^k(I;X)$ denotes the space of k-times continuously differential functions on I with values in X.

Throughout the paper, we denote every positive constant by the same symbol C or c without confusion. $[\cdot]$ is the Gauss symbol.

2 Decay property

The aim of this section is to derive the solution formula to the Cauchy problem (1.1), (1.2). Without loss of generality, we take $a=1$. We first study the linearized equation of (1.1),

$$u_{tt} + \Delta^2 u + u_t = 0, \tag{2.1}$$

with the initial data in (1.2). Taking the Fourier transform, we have

$$\hat{u}_{tt} + \hat{u}_t + |\xi|^4 \hat{u} = 0. \tag{2.2}$$

The corresponding initial value are given as

$$t = 0 : \hat{u} = \hat{u}_0(\xi), \qquad \hat{u}_t = \hat{u}_1(\xi). \tag{2.3}$$

The characteristic equation of (2.2) is

$$\lambda^2 + \lambda + |\xi|^4 = 0.$$

Let $\lambda = \lambda_\pm(\xi)$ be the corresponding eigenvalues, we obtain

$$\lambda_\pm(\xi) = \frac{-1 \pm \sqrt{1 - 4|\xi|^4}}{2}. \tag{2.4}$$

The solution to the problem (2.2), (2.3) is given in the form

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)\hat{u}_1(\xi) + \hat{H}(\xi, t)\hat{u}_0(\xi), \tag{2.5}$$

where

$$\hat{G}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)}\left(e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}\right) \tag{2.6}$$

and

$$\hat{H}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)}\left(\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}\right). \tag{2.7}$$

We define $G(x,t)$ and $H(x,t)$ by $G(x,t) = \mathcal{F}^{-1}[\hat{G}(\xi,t)](x)$ and $H(x,t) = \mathcal{F}^{-1}[\hat{H}(\xi,t)](x)$, respectively, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then, applying $\mathcal{F}^{-1}$ to (2.5), we obtain

$$u(t) = G(t) * u_1 + H(t) * u_0. \tag{2.8}$$

By the Duhamel principle, we obtain the solution formula to (1.1), (1.2)

$$u(t) = G(t) * u_1 + H(t) * u_0 + \int_0^t G(t - \tau) * \Delta f(u)(\tau)\, d\tau. \tag{2.9}$$

The aim of this section is to establish decay estimates of the solution operators $G(t)$ and $H(t)$ appearing in the solution formula (2.8).

Lemma 2.1 *The solution of the problem* (2.2), (2.3) *verifies the estimate*

$$\left(1 + |\xi|^2\right)^2\left|\hat{u}(\xi, t)\right|^2 + \left|\hat{u}_t(\xi, t)\right|^2 \le Ce^{-c\omega(\xi)t}\left(\left(1 + |\xi|^2\right)^2\left|\hat{u}_0(\xi)\right|^2 + \left|\hat{u}_1(\xi)\right|^2\right), \tag{2.10}$$

for $\xi \in \mathbb{R}^n$ *and* $t \ge 0$, *where* $\omega(\xi) = \frac{|\xi|^4}{(1+|\xi|^2)^2}$.

Proof We apply the energy method in the Fourier space to prove (2.10). Such an energy method was first developed in [15]. We multiply (2.2) by $\bar{\hat{u}}_t$ and take the real part. This yields

$$\frac{1}{2}\frac{d}{dt}\left\{|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2\right\} + |\hat{u}_t|^2 = 0. \tag{2.11}$$

Multiplying (2.2) by $\bar{\hat{u}}$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|\hat{u}|^2 + 2\,\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\right\} + |\xi|^4|\hat{u}|^2 - |\hat{u}_t|^2 = 0. \tag{2.12}$$

Combining (2.11) and (2.12) yields

$$\frac{d}{dt}E + F = 0, \tag{2.13}$$

where

$$E = |\hat{u}_t|^2 + \left[\frac{1}{2} + |\xi|^4\right]|\hat{u}|^2 + \mathrm{Re}(\hat{u}_t\bar{\hat{u}})$$

and

$$F = |\xi|^4|\hat{u}|^2 + |\hat{u}_t|^2.$$

A simple computation implies that

$$CE_0 \le E \le CE_0, \tag{2.14}$$

where

$$E_0 = \left(1 + |\xi|^2\right)^2|\hat{u}|^2 + |\hat{u}_t|^2.$$

Note that

$$F \ge \frac{|\xi|^4}{(1+|\xi|^2)^2}E_0.$$

It follows from (2.14) that

$$F \ge c\omega(\xi)E, \tag{2.15}$$

where

$$\omega(\xi) = \frac{|\xi|^4}{(1+|\xi|^2)^2}.$$

Using (2.13) and (2.15), we get

$$\frac{d}{dt}E + c\omega(\xi)E \le 0.$$

Thus

$$E(\xi,t) \le e^{-cw(\xi)t}E(\xi,0),$$

which together with (2.14) proves the desired estimates (2.10). Then we have completed the proof of the lemma. □

Lemma 2.2 *Let $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ be the fundamental solutions to* (2.1) *in the Fourier space, which are given in* (2.6) *and* (2.7), *respectively. Then we have the estimates*

$$\left(1+|\xi|^2\right)^2\left|\hat{G}(\xi,t)\right|^2 + \left|\hat{G}_t(\xi,t)\right|^2 \le Ce^{-c\omega(\xi)t} \tag{2.16}$$

and

$$\left(1+|\xi|^2\right)^2\left|\hat{H}(\xi,t)\right|^2 + \left|\hat{H}_t(\xi,t)\right|^2 \le C\left(1+|\xi|^2\right)^2e^{-c\omega(\xi)t} \tag{2.17}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$, where $\omega(\xi) = \frac{|\xi|^4}{(1+|\xi|^2)^2}$.

