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Bifurcation from interval and positive solutions for a class of fourth-order two-point boundary value problem

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Abstract
We consider the fourth-order two-point boundary value problem $x'''' + kx'' + lx = f(t,x)$, $0 < t < 1$, $x(0) = x(1) = x'(0) = x'(1) = 0$, which is not necessarily linearizable. We give conditions on the parameters k, l and $f(t,x)$ that guarantee the existence of positive solutions. The proof of our main result is based upon topological degree theory and global bifurcation techniques.
MSC: 34B15

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1 Introduction

The deformations of an elastic beam in an equilibrium state with fixed both endpoints can be described by the fourth-order boundary value problem

$$\begin{aligned} &x'''' + lx = \lambda h(t)f(x), \quad 0 < t < 1, \\ &x(0) = x(1) = x'(0) = x'(1) = 0, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\lambda \in \mathbb{R}$ is a parameter and l is a given constant. Since problem (1.1) cannot transform into a system of second-order equations, the treatment method of the second-order system does not apply to it. Thus, the existing literature on problem (1.1) is limited. When $l = 0$, the existence of positive solutions of problem (1.1) has been studied by several authors, see [1–5]. Especially, when $l \neq 0$, Xu and Han [6] studied the existence of nodal solutions of problem (1.1) by applying disconjugate operator theory and bifurcation techniques.

Recently, motivated by [6], when k, l satisfy (A1), Shen [7] studied the existence of nodal solutions of a general fourth-order boundary value problem by applying disconjugate operator theory [8, 9] and Rabinowitz's global bifurcation theorem

$$\begin{aligned} &x'''' + kx'' + lx = f(t,x), \quad 0 < t < 1, \\ &x(0) = x(1) = x'(0) = x'(1) = 0, \end{aligned} \tag{1.2}$$

where

(A1) one of following conditions holds:

(i) k, l satisfying $(k,l) \in \{(k,l)|k \in (-\infty,0], l \in (0,\infty)\} \setminus \{(0,\frac{\pi^4}{64})\} \cup \{(k,l)|k \in (-\infty,\pi^2), l \in (-\infty,0]\}$ are given constants with

$$\pi^2(k-\pi^2) < l \le \frac{1}{4}\left(k - \frac{\pi^2}{4}\right)^2; \tag{1.3}$$

(ii) k, l satisfying $(k,l) \in \{(k,l)|k \in (0,\frac{\pi^2}{2}), l \in (0,\infty)\}$ are given constants with

$$\frac{1}{4}\left(\pi^2 k - \frac{\pi^4}{4}\right) < l \le \frac{1}{4}k^2. \tag{1.4}$$

In this paper, we consider bifurcation from interval and positive solutions for problem (1.2). In order to prove our main result, condition (A1) and the following weaker conditions are satisfied throughout this paper:

(H1) $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous and there exist functions $a_0(t)$, $a^0(t)$, $b_\infty(t)$, and $b^\infty(t) \in C([0,1],[0,\infty))$ such that

$$a_0(t)x - \xi_1(t,x) \le f(t,x) \le a^0(t)x + \xi_2(t,x) \tag{1.5}$$

for some functions ξ_1, ξ_2 defined on $[0,1]\times[0,\infty)$ with

$$\xi_1(t,x) = o(x), \qquad \xi_2(t,x) = o(x) \quad \text{as } x \to 0 \tag{1.6}$$

uniformly for $t \in [0,1]$, and

$$b_\infty(t)x - \zeta_1(t,x) \le f(t,x) \le b^\infty(t)x + \zeta_2(t,x) \tag{1.7}$$

for some functions ζ_1, ζ_2 defined on $[0,1]\times[0,\infty)$ with

$$\zeta_1(t,x) = o(x), \qquad \zeta_2(t,x) = o(x) \quad \text{as } x \to \infty \tag{1.8}$$

uniformly for $t \in [0,1]$.

(H2) $f(t,x) > 0$ for $t \in [0,1]$ and $x \in (0,\infty)$.

(H3) There exists a function $c(t) \in C([0,1],[0,\infty))$ with $c(t) \not\equiv 0$ in any subinterval of $[0,1]$ such that

$$f(t,x) \ge c(t)x, \quad (t,x) \in [0,1]\times[0,\infty). \tag{1.9}$$

It is the purpose of this paper to study the existence of positive solutions of (1.2) under conditions (A1), (H1), (H2) and (H3). The main tool we use is the following global bifurcation theorem for the problem which is not necessarily linearizable.

