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Existence of a positive solution for quasilinear elliptic equations with nonlinearity including the gradient

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Abstract
We provide the existence of a positive solution for the quasilinear elliptic equation

$$-\operatorname{div}(a(x,|\nabla u|)\nabla u) = f(x,u,\nabla u)$$

in Ω under the Dirichlet boundary condition. As a special case ($a(x,t)=t^{p-2}$), our equation coincides with the usual p-Laplace equation. The solution is established as the limit of a sequence of positive solutions of approximate equations. The positivity of our solution follows from the behavior of $f(x,t\xi)$ as t is small. In this paper, we do not impose the sign condition to the nonlinear term f.

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1 Introduction

In this paper, we consider the existence of a positive solution for the following quasilinear elliptic equation:

$$\begin{cases} -\operatorname{div} A(x,\nabla u) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$. Here, $A\colon \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption (A)). Equation (P) contains the corresponding p-Laplacian problem as a special case. However, in general, we do not suppose that this operator is $(p-1)$-homogeneous in the second variable.

Throughout this paper, we assume that the map A and the nonlinear term f satisfy the following assumptions (A) *and* (f), *respectively*.

(A) $A(x,y) = a(x,|y|)y$, where $a(x,t) > 0$ for all $(x,t) \in \overline{\Omega}\times(0,+\infty)$, and there exist positive constants C_0, C_1, C_2, C_3, $0 < t_0 \le 1$ and $1 < p < \infty$ such that
  - (i) $A \in C^0(\overline{\Omega}\times\mathbb{R}^N,\mathbb{R}^N)\cap C^1(\overline{\Omega}\times(\mathbb{R}^N\setminus\{0\}),\mathbb{R}^N)$;
  - (ii) $|D_yA(x,y)| \le C_1|y|^{p-2}$ for every $x\in\overline{\Omega}$, and $y\in\mathbb{R}^N\setminus\{0\}$;
  - (iii) $D_yA(x,y)\xi\cdot\xi \ge C_0|y|^{p-2}|\xi|^2$ for every $x\in\overline{\Omega}$, $y\in\mathbb{R}^N\setminus\{0\}$ and $\xi\in\mathbb{R}^N$;

(iv) $|D_x A(x,y)| \le C_2(1+|y|^{p-1})$ for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N \setminus \{0\}$;
(v) $|D_x A(x,y)| \le C_3|y|^{p-1}(-\log|y|)$ for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

(f) f is a continuous function on $\Omega \times [0,\infty) \times \mathbb{R}^N$ satisfying $f(x,0,\xi) = 0$ for every $(x,\xi) \in \Omega \times \mathbb{R}^N$ and the following growth condition: there exist $1 < q < p$, $b_1 > 0$ and a continuous function f_0 on $\Omega \times [0,\infty)$ such that

$$-b_1\bigl(1+t^{q-1}\bigr) \le f_0(x,t) \le f(x,t,\xi) \le b_1\bigl(1+t^{q-1}+|\xi|^{q-1}\bigr) \tag{1}$$

for every $(x,t,\xi) \in \Omega \times [0,\infty) \times \mathbb{R}^N$.

In this paper, we say that $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of (P) if

$$\int_\Omega A(x,\nabla u)\nabla\varphi\,dx = \int_\Omega f(x,u,\nabla u)\varphi\,dx$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

A similar hypothesis to (A) is considered in the study of quasilinear elliptic problems (see [1, Example 2.2.], [2–5] and also refer to [6, 7] for the generalized p-Laplace operators). *From now on, we assume that $C_0 \le p-1 \le C_1$*, which is without any loss of generality as can be seen from assumptions (A)(ii), (iii).

In particular, for $A(x,y) = |y|^{p-2}y$, that is, $\operatorname{div} A(x,\nabla u)$ stands for the usual p-Laplacian $\Delta_p u$, we can take $C_0 = C_1 = p-1$ in (A). Conversely, in the case where $C_0 = C_1 = p-1$ holds in (A), by the inequalities in Remark 3(ii) and (iii), we see that $a(x,t) = |t|^{p-2}$ whence $A(x,y) = |y|^{p-2}y$. Hence, our equation contains the p-Laplace equation as a special case.

In the case where f does not depend on the gradient of u, there are many existence results because our equation has the variational structure (*cf.* [1, 4, 8]). Although there are a few results for our equation (P) with f including ∇u, we can refer to [7, 9] and [10] for the existence of a positive solution in the case of the (p,q)-Laplacian or m-Laplacian $(1 < m < N)$. In particular, in [9] and [7], the nonlinear term f is imposed to be nonnegative. The results in [7] and [10] are applied to the m-Laplace equation with an $(m-1)$-superlinear term f w.r.t. u. Here, we mention the result in [9] for the p-Laplacian. Faria, Miyagaki and Motreanu considered the case where f is $(p-1)$-sublinear w.r.t. u and ∇u, and they supposed that $f(x,u,\nabla u) \ge cu^r$ for some $c > 0$ and $0 < r < p-1$. The purpose of this paper is to remove the sign condition and to admit the condition like $f(x,u,\nabla u) \ge \lambda u^{p-1} + o(u^{p-1})$ for large $\lambda > 0$ as $u \to 0+$. Concerning the condition for f as $|u| \to 0$, Zou in [10] imposed that there exists an $L > 0$ satisfying $f(x,u,\nabla u) = Lu^{m-1} + o(|u|^{m-1} + |\nabla u|^{m-1})$ as $|u|, |\nabla u| \to 0$ for the m-Laplace problem. Hence, we cannot apply the result of [10] and [9] to the case of $f(x,u,\nabla u) = \lambda m(x)u^{p-1} + (1-u^{p-1})|\nabla u|^{r-1} + o(u^{p-1})$ as $u \to 0+$ for $1 < r < p$ and $m \in L^\infty(\Omega)$ (admitting sign changes), but we can do our result if $\lambda > 0$ is large.