Proof If $\hat{u}_0(\xi) = 0$, from (2.5), we obtain

$$\hat{u}(\xi,t) = \hat{G}(\xi,t)\hat{u}_1(\xi), \qquad \hat{u}_t(\xi,t) = \hat{G}_t(\xi,t)\hat{u}_1(\xi).$$

Substituting the equalities into (2.10) with $\hat{u}_0(\xi) = 0$, we get (2.16).

In what follows, we consider $\hat{u}_1(\xi) = 0$, it follows from (2.5) that

$$\hat{u}(\xi,t) = \hat{H}(\xi,t)\hat{u}_0(\xi), \qquad \hat{u}_t(\xi,t) = \hat{H}_t(\xi,t)\hat{u}_0(\xi).$$

Substituting the equalities into (2.10) with $\hat{u}_1(\xi) = 0$, we get (2.17). The lemma is proved. □

Lemma 2.3 *Let $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ be the fundamental solutions to* (2.1) *in the Fourier space, which are given in* (2.6) *and* (2.7), *respectively. Then there exists a small positive number R_0 such that if $|\xi| \le R_0$ and $t \ge 0$, we have the following estimate*:

$$\left|\hat{G}_t(\xi,t)\right| \le C|\xi|^4e^{-c|\xi|^4t} + Ce^{-ct} \tag{2.18}$$

and

$$\left|\hat{H}_t(\xi,t)\right| \le C|\xi|^4e^{-c|\xi|^4t} + Ce^{-ct}. \tag{2.19}$$

Proof For sufficiently small ξ, using the Taylor formula, we get

$$\lambda_+(\xi) = -|\xi|^4 + O\left(|\xi|^8\right), \qquad \lambda_-(\xi) = -1 + |\xi|^4 + O\left(|\xi|^8\right) \tag{2.20}$$

and

$$\frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} = 1 + 2|\xi|^4 + O\left(|\xi|^8\right). \tag{2.21}$$

It follows from (2.6) and (2.7) that

$$\begin{cases} \hat{G}_t(\xi,t) = \frac{\lambda_+(\xi)e^{\lambda_+ t} - \lambda_-(\xi)e^{\lambda_- t}}{\lambda_+(\xi)-\lambda_-(\xi)}, \\ \hat{H}_t(\xi,t) = \frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(\lambda_+(\xi)\lambda_-(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)\lambda_+(\xi)e^{\lambda_+(\xi)t}). \end{cases} \tag{2.22}$$

Equations (2.18) and (2.19) follow from (2.20)-(2.22). The proof of Lemma 2.3 is completed. □

Lemma 2.4 *Let* $1 \le p \le 2$ *and* $k \ge 0$. *Then we have*

$$\left\|\partial_x^k G(t) * \phi\right\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k+l}{4}}\|\phi\|_{\dot{W}^{-l,p}} + Ce^{-ct}\left\|\partial_x^{(k-2)_+}\phi\right\|_{L^2}, \tag{2.23}$$

$$\left\|\partial_x^k H(t) * \phi\right\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k+l}{4}}\|\phi\|_{\dot{W}^{-l,p}} + Ce^{-ct}\left\|\partial_x^k\phi\right\|_{L^2}, \tag{2.24}$$

$$\left\|\partial_x^k G_t(t) * \phi\right\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k+l}{4}-1}\|\phi\|_{\dot{W}^{-l,p}} + Ce^{-ct}\left\|\partial_x^k\phi\right\|_{L^2}, \tag{2.25}$$

and

$$\left\|\partial_x^k H_t(t) * \phi\right\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k+l}{4}-1}\|\phi\|_{\dot{W}^{-l,p}} + Ce^{-ct}\left\|\partial_x^{k+2}\phi\right\|_{L^2}, \tag{2.26}$$

$$\left\|\partial_x^k G(t) * \Delta g\right\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}-\frac{1}{2}}\|g\|_{L^p} + Ce^{-ct}\left\|\partial_x^k g\right\|_{L^2}, \tag{2.27}$$

$$\left\|\partial_x^k G_t(t) * \Delta g\right\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}-\frac{3}{2}}\|g\|_{L^p} + Ce^{-ct}\left\|\partial_x^{k+2} g\right\|_{L^2}, \tag{2.28}$$

where $(k-2)_+ = \max\{0, k-2\}$ *in* (2.23).