Theorem A (Rabinowitz [10]) *Let V be a real reflexive Banach space. Let $F:\mathbb{R}\times V\to V$ be completely continuous such that $F(\lambda,0)=0$, $\forall\lambda\in\mathbb{R}$. Let $a,b\in\mathbb{R}$ $(a<b)$ be such that $u=0$ is an isolated solution of the following equation*:

$$u - F(\lambda,u) = 0, \quad u \in V \tag{1.10}$$

for $\lambda = a$ *and* $\lambda = b$, *where* $(a,0)$, $(b,0)$ *are not bifurcation points of* (1.10). *Furthermore, assume that*

$$d\big(I - F(a,\cdot), B_r(0), 0\big) \neq d\big(I - F(b,\cdot), B_r(0), 0\big), \tag{1.11}$$

where $B_r(0)$ *is an isolating neighborhood of the trivial solution. Let*

$$\mathcal{S} = \overline{\big\{(\lambda,u) : (\lambda,u) \textit{ is a solution of } (1.10) \textit{ with } u \neq 0\big\}} \cup \big([a,b] \times \{0\}\big).$$

Then there exists a continuum (*i.e., a closed connected set*) $\mathcal{C}$ *of* $\mathcal{S}$ *containing* $[a,b] \times \{0\}$, *and either*

(i) $\mathcal{C}$ *is unbounded in* $V \times \mathbb{R}$, *or*
(ii) $\mathcal{C} \cap [(\mathbb{R} \setminus [a,b]) \times \{0\}] \neq \emptyset$.

Remark 1.1 For other results on the existence and multiplicity of positive solutions and nodal solutions for boundary value problems of fourth-order ordinary differential equations based on bifurcation techniques, see [11–20].

2 Hypotheses and lemmas

Let

$$L[x] := x'''' + kx'' + lx. \tag{2.1}$$

Theorem 2.1 (see [7, Theorem 2.4]) *Let* (A1) *hold. Then*

(i) $L[x] = 0$ *is disconjugate on* $[0,1]$, *and* $L[x]$ *has a factorization*

$$L[x] := \rho_4\big(\rho_3\big(\rho_2\big(\rho_1(\rho_0 x)'\big)'\big)'\big)', \tag{2.2}$$

where $\rho_k \in C^{4-k}[0,1]$ *with* $\rho_k > 0$ $(k = 0,1,2,3,4)$;

(ii) $x(0) = x(1) = x'(0) = x'(1) = 0$ *if and only if*

$$(L_0x)(0) = (L_0x)(1) = (L_1x)(0) = (L_1x)(1) = 0, \tag{2.3}$$

where

$$\begin{aligned} &L_0x = \rho_0 x, \\ &L_ix = \rho_i(L_{i-1}x)', \quad i = 1,2,3,4. \end{aligned} \tag{2.4}$$

Theorem 2.2 (see [7, Theorem 2.7]) *Let* (A1) *hold and* $h \in C([0,1],[0,\infty))$ *with* $h(t) \not\equiv 0$ *on any subinterval of* $[0,1]$. *Then*

(i) *the problem*

$$\begin{cases} x''''(t) + kx''(t) + lx(t) = \lambda h(t)x, & 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) = 0 \end{cases} \tag{2.5}$$

has an infinite sequence of positive eigenvalues

$$0 < \lambda_1(h) < \lambda_2(h) < \cdots < \lambda_k(h) < \lambda_{k+1}(h) < \cdots; \tag{2.6}$$

(ii) $\lambda_k(h) \to +\infty$ *as* $k \to +\infty$;
(iii) *to each eigenvalue* $\lambda_k(h)$, *there corresponds an essential unique eigenfunction* ψ_k *which has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* 0;
(iv) *given an arbitrary subinterval of* $[0,1]$, *an eigenfunction that belongs to a sufficiently large eigenvalue changes its sign in that subinterval*;
(v) *for each* $k \in \mathbb{N}$, *the algebraic multiplicity of* $\lambda_k(h)$ *is* 1.

Theorem 2.3 (see [7, Theorem 2.8]) (Maximum principle) *Let* (A1) *hold*. *Let* $e \in C[0,1]$ *with* $e \geq 0$ *on* $[0,1]$ *and* $e \not\equiv 0$ *in* $[0,1]$. *If* $x \in C^4[0,1]$ *satisfies*

$$\begin{cases} x''''(t) + kx''(t) + lx = e(t), \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) = 0, \end{cases} \tag{2.7}$$

then $x > 0$ *on* $(0,1)$.

Let $Y = C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x|$. Let $E = C^2[0,1]$ with its usual norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$. By a positive solution of (1.2), we mean x is a solution of (1.2) with $x > 0$ (*i.e.*, $x \geq 0$ in $(0,1)$ and $x \not\equiv 0$).

Let $H := L^2(0,1)$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|_{L^2}$. Further, define the linear operator $\widehat{L} : D(\widehat{L}) \subset E \to Y$

$$\widehat{L}x = x'''' + kx'' + lx, \quad x \in D(\widehat{L}) \tag{2.8}$$

with

$$D(\widehat{L}) = \{x \in C^4[0,1] | x(0) = x(1) = x'(0) = x'(1) = 0\}. \tag{2.9}$$

Then $\widehat{L}$ is a closed operator and $\widehat{L}^{-1} : Y \to E$ is completely continuous.

Lemma 2.4 *Let* ψ_1 *be the first eigenfunction of* (2.5). *Then*, *for all* $x \in D(\widehat{L})$, *we get*

$$\langle \widehat{L}x, \psi_1 \rangle = \langle x, \widehat{L}\psi_1 \rangle. \tag{2.10}$$

Proof Obviously, $\forall x \in D(\widehat{L})$, we have

$$\psi_1(0) = \psi_1(1) = \psi_1'(0) = \psi_1'(1) = 0, \qquad x(0) = x(1) = x'(0) = x'(1) = 0.$$

Integrating by parts, we obtain

$$\begin{aligned} \langle \widehat{L}x, \psi_1 \rangle &= \int_0^1 \left[x''''(t) + kx''(t) + lx(t)\right]\psi_1(t)\,dt \\ &= \int_0^1 x(t)\left[\psi_1''''(t) + k\psi_1''(t) + l\psi_1(t)\right]dt = \langle x, \widehat{L}\psi_1 \rangle. \end{aligned}$$

□

Let $\Sigma \subset \mathbb{R}^+ \times E$ be the closure of the set of positive solutions of the problem

$$\widehat{L}x = \lambda f(t,x). \tag{2.11}$$

We extend the function f to a continuous function $\bar{f}$ defined on $[0,1] \times \mathbb{R}$ by

$$\bar{f}(t,x) = \begin{cases} f(t,x), & (t,x) \in [0,1] \times [0,\infty], \\ f(t,0), & (t,x) \in [0,1] \times (-\infty,0]. \end{cases} \tag{2.12}$$

Then $\bar{f}(t,x) \geq 0$ for $(t,x) \in [0,1] \times \mathbb{R}$. For $\lambda \geq 0$, let x be an arbitrary solution of the problem

$$\widehat{L}x = \lambda \bar{f}(t,x). \tag{2.13}$$

Since $\lambda \bar{f}(t,x(t)) \geq 0$ for $t \in [0,1]$, we have $x \geq 0$ for $t \in [0,1]$. Thus x is a nonnegative solution of (2.11), and the closure of the set of nontrivial solutions (λ,x) of (2.13) in $\mathbb{R}^+ \times E$ is exactly Σ.

Let $N : E \to Y$ be the Nemytskii operator associated with the function $\bar{f}$

$$N(x)(t) = \bar{f}(t,x), \quad x \in E. \tag{2.14}$$

Then (2.13), with $\lambda \geq 0$, is equivalent to the operator equation

$$x = \lambda \widehat{L}^{-1} N(x), \quad x \in E. \tag{2.15}$$

In the following, we shall apply the Leray-Schauder degree theory, mainly to the mapping $\Phi_\lambda : E \to E$,

$$\Phi_\lambda(x) = x - \lambda \widehat{L}^{-1} N(x). \tag{2.16}$$

For $R > 0$, let $B_R = \{x \in E : \|x\| \leq R\}$, and let $\deg(\Phi_\lambda, B_R, 0)$ denote the degree of Φ_λ on B_R with respect to 0.