Proof By the property of the Fourier transform and (2.16), we obtain

$$\begin{aligned} \left\|\partial_x^k G(t) * \phi\right\|_{L^2}^2 &= \int_{\mathbb{R}^n} |\xi|^{2k}\left|\hat{G}(\xi,t)\right|^2\left|\hat{\phi}(\xi)\right|^2 d\xi \\ &\le \int_{|\xi|\le R_0} |\xi|^{2k}\left(1+|\xi|^2\right)^{-2} e^{-c\omega(\xi)t}\left|\hat{\phi}(\xi)\right|^2 d\xi \\ &\quad + C\int_{|\xi|\ge R_0} |\xi|^{2k}\left(1+|\xi|^2\right)^{-2} e^{-c\omega(\xi)t}\left|\hat{\phi}(\xi)\right|^2 d\xi \\ &\le C\int_{|\xi|\le R_0} |\xi|^{2k} e^{-c|\xi|^4 t}\left|\hat{\phi}(\xi)\right|^2 d\xi + C\int_{|\xi|\ge R_0} |\xi|^{2k}|\xi|^{-4} e^{-ct}\left|\hat{\phi}(\xi)\right|^2 d\xi \\ &\le C\left\||\xi|^{2k+2l} e^{-c|\xi|^4 t}\right\|_{L^q}\left\||\xi|^{-l}\hat{\phi}(\xi)\right\|_{L^{p'}}^2 + Ce^{-ct}\left\|\partial_x^{(k-2)_+}\phi\right\|_{L^2}, \end{aligned} \tag{2.29}$$

where R_0 is a positive constant in Lemma 2.3, and $\frac{1}{q} + \frac{2}{p'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

By a straight computation, we get

$$\begin{aligned} \left\||\xi|^{2k+2l} e^{-c|\xi|^4 t}\right\|_{L^q(|\xi|\le R_0)} &\le C(1+t)^{-\frac{n}{4q}-\frac{k+l}{2}} \\ &\le C(1+t)^{-\frac{n}{4}(\frac{2}{p}-1)-\frac{k+l}{2}}. \end{aligned} \tag{2.30}$$

It follows from the Hausdorff-Young inequality that

$$\left\||\xi|^{-l}\hat{\phi}(\xi)\right\|_{L^{p'}} \le C\left\|(-\Delta)^{\frac{-l}{2}}\phi\right\|_{L^p} \le C\|\phi\|_{\dot{W}^{-l,p}}. \tag{2.31}$$

Combining (2.29)-(2.31) yields (2.23). Similarly, we can prove (2.24)-(2.28). Thus we have completed the proof of the lemma. □

3 Global existence and decay estimate (I)

The purpose of this section is to prove global existence and asymptotic behavior of solutions to the Cauchy problem (1.1), (1.2) with L^1 data. We need the following lemma, which comes from [16] (see also [17]).

Lemma 3.1 *Assume that $f = f(v)$ is smooth, where $v = (v_1, \dots, v_n)$ is a vector function. Suppose that $f(v) = O(|v|^{1+\theta})$ ($\theta \geq 1$ is an integer) when $|v| \leq v_0$. Then, for the integer $m \geq 0$, if $v, w \in W^{m,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\|v\|_{L^\infty} \leq v_0$, $\|w\|_{L^\infty} \leq v_0$, then $f(v) - f(w) \in W^{m,r}(\mathbb{R}^n)$. Furthermore, the following inequalities hold*:

$$\left\|\partial_x^m f(v)\right\|_{L^r} \leq C\|v\|_{L^p}\left\|\partial_x^m v\right\|_{L^q}\|v\|_{L^\infty}^{\theta-1} \tag{3.1}$$

and

$$\begin{aligned}\left\|\partial_x^m\big(f(v)-f(w)\big)\right\|_{L^r} \leq C\big\{&\big(\left\|\partial_x^m v\right\|_{L^q} + \left\|\partial_x^m w\right\|_{L^q}\big)\|v-w\|_{L^p} + \\ &+ \big(\|v\|_{L^p} + \|w\|_{L^p}\left\|\partial_x^m (v-w)\right\|_{L^q}\big)\big\}\big(\|v\|_{L^\infty} + \|w\|_{L^\infty}\big)^{\theta-1},\end{aligned} \tag{3.2}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 \leq p, q, r \leq +\infty$.

Based on the decay estimates of solutions to the linear problem (2.1), (1.2), we define the following solution space:

$$X = \left\{u \in C\big([0,\infty); H^{s+2}(\mathbb{R}^n)\big) \cap C^1\big([0,\infty); H^s(\mathbb{R}^n)\big) : \|u\|_X < \infty\right\},$$

where

$$\|u\|_X = \sup_{t\geq 0}\left\{\sum_{k\leq s+2}(1+t)^{\frac{n}{8}+\frac{k}{4}+\frac{1}{2}}\left\|\partial_x^k u(t)\right\|_{L^2} + \sum_{h\leq s}(1+t)^{\frac{n}{8}+\frac{h}{4}+\frac{3}{2}}\left\|\partial_x^h u_t(t)\right\|_{L^2}\right\}.$$

For $R > 0$, we define

$$X_R = \left\{u \in X : \|u\|_X \leq R\right\}.$$

The Gagliardo-Nirenberg inequality gives

$$\left\|u(t)\right\|_{L^\infty} \leq C\left\|u(t)\right\|_{L^2}^{1-\frac{n}{2s_0}}\left\|\partial_x^{s_0} u(t)\right\|_{L^2}^{\frac{n}{2s_0}} \leq C(1+t)^{-(\frac{n}{4}+\frac{1}{2})}\|u\|_X, \tag{3.3}$$

where $s_0 = [\frac{n}{2}] + 1$, $s_0 \leq s + 2$ (*i.e.*, $s \geq [\frac{n}{2}] - 1$).