Lemma 2.5 *Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $[\lambda_1(a^0), \lambda_1(a_0)] \cap \Lambda = \emptyset$. Then there exists a number $\delta_1 > 0$ with the property*

$$\Phi_\lambda(x) \neq 0, \quad \forall x \in E : 0 < \|x\| \leq \delta_1, \forall \lambda \in \Lambda. \tag{2.17}$$

Proof Suppose to the contrary that there exist sequences $\{\mu_n\} \subset \Lambda$ and $\{x_n\}$ in $E : \mu_n \to \mu^* \in \Lambda$, $x_n \to 0$ in E, such that $\Phi_{\mu_n}(x_n) = 0$ for all $n \in \mathbb{N}$, then $x_n \geq 0$ in $[0,1]$.

Set $y_n = x_n/\|x_n\|$. Then $Ly_n = \mu_n \|x_n\|^{-1} N(x_n) = \mu_n \|x_n\|^{-1} f(t,x_n)$ and $\|y_n\| = 1$. Now, from condition (H1), we have the following:

$$a_0(t)x_n - \xi_1(t,x_n) \leq f(t,x_n) \leq a^0(t)x_n + \xi_2(t,x_n), \tag{2.18}$$

and, accordingly,

$$\mu_n\left(a_0(t)y_n - \frac{\xi_1(t,x_n)}{\|x_n\|}\right) \leq \mu_n \frac{f(t,x_n)}{\|x_n\|} \leq \mu_n\left(a^0(t)y_n + \frac{\xi_2(t,x_n)}{\|x_n\|}\right). \tag{2.19}$$

Let φ^0 and φ_0 denote the nonnegative eigenfunctions corresponding to $\lambda_1(a^0)$ and $\lambda_1(a_0)$, respectively. Then we have, from the first inequality in (2.19),

$$\left\langle \mu_n \left(a_0(t) y_n - \frac{\xi_1(t, x_n)}{\|x_n\|} \right), \varphi_0 \right\rangle \le \left\langle \mu_n \frac{f(t, x_n)}{\|x_n\|}, \varphi_0 \right\rangle = \langle \widehat{L} y_n, \varphi_0 \rangle. \tag{2.20}$$

From Lemma 2.4, we have

$$\langle \widehat{L} y_n, \varphi_0 \rangle = \langle y_n, \widehat{L} \varphi_0 \rangle = \lambda_1(a_0) \bigl\langle y_n, a_0(t) \varphi_0 \bigr\rangle. \tag{2.21}$$

Since $x_n \to 0$ in E, from (1.6) we have

$$\frac{\xi_1(t, x_n)}{\|x_n\|} \to 0 \quad \text{as } \|x_n\| \to 0. \tag{2.22}$$

By the fact that $\|y_n\| = 1$, we conclude that $y_n \to y$ in E. Thus,

$$\bigl\langle y_n, a_0(t) \varphi_0 \bigr\rangle \to \bigl\langle y, a_0(t) \varphi_0 \bigr\rangle. \tag{2.23}$$

Combining this and (2.21) and letting $n \to \infty$ in (2.20), we get

$$\bigl\langle \mu^* a_0(t) y, \varphi_0 \bigr\rangle \le \lambda_1(a_0) \bigl\langle a_0(t) \varphi_0, y \bigr\rangle, \tag{2.24}$$

and consequently

$$\mu^* \le \lambda_1(a_0). \tag{2.25}$$

Similarly, we deduce from the second inequality in (2.19) that

$$\lambda_1\bigl(a^0\bigr) \le \mu^*. \tag{2.26}$$

Thus, $\lambda_1(a^0) \le \mu^* \le \lambda_1(a_0)$. This contradicts $\mu^* \in \Lambda$. □

Corollary 2.6 *For* $\lambda \in (0, \lambda_1(a^0))$ *and* $\delta \in (0, \delta_1)$, $\deg(\Phi_\lambda, B_\delta, 0) = 1$.

Proof Lemma 2.5, applied to the interval $\Lambda = [0, \lambda]$, guarantees the existence of $\delta_1 > 0$ such that for $\delta \in (0, \delta_1)$,

$$x - \tau \lambda \widehat{L}^{-1} N(x) \ne 0, \quad x \in E : 0 < \|x\| \le \delta, \tau \in [0, 1]. \tag{2.27}$$

Hence, for any $\delta \in (0, \delta_1)$,

$$\deg(\Phi_\lambda, B_\delta, 0) = \deg(I, B_\delta, 0) = 1, \tag{2.28}$$

which ends the proof. □

Lemma 2.7 *Suppose $\lambda > \lambda_1(a_0)$. Then there exists $\delta_2 > 0$ such that $\forall x \in E$ with $0 < \|x\| \le \delta_2$, $\forall \tau \ge 0$,*

$$\Phi_\lambda(x) \ne \tau\varphi_0, \tag{2.29}$$

where φ_0 is the nonnegative eigenfunction corresponding to $\lambda_1(a_0)$.

Proof We assume to the contrary that there exist $\tau_n \ge 0$ and a sequence $\{x_n\}$, with $\|x_n\| > 0$ and $\|x_n\| \to 0$ in E, such that $\Phi_\lambda(x_n) = \tau_n\varphi_0$ for all $n \in \mathbb{N}$. As

$$\widehat{L}x_n = \lambda N(x_n) + \tau_n\lambda_1(a_0)a_0(t)\varphi_0 \tag{2.30}$$

and $\tau_n\lambda_1(a_0)a_0(t)\varphi_0 \ge 0$ in $(0,1)$, it follows that

$$x_n \ge 0, \quad t \in [0,1]. \tag{2.31}$$

Notice that $x_n \in D(\hat{L})$ has a unique decomposition

$$x_n = \omega_n + s_n\varphi_0, \tag{2.32}$$

where $s_n \in \mathbb{R}$ and $\langle \omega_n, a_0(t)\varphi_0\rangle = 0$. Since $x_n \ge 0$ on $[0,1]$ and $\|x_n\| \ge 0$, we have from (2.32) that $s_n > 0$.

Choose $\sigma > 0$ such that

$$\sigma < \frac{\lambda - \lambda_1(a_0)}{\lambda}. \tag{2.33}$$

By (H1), there exists $r_1 > 0$ such that

$$\left|\xi_1(t,x)\right| \le \sigma a_0(t)x, \quad t \in [0,1], x \in [0,r_1]. \tag{2.34}$$

Therefore, for $t \in [0,1]$, $x \in [0,r_1]$,

$$f(t,x) \ge a_0(t)x - \xi_1(t,x) \ge (1-\sigma)a_0(t)x. \tag{2.35}$$

Since $\|x_n\| \to 0$, there exists $N^* > 0$ such that

$$0 \le x_n \le r_1, \quad \forall n \ge N^*, \tag{2.36}$$

and consequently

$$f(t,x_n) \ge (1-\sigma)a_0(t)x_n, \quad \forall n \ge N^*. \tag{2.37}$$

Applying Lemma 2.4 and (2.37), it follows that

$$\begin{aligned} s_n\lambda_1(a_0)\bigl\langle\varphi_0, a_0(t)\varphi_0\bigr\rangle &= \langle x_n, \widehat{L}\varphi_0\rangle = \langle\widehat{L}x_n, \varphi_0\rangle \\ &= \lambda\bigl\langle N(x_n), \varphi_0\bigr\rangle + \tau_n\lambda_1(a_0)\bigl\langle a_0(t)\varphi_0, \varphi_0\bigr\rangle \end{aligned} \tag{2.38}$$

$$\begin{aligned}&\geq \lambda\big\langle N(x_n),\varphi_0\big\rangle \geq \lambda\big\langle(1-\sigma)a_0(t)x_n,\varphi_0\big\rangle\\&= \lambda\big\langle(1-\sigma)a_0(t)\varphi_0,x_n\big\rangle\\&= \lambda(1-\sigma)s_n\big\langle a_0(t)\varphi_0,\varphi_0\big\rangle.\end{aligned}\tag{2.39}$$

Thus,

$$\lambda_1(a_0) \geq \lambda(1-\sigma).\tag{2.40}$$

This contradicts (2.33). □

Corollary 2.8 *For* $\lambda > \lambda_1(a_0)$ *and* $\delta \in (0,\delta_2)$, $\deg(\Phi_\lambda, B_\delta, 0) = 0$.