Theorem 3.1 *Assume that $u_0 \in \dot{W}^{-2,1}(\mathbb{R}^n) \cap H^{s+2}(\mathbb{R}^n)$, $u_1 \in \dot{W}^{-2,1}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ ($s \geq \max\{0, [n/2] - 1\}$). Put*

$$E_0 = \|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}} + \|u_1\|_{\dot{W}^{-2,1}\cap H^s}.$$

If E_0 is suitably small, the Cauchy problem (1.1), (1.2) *has a unique global solution* $u(x,t)$ *satisfying*

$$u \in C\big([0,\infty); H^{s+2}(\mathbb{R}^n)\big) \cap C^1\big([0,\infty); H^s(\mathbb{R}^n)\big).$$

Moreover, the solution satisfies the decay estimate

$$\big\|\partial_x^k u(t)\big\|_{L^2} \le CE_0(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}} \tag{3.4}$$

and

$$\big\|\partial_x^h u_t(t)\big\|_{L^2} \le CE_0(1+t)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}} \tag{3.5}$$

for $0 \le k \le s+2$ *and* $0 \le h \le s$.

Proof Let us define the mapping

$$\Phi(u) = G(t) * u_1 + H(t) * u_0 + \int_0^t G(t-\tau) * \Delta f\big(u(\tau)\big)\, d\tau. \tag{3.6}$$

Using (2.23), (2.24), (2.27), (3.1) and (3.3), for $0 \le k \le s+2$, we obtain

$$\begin{aligned}
\big\|\partial_x^k \Phi(u)\big\|_{L^2} &\le \big\|\partial_x^k G(t) * u_1\big\|_{L^2} + C\big\|\partial_x^k H(t) * u_0\big\|_{L^2} \\
&\quad + C\int_0^t \big\|\partial_x^k G(t-\tau) * \Delta f\big(u(\tau)\big)\big\|_{L^2}\, d\tau \\
&\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\big(\|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}} + \|u_1\|_{\dot{W}^{-2,1}\cap H^s}\big) \\
&\quad + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}} \big\|f(u)\big\|_{L^1}\, d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{2}} \big\|\partial_x^k f(u)\big\|_{L^2}\, d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)} \big\|\partial_x^k f(u)\big\|_{L^2}\, d\tau \\
&\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\big(\|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}} + \|u_1\|_{\dot{W}^{-2,1}\cap H^s}\big) \\
&\quad + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}} \|u\|_{L^2}^2\, d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{2}} \big\|\partial_x^k u\big\|_{L^2}\|u\|_{L^\infty}\, d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)} \big\|\partial_x^k u\big\|_{L^2}\|u\|_{L^\infty}\, d\tau \\
&\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\big(\|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}} + \|u_1\|_{\dot{W}^{-2,1}\cap H^s}\big) \\
&\quad + CR^2\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{4}-1}\, d\tau \\
&\quad + CR^2\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3n}{8}-\frac{k}{4}-1}\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ CR^2 \int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{3n}{8}-\frac{k}{4}-1}\,d\tau \\
&\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\bigl\{\bigl(\|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}}+\|u_1\|_{\dot{W}^{-2,1}\cap H^{s}}\bigr)+R^2\bigr\}.
\end{aligned}
$$

Thus

$$
(1+t)^{\frac{n}{8}+\frac{k}{4}+\frac{1}{2}}\bigl\|\partial_x^k\Phi(u)\bigr\|_{L^2}\le CE_0+CR^2. \tag{3.7}
$$

It follows from (3.6) that

$$
\Phi(u)_t = G_t(t)*u_1 + H_t(t)*u_0 + \int_0^t G_t(t-\tau)*\Delta f\bigl(u(\tau)\bigr)\,d\tau. \tag{3.8}
$$

By exploiting (3.8), (2.25), (2.26), (2.28), (3.1) and (3.3), for $h \le s$, we have

$$
\begin{aligned}
\bigl\|\partial_x^h\Phi(u)_t\bigr\|_{L^2} &\le C\bigl\|\partial_x^h G_t(t)*u_1\bigr\|_{L^2} + C\bigl\|\partial_x^h H_t(t)*u_0\bigr\|_{L^2} \\
&\quad + C\int_0^t \bigl\|\partial_x^h G_t(t-\tau)*\Delta f\bigl(u(\tau)\bigr)\bigr\|_{L^2}\,d\tau \\
&\le C(1+t)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}}\bigl(\|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}}+\|u_1\|_{\dot{W}^{-2,1}\cap H^{s}}\bigr) \\
&\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}}\bigl\|f(u)\bigr\|_{L^1}\,d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{2}}\bigl\|\partial_x^h f(u)\bigr\|_{L^2}\,d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)}\bigl\|\partial_x^h f(u)\bigr\|_{L^2}\,d\tau \\
&\le C(1+t)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}}\bigl\{\bigl(\|u_0\|_{\dot{W}^{-2,1}\cap H^{s+2}}+\|u_1\|_{\dot{W}^{-2,1}\cap H^{s}}\bigr)+R^2\bigr\}.
\end{aligned}
$$

The above inequality implies

$$
(1+t)^{\frac{n}{8}+\frac{h}{4}+\frac{3}{2}}\bigl\|\partial_x^h\Phi(u)_t\bigr\|_{L^2}\le CE_0+CR^2. \tag{3.9}
$$

Combining (3.7) and (3.9) and taking E_0 and R suitably small, we get

$$
\bigl\|\Phi(u)\bigr\|_X \le R. \tag{3.10}
$$

For $\tilde{u},\bar{u}\in X_R$, (3.6) gives

$$
\Phi(\tilde{u})-\Phi(\bar{u}) = \int_0^t G(t-\tau)*\Delta\bigl[f(\tilde{u})-f(\bar{u})\bigr]\,d\tau. \tag{3.11}
$$

By (2.27), (3.2) and (3.3), for $0\le k\le s+2$, we infer that

$$
\begin{aligned}
\bigl\|\partial_x^k\bigl(\Phi(\tilde{u})-\Phi(\bar{u})\bigr)\bigr\|_{L^2} &\le \int_0^t \bigl\|\partial_x^k G(t-\tau)*\Delta\bigl(f(\tilde{u})-f(\bar{u})\bigr)\bigr\|_{L^2}\,d\tau \\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\bigl\|f(\tilde{u})-f(\bar{u})\bigr\|_{L^1}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\big\|\partial_x^k\big(f(\tilde{u})-f(\bar{u})\big)\big\|_{L^2}\,d\tau \\
&+ C\int_0^{t}e^{-c(t-\tau)}\big\|\partial_x^k\big(f(\tilde{u})-f(\bar{u})\big)\big\|_{L^2}\,d\tau \\
\le{}& C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\big(\|\tilde{u}\|_{L^2}+\|\bar{u}\|_{L^2}\big)\|\tilde{u}-\bar{u}\|_{L^2}\,d\tau \\
&+ C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\big\{\big(\big\|\partial_x^k\tilde{u}\big\|_{L^2}+\big\|\partial_x^k\tilde{u}\big\|_{L^2}\big)\|\tilde{u}-\bar{u}\|_{L^\infty} \\
&+ \big(\|\tilde{u}\|_{L^2}+\|\bar{u}\|_{L^2}\big)\big\|\partial_x^k(\tilde{u}-\bar{u})\big\|_{L^2}\big\}\,d\tau \\
&+ C\int_0^{t}e^{-c(t-\tau)}\big\{\big(\big\|\partial_x^k\tilde{u}\big\|_{L^2}+\big\|\partial_x^k\tilde{u}\big\|_{L^2}\big)\|\tilde{u}-\bar{u}\|_{L^\infty} \\
&+ \big(\|\tilde{u}\|_{L^\infty}+\|\bar{u}\|_{L^\infty}\big)\big\|\partial_x^k(\tilde{u}-\bar{u})\big\|_{L^2}\big\}\,d\tau \\
\le{}& CR\|\tilde{u}-\bar{u}\|_X\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{4}-\frac{1}{2}}\,d\tau \\
&+ CR\|\tilde{u}-\bar{u}\|_X\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{8}-\frac{1}{2}}(1+\tau)^{-\frac{3n}{8}-\frac{k}{4}-1}\,d\tau \\
&+ CR|\tilde{u}-\bar{u}\|_X\int_0^{t}e^{-c(t-\tau)}(1+\tau)^{-\frac{3n}{8}-\frac{k}{4}-1}\,d\tau \\
\le{}& CR(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\|\tilde{u}-\bar{u}\|_X,
\end{aligned}
$$

which implies

$$
(1+t)^{\frac{n}{8}+\frac{k}{4}+\frac{1}{2}}\big\|\partial_x^k\big(\Phi(\tilde{u})-\Phi(\bar{u})\big)\big\|_{L^2}\le CR\|\tilde{u}-\bar{u}\|_X. \tag{3.12}
$$

Similarly, for $0\le h\le s$, from (3.11), (2.28) and (3.2), (3.3), we deduce that

$$
\begin{aligned}
\big\|\partial_x^h\big(\Phi(\tilde{u})-\Phi(\bar{u})\big)_t\big\|_{L^2}\le{}& \int_0^t\big\|\partial_x^hG_t(t-\tau)*\Delta\big[f(\tilde{u})-f(\bar{u})\big]\big\|_{L^2}\,d\tau \\
\le{}& C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}}\big\|\big(f(\tilde{u})-f(\bar{u})\big)\big\|_{L^1}\,d\tau \\
&+ C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{2}}\big\|\partial_x^h\big(f(\tilde{u})-f(\bar{u})\big)\big\|_{L^2}\,d\tau \\
&+ C\int_0^{t}e^{-c(t-\tau)}\big\|\partial_x^h\big(f(\tilde{u})-f(\bar{u})\big)\big\|_{L^2}\,d\tau \\
\le{}& CR(1+t)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}}\|\tilde{u}-\bar{u}\|_X,
\end{aligned}
$$

which gives

$$
(1+t)^{\frac{n}{8}+\frac{h}{4}+\frac{3}{2}}\big\|\partial_x^h\big(\Phi(\tilde{u})-\Phi(\bar{u})\big)_t\big\|_{L^2}\le CR\|\tilde{u}-\bar{u}\|_X. \tag{3.13}
$$