Proof Let $0 < \delta \leq \delta_2$, where δ_2 is the number asserted in Lemma 2.7. As Φ_λ is bounded in $\bar{B}_\delta$, there exists $c > 0$ such that $\Phi_\lambda(x) \neq c\varphi_0$ for all $x \in \bar{B}_\delta$. By Lemma 2.7, one has

$$\Phi_\lambda(x) \neq \tau c\varphi_0, \quad x \in \partial B_\delta, \tau \in [0,1].\tag{2.41}$$

Hence

$$\deg(\Phi_\lambda, B_\delta, 0) = \deg(\Phi_\lambda - c\varphi_0, B_\delta, 0) = 0.\tag{2.42}$$

□

Now, using Theorem A, we may prove the following.

Proposition 2.9 $[\lambda_1(a^0), \lambda_1(a_0)]$ *is a bifurcation interval from the trivial solution for* (2.15). *There exists an unbounded component C of a positive solution of* (2.15), *which meets* $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$. *Moreover,*

$$C \cap \big[\big(\mathbb{R} \setminus \big[\lambda_1\big(a^0\big), \lambda_1(a_0)\big]\big) \times \{0\}\big] = \emptyset.\tag{2.43}$$

Proof For fixed $n \in \mathbb{N}$ with $\lambda_1(a^0) - \frac{1}{n} > 0$, let us take that $a_n = \lambda_1(a^0) - \frac{1}{n}$, $b_n = \lambda_1(a_0) + \frac{1}{n}$ and $\bar{\delta} = \min\{\delta_1, \delta_2\}$. It is easy to check that for $0 < \delta < \bar{\delta}$, all of the conditions of Theorem A are satisfied. So, there exists a connected component C_n of solutions of (2.15) containing $[a_n, b_n] \times \{0\}$, and either

(i) C_n is unbounded, or

(ii) $C_n \cap [(\mathbb{R} \setminus [a_n, b_n]) \times \{0\}] \neq \emptyset$.

By Lemma 2.5, the case (ii) cannot occur. Thus C_n is unbounded bifurcated from $[a_n, b_n] \times \{0\}$ in $\mathbb{R} \times E$. Furthermore, we have from Lemma 2.5 that for any closed interval $I \subset [a_n, b_n] \setminus [\lambda_1(a^0), \lambda_1(a_0)]$, if $x \in \{x \in E | (\lambda, x) \in \Sigma, \lambda \in I\}$, then $\|x\| \to 0$ in E is impossible. So, C_n must be bifurcated from $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ in $\mathbb{R} \times E$. □

3 Main results

Theorem 3.1 *Let* (A1), (H1), (H2), (H3) *hold. Assume that either*

$$\lambda_1(b_\infty) < 1 < \lambda_1\big(a^0\big)\tag{3.1}$$

or

$$\lambda_1(a_0) < 1 < \lambda_1\big(b^\infty\big), \tag{3.2}$$

then problem (1.2) *has at least one positive solution.*

Proof of Theorem 3.1 It is clear that any solution of (2.15) of the form $(1,x)$ yields a solution x of (1.2). We will show that C crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. To do this, it is enough to show that C joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \{\infty\}$. Let $(\mu_n, x_n) \in C$ satisfy

$$\mu_n + \|x_n\| \to \infty. \tag{3.3}$$

We note that $\mu_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (2.15) for $\lambda = 0$ and $C \cap (\{0\} \times E) = \emptyset$.

Case 1. $\lambda_1(b_\infty) < 1 < \lambda_1(a^0)$.

In this case, we show that

$$\big(\lambda_1(b_\infty), \lambda_1\big(a^0\big)\big) \subseteq \big\{\lambda \in \mathbb{R} | (\lambda, x) \in C\big\}.$$

We divide the proof into two steps.

Step 1. We show that $\{\mu_n\}$ is bounded.

Since $(\mu_n, x_n) \in C$, $Lx_n = \mu_n f(t, x_n)$. From (H3), we have

$$Lx_n \geq \mu_n c(t) x_n. \tag{3.4}$$

Let $\bar{\varphi}$ denote the nonnegative eigenfunction corresponding to $\lambda_1(c)$.

From (3.4), we have

$$\langle Lx_n, \bar{\varphi}\rangle \geq \mu_n \big\langle c(t) x_n, \bar{\varphi}\big\rangle. \tag{3.5}$$

By Lemma 2.4, we have

$$\lambda_1(c)\big\langle x_n, c(t)\bar{\varphi}\big\rangle = \langle x_n, L\bar{\varphi}\rangle \geq \mu_n \big\langle c(t)\bar{\varphi}, x_n\big\rangle. \tag{3.6}$$

Thus

$$\mu_n \leq \lambda_1(c). \tag{3.7}$$

Step 2. We show that C joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \{\infty\}$.

From (3.3) and (3.7), we have that $\|x_n\| \to \infty$. Notice that (2.15) is equivalent to the integral equation

$$x_n(t) = \mu_n \int_0^1 G(t,s) f\big(s, x_n(s)\big)\, ds, \tag{3.8}$$

which implies that

$$\begin{aligned}
&\mu_n \int_0^1 G(t,s)\big[b_\infty(s)x_n(s) - \zeta_1\big(s,x_n(s)\big)\big]\,ds \\
&\quad \le x_n(t) \\
&\quad \le \mu_n \int_0^1 G(t,s)\big[b^\infty(s)x_n(s) + \zeta_2\big(s,x_n(s)\big)\big]\,ds.
\end{aligned} \tag{3.9}$$

We divide both of (3.9) by $\|x_n\|$ and set $y_n = \frac{x_n}{\|x_n\|}$. Since y_n is bounded in E, there exists a subsequence of $\{y_n\}$ and $y^* \in E$, with $y^* \ge 0$ and $y^* \not\equiv 0$ on $(0,1)$, such that

$$\mu_n \to \mu^*, \qquad y_n \to y^* \quad \text{in } E, \tag{3.10}$$

relabeling if necessary. Thus, (3.9) yields that

$$\mu^* \int_0^1 G(t,s)b_\infty(s)y^*(s)\,ds \le y^*(t) \le \mu^* \int_0^1 G(t,s)b^\infty(s)y^*(s)\,ds, \tag{3.11}$$

which implies that

$$\mu^* b_\infty(t)y^* \le Ly^* \le \mu^* b^\infty(t)y^* \le Ly^*. \tag{3.12}$$

Let φ^∞ and φ_∞ denote the nonnegative eigenfunction corresponding to $\lambda_1(b^\infty)$ and $\lambda_1(b_\infty)$, respectively. Then we have, from the first inequality in (3.12),

$$\langle \mu^* b_\infty(t)y^*, \varphi_\infty\rangle \le \langle Ly^*, \varphi_\infty\rangle.$$

From Lemma 2.4, integrating by parts, we obtain that

$$\mu^*\langle b_\infty(t)y^*, \varphi_\infty\rangle \le \langle Ly^*, \varphi_\infty\rangle = \langle L\varphi_\infty, y^*\rangle = \lambda_1(b_\infty)\langle b_\infty(t)\varphi_\infty, y^*\rangle,$$

and consequently

$$\mu^* \le \lambda_1(b_\infty). \tag{3.13}$$

Similarly, we deduce from the second inequality in (3.12) that

$$\lambda_1\big(b^\infty\big) \le \mu^*. \tag{3.14}$$

Thus

$$\lambda_1\big(b^\infty\big) \le \mu^* \le \lambda_1(b_\infty). \tag{3.15}$$

So, C joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \{\infty\}$.

Case 2. $\lambda_1(a_0) < 1 < \lambda_1(b^\infty)$.

In this case, if $(\mu_n, x_n) \in C$ is such that

$$\lim_{n\to\infty}\big(\mu_n + \|x_n\|\big) = \infty$$

and

$$\lim_{n\to\infty} \mu_n = \infty,$$

then

$$\bigl(\lambda_1(a_0), \lambda_1\bigl(b^\infty\bigr)\bigr) \subseteq \bigl\{\lambda \in (0,\infty) | (\lambda, x) \in C\bigr\} \tag{3.16}$$

and, moreover,

$$\bigl(\{1\} \times E\bigr) \cap C \neq \emptyset. \tag{3.17}$$

Assume that $\{\mu_n\}$ is bounded; applying a similar argument to that used in *Step* 2 of *Case* 1, after taking a subsequence and relabeling if necessary, we obtain

$$\mu_n \to \mu^* \in \bigl[\lambda_1(a_0), \lambda_1\bigl(b^\infty\bigr)\bigr], \qquad \|x_n\| \to \infty \quad \text{as } n \to \infty. \tag{3.18}$$

Again C joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \{\infty\}$ and the result follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WS conceived of the study, and participated in its design and coordination and helped to draft the manuscript. TH drafted the manuscript. All authors read and approved the final manuscript.

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