Combining (3.12) and (3.13) and taking R suitably small yields

$$
\big\|\Phi(\tilde{u})-\Phi(\bar{u})\big\|_X\le\frac{1}{2}\|\tilde{u}-\bar{u}\|_X. \tag{3.14}
$$

From (3.10) and (3.14), we know that Φ is a strictly contracting mapping. Consequently, we conclude that there exists a fixed point $u \in X_R$ of the mapping Φ, which is a solution to (1.1), (1.2). We have completed the proof of the theorem. □

4 Global existence and decay estimate (II)

In the previous section, we have proved global existence and asymptotic behavior of solutions to the Cauchy problem (1.1), (1.2) with L^1 data. The purpose of this section is to establish global existence and asymptotic behavior of solutions to the Cauchy problem (1.1), (1.2) with L^2 data. Based on the decay estimates of solutions to the linear problem (2.1), (1.2), we define the following solution space:

$$X = \{u \in C([0,\infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0,\infty); H^s(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sup_{t\geq 0}\left\{\sum_{k\leq s+2}(1+t)^{\frac{k}{4}+\frac{1}{2}}\left\|\partial_x^k u(t)\right\|_{L^2} + \sum_{h\leq s}(1+t)^{\frac{h}{4}+\frac{3}{2}}\left\|\partial_x^h u_t(t)\right\|_{L^2}\right\}.$$

For $R > 0$, we define

$$X_R = \{u \in X : \|u\|_X \leq R\}.$$

Thanks to the Gagliardo-Nirenberg inequality, we get

$$\left\|u(t)\right\|_{L^\infty} \leq C(1+t)^{-(\frac{n}{8}+\frac{1}{2})}\|u\|_X. \tag{4.1}$$

Theorem 4.1 *Suppose that* $u_0 \in \dot{H}^{-2}(\mathbb{R}^n) \cap H^{s+2}(\mathbb{R}^n)$, $u_1 \in \dot{H}^{-2}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ $(s \geq \max\{0, [n/2]-1\})$. *Put*

$$E_1 = \|u_0\|_{\dot{H}^{-2}\cap H^{s+2}} + \|u_1\|_{\dot{H}^{-2}\cap H^s}.$$

If E_0 *is suitably small, the Cauchy problem* (1.1), (1.2) *has a unique global solution* $u(x,t)$ *satisfying*

$$u \in C([0,\infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0,\infty); H^s(\mathbb{R}^n)).$$

Moreover, the solution satisfies the decay estimate

$$\left\|\partial_x^k u(t)\right\|_{L^2} \leq CE_1(1+t)^{-\frac{k}{4}-\frac{1}{2}} \tag{4.2}$$

and

$$\left\|\partial_x^h u_t(t)\right\|_{L^2} \leq CE_1(1+t)^{-\frac{h}{4}-\frac{3}{2}} \tag{4.3}$$

for $0 \leq k \leq s+2$ *and* $0 \leq h \leq s$.

Proof Let the mapping Φ be defined in (3.6).

For $k \le s+2$, (2.23), (2.24), (2.27), (3.1) and (4.1) give

$$\begin{aligned}
\left\|\partial_x^k \Phi(u)\right\|_{L^2} \le{}& \left\|\partial_x^k G(t) * u_1\right\|_{L^2} + C\left\|\partial_x^k H(t) * u_0\right\|_{L^2} \\
&+ C\int_0^t \left\|\partial_x^k G(t-\tau) * \Delta f\bigl(u(\tau)\bigr)\right\|_{L^2} d\tau \\
\le{}& C(1+t)^{-\frac{k}{4}-\frac{1}{2}}\bigl(\|u_0\|_{\dot H^{-2}\cap H^{s+2}} + \|u_1\|_{\dot H^{-2}\cap H^s}\bigr) \\
&+ C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}} \left\|f(u)\right\|_{L^1} d\tau \\
&+ C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{2}} \left\|\partial_x^k f(u)\right\|_{L^2} d\tau \\
&+ C\int_0^t e^{-c(t-\tau)} \left\|\partial_x^k f(u)\right\|_{L^2} d\tau \\
\le{}& C(1+t)^{-\frac{k}{4}-\frac{1}{2}}\bigl(\|u_0\|_{\dot H^{-2}\cap H^{s+2}} + \|u_1\|_{\dot H^{-2}\cap H^s}\bigr) \\
&+ C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}} \|u\|_{L^2}^2 d\tau \\
&+ C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{2}} \left\|\partial_x^k u\right\|_{L^2} \|u\|_{L^\infty} d\tau \\
&+ C\int_0^t e^{-c(t-\tau)} \left\|\partial_x^k u\right\|_{L^2} \|u\|_{L^\infty} d\tau \\
\le{}& C(1+t)^{-\frac{k}{4}-\frac{1}{2}}\bigl(\|u_0\|_{\dot H^{-2}\cap H^{s+2}} + \|u_1\|_{\dot H^{-2}\cap H^s}\bigr) \\
&+ CR^2\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}} (1+\tau)^{-1} d\tau \\
&+ CR^2\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{n}{8}-\frac{k}{4}-1} d\tau \\
&+ CR^{\theta+1}\int_0^t e^{-c(t-\tau)} (1+\tau)^{-\frac{n}{8}-\frac{k}{4}-1} d\tau \\
\le{}& C(1+t)^{-\frac{k}{4}-\frac{1}{2}}\bigl\{\bigl(\|u_0\|_{\dot H^{-2}\cap H^{s+2}} + \|u_1\|_{\dot H^{-2}\cap H^s}\bigr) + R^2\bigr\}.
\end{aligned}$$

Thus we get

$$(1+t)^{\frac{k}{4}+\frac{1}{2}} \left\|\partial_x^k \Phi(u)\right\|_{L^2} \le CE_1 + CR^2. \tag{4.4}$$

Applying ∂_t to (3.6), we obtain

$$\Phi(u)_t = G_t(t) * u_1 + H_t(t) * u_0 + \int_0^t G_t(t-\tau) * \Delta f\bigl(u(\tau)\bigr) d\tau. \tag{4.5}$$

By using (2.25), (2.26), (2.28), (3.1), (4.1), for $0 \le h \le s$, we have

$$\begin{aligned}
\left\|\partial_x^h \Phi(u)_t\right\|_{L^2} \le{}& C\left\|\partial_x^h G_t(t) * u_1\right\|_{L^2} + C\left\|\partial_x^h H_t(t) * u_0\right\|_{L^2} \\
&+ C\int_0^t \left\|\partial_x^h G_t(t-\tau) * \Delta f\bigl(u(\tau)\bigr)\right\|_{L^2} d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{h}{4}-\frac{3}{2}}\left(\|u_0\|_{\dot H^{-2}\cap H^{s+2}}+\|u_1\|_{\dot H^{-2}\cap H^{s}}\right)\\
&\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}}\big\|f(u)\big\|_{L^1}\,d\tau\\
&\quad + C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{2}}\big\|\partial_x^h f(u)\big\|_{L^2}\,d\tau\\
&\quad + C\int_0^{t}e^{-c(t-\tau)}\big\|\partial_x^h f(u)\big\|_{L^2}\,d\tau\\
&\le C(1+t)^{-\frac{h}{4}-\frac{3}{2}}\left\{\left(\|u_0\|_{\dot H^{-2}\cap H^{s+2}}+\|u_1\|_{\dot H^{-2}\cap H^{s}}\right)+R^2\right\}.
\end{aligned}
$$

This yields

$$
(1+t)^{\frac{h}{4}+\frac{3}{2}}\big\|\partial_x^h\Phi(u)_t\big\|_{L^2}\le CE_1+CR^2. \tag{4.6}
$$

Combining (4.4) and (4.6) and taking E_1 and R suitably small, we obtain

$$
\big\|\Phi(u)\big\|_X\le R. \tag{4.7}
$$

For $\tilde u,\bar u\in X_R$, by using (3.6), we have

$$
\Phi(\tilde u)-\Phi(\bar u)=\int_0^t G(t-\tau)*\Delta\big[f(\tilde u)-f(\bar u)\big]\,d\tau. \tag{4.8}
$$

It follows from (2.27), (3.2) and (4.1) for $0\le k\le s+2$ that

$$
\begin{aligned}
\big\|\partial_x^k\big(\Phi(\tilde u)-\Phi(\bar u)\big)\big\|_{L^2}&\le\int_0^t\big\|\partial_x^k G(t-\tau)*\Delta\big(f(\tilde u)-f(\bar u)\big)\big\|_{L^2}\,d\tau\\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\big\|f(\tilde u)-f(\bar u)\big\|_{L^1}\,d\tau\\
&\quad + C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\big\|\partial_x^k\big(f(\tilde u)-f(\bar u)\big)\big\|_{L^2}\,d\tau\\
&\quad + C\int_0^{t}e^{-c(t-\tau)}\big\|\partial_x^k\big(f(\tilde u)-f(\bar u)\big)\big\|_{L^2}\,d\tau\\
&\le CR(1+t)^{-\frac{k}{4}-\frac{1}{2}}\|\tilde u-\bar u\|_X,
\end{aligned}
$$

which implies

$$
(1+t)^{\frac{k}{4}+\frac{1}{2}}\big\|\partial_x^k\big(\Phi(\tilde u)-\Phi(\bar u)\big)\big\|_{L^2}\le CR\|\tilde u-\bar u\|_X. \tag{4.9}
$$

Similarly, for $0\le h\le s$, from (4.5), (2.28), (3.2) and (4.1), we infer that

$$
\begin{aligned}
\big\|\partial_x^h\big(\Phi(\tilde u)-\Phi(\bar u)\big)_t\big\|_{L^2}&\le\int_0^t\big\|\partial_x^h G_t(t-\tau)*\Delta\big[f(\tilde u)-f(\bar u)\big]\big\|_{L^2}\,d\tau\\
&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{h}{4}-\frac{3}{2}}\big\|\big(f(\tilde u)-f(\bar u)\big)\big\|_{L^1}\,d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{2}}\left\|\partial_x^h\big(f(\tilde{u})-f(\bar{u})\big)\right\|_{L^2}d\tau \\
&+ C\int_0^t e^{-c(t-\tau)}\left\|\partial_x^h\big(f(\tilde{u})-f(\bar{u})\big)\right\|_{L^2}d\tau \\
\leq{}& CR(1+t)^{-\frac{h}{4}-\frac{3}{2}}\|\tilde{u}-\bar{u}\|_X,
\end{aligned}
$$

which implies

$$
(1+t)^{\frac{h}{4}+\frac{3}{2}}\left\|\partial_x^h\big(\Phi(\tilde{u})-\Phi(\bar{u})\big)_t\right\|_{L^2} \leq CR\|\tilde{u}-\bar{u}\|_X. \tag{4.10}
$$

Using (4.9) and (4.10) and taking R suitably small yields

$$
\left\|\Phi(\tilde{u})-\Phi(\bar{u})\right\|_X \leq \frac{1}{2}\|\tilde{u}-\bar{u}\|_X. \tag{4.11}
$$

It follows from (4.7) and (4.11) that Φ is a strictly contracting mapping. Consequently, we infer that there exists a fixed point $u \in X_R$ of the mapping Φ, which is a solution to (1.1), (1.2). We have completed the proof of the theorem. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final manuscript.

Received: 9 May 2013 Accepted: 27 June 2013 Published: 16 July 2013

References

1. Cahn, JW, Hilliard, JE: Free energy of a nonuniform system, I. Interfacial free energy. J. Chem. Phys. **28**, 258-267 (1958)
2. Galenko, P: Phase-field model with relaxation of the diffusion flux in nonequilibrium solidification of a binary system. Phys. Lett. A **287**, 190-197 (2001)
3. Galenko, P, Jou, D: Diffuse-interface model for rapid phase transformations in nonequilibrium systems. Phys. Rev. E **71**, 046125 (2005)
4. Galenko, P, Lebedev, V: Analysis of the dispersion relation in spinodal decomposition of a binary system. Philos. Mag. Lett. **87**, 821-827 (2007)
5. Galenko, P, Lebedev, V: Local nonequilibrium effect on spinodal decomposition in a binary system. Int. J. Appl. Thermodyn. **11**, 21-28 (2008)
6. Gatti, S, Grasselli, M, Miranville, A, Pata, V: Hyperbolic relaxation of the viscous Cahn-Hilliard equation in 3D. Math. Models Methods Appl. Sci. **15**, 165-198 (2005)
7. Wang, Y-X, Wei, Z: Global existence and asymptotic behavior of solution to Cahn-Hilliard equation with inertial term. Int. J. Math. **23**, Article ID 1250087 (2012)
8. Wang, Y-X: Existence and asymptotic behavior of solutions to the generalized damped Boussinesq equation. Electron. J. Differ. Equ. **2012**(96), 1-11 (2012)
9. Wang, W, Wu, Z: Optimal decay rate of solutions for Cahn-Hilliard equation with inertial term in multi-dimensions. J. Math. Anal. Appl. **387**, 349-358 (2012)
10. Wang, Y: Global existence and asymptotic behaviour of solutions for the generalized Boussinesq equation. Nonlinear Anal. **70**, 465-482 (2009)
11. Wang, Y, Wang, Y: Existence and nonexistence of global solutions for a class of nonlinear wave equations of higher order. Nonlinear Anal. **72**, 4500-4507 (2010)
12. Wang, Y, Wang, Y: Global existence and asymptotic behavior of solutions to a nonlinear wave equation of fourth-order. J. Math. Phys. **53**, Article ID 013512 (2012)
13. Wang, Y, Liu, F, Zhang, Y: Global existence and asymptotic of solutions for a semi-linear wave equation. J. Math. Anal. Appl. **385**, 836-853 (2012)
14. Wang, S, Xu, H: On the asymptotic behavior of solution for the generalized IBq equation with hydrodynamical damped term. J. Differ. Equ. **252**, 4243-4258 (2012)
15. Umeda, T, Kawashima, S, Shizuta, Y: On the decay of solutions to the linearized equations of electro-magneto-fluid dynamics. Jpn. J. Appl. Math. **1**, 435-457 (1984)
16. Li, TT, Chen, YM: Nonlinear Evolution Equations. Science Press, Beijing (1989) (in Chinese)
17. Zheng, SM: Nonlinear Evolution Equations. CRC Press, Boca Raton (2004)

doi:10.1186/1687-2770-2013-168
Cite this article as: Zhuang and Zhang: Global existence and asymptotic behavior of solutions to a class of fourth-order wave equations. *Boundary Value Problems* 2013 **2013**:168.

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Immediate publication on acceptance
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com