In [9], the positivity of a solution is proved by the comparison principle. However, since we are not able to do it for our operator in general, after we provide a non-negative and non-trivial solution as a limit of positive approximate solutions (in Section 2), we obtain the positivity of it due to the strong maximum principle for our operator.

1.1 Statements

To state our first result, we define a positive constant A_p by

$$A_p := \frac{C_1}{p-1}\left(\frac{C_1}{C_0}\right)^{p-1} \ge 1, \tag{2}$$

which is equal to 1 in the case of $A(x,y) = |y|^{p-2}y$ (*i.e.*, the case of the p-Laplacian) because we can choose $C_0 = C_1 = p-1$. Then, we introduce the hypothesis (f1) to the function $f_0(x,t)$ in (f) as t is small.

(f1) There exist $m \in L^\infty(\Omega)$ and $b_0 > \mu_1(m)A_p$ such that the Lebesgue measure of $\{x \in \Omega; m(x) > 0\}$ is positive and

$$\liminf_{t\to 0+} \frac{f_0(x,t)}{t^{p-1}} \geq b_0 m(x) \quad \text{uniformly in } x \in \Omega, \tag{3}$$

where f_0 is the continuous function in (f) and $\mu_1(m)$ is the first positive eigenvalue of the p-Laplacian with the weight function m obtained by

$$\mu_1(m) := \inf\left\{\int_\Omega |\nabla u|^p\,dx; u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega m|u|^p\,dx = 1\right\}. \tag{4}$$

Theorem 1 *Assume* (f1). *Then equation* (P) *has a positive solution* $u \in \operatorname{int} P$, *where*

$$P := \{u \in C_0^1(\overline{\Omega}); u(x) \geq 0 \textit{ in } \Omega\},$$
$$\operatorname{int} P := \{u \in C_0^1(\overline{\Omega}); u(x) > 0 \textit{ in } \Omega \textit{ and } \partial u/\partial\nu < 0 \textit{ on } \partial\Omega\},$$

and ν *denotes the outward unit normal vector on* $\partial\Omega$.

Next, we consider the case where A is asymptotically $(p-1)$-homogeneous near zero in the following sense:

(AH0) There exist a positive function $a_0 \in C(\overline{\Omega},(0,+\infty))$ and $\widetilde{a}_0(x,t) \in C(\overline{\Omega}\times[0,+\infty),\mathbb{R})$ such that

$$A(x,y) = a_0(x)|y|^{p-2}y + \widetilde{a}_0(x,|y|)y \quad \text{for every } x \in \Omega, y \in \mathbb{R}^N \quad \text{and} \tag{5}$$

$$\lim_{t\to 0+} \frac{\widetilde{a}_0(x,t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}. \tag{6}$$

Under (AH0), we can replace the hypothesis (f1) with the following (f2):

(f2) There exist $m \in L^\infty(\Omega)$ and $b_0 > \lambda_1(m)$ such that (3) and the Lebesgue measure of $\{x \in \Omega; m(x) > 0\}$ is positive, where $\lambda_1(m)$ is the first positive eigenvalue of $-\operatorname{div}(a_0(x)|\nabla u|^{p-2}\nabla u)$ with a weight function m obtained by

$$\lambda_1(m) := \inf\left\{\int_\Omega a_0(x)|\nabla u|^p\,dx; u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega m|u|^p\,dx = 1\right\}. \tag{7}$$

Theorem 2 *Assume* (AH0) *and* (f2). *Then equation* (P) *has a positive solution* $u \in \operatorname{int} P$.

Throughout this paper, we may assume that $f(x,t,\xi) = 0$ for every $t \leq 0$, $x \in \Omega$ and $\xi \in \mathbb{R}^N$ because we consider the existence of a positive solution only. In what follows, the norm on $W_0^{1,p}(\Omega)$ is given by $\|u\| := \|\nabla u\|_p$, where $\|u\|_q$ denotes the usual norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ $(1 \leq q \leq \infty)$. Moreover, we denote $u_\pm := \max\{\pm u, 0\}$.

1.2 Properties of the map A

Remark 3 The following assertions hold under condition (A):

(i) for all $x \in \overline{\Omega}$, $A(x,y)$ is maximal monotone and strictly monotone in y;
(ii) $|A(x,y)| \leq \frac{C_1}{p-1}|y|^{p-1}$ for every $(x,y) \in \overline{\Omega}\times\mathbb{R}^N$;

(iii) $A(x,y)y \geq \frac{C_0}{p-1}|y|^p$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$,
where C_0 and C_1 are the positive constants in (A).

Proposition 4 ([3, Proposition 1]) *Let* $A\colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ *be a map defined by*

$$\langle A(u), v\rangle = \int_\Omega A(x, \nabla u)\nabla v\, dx$$

for $u, v \in W_0^{1,p}(\Omega)$. *Then* A *is maximal monotone, strictly monotone and has* $(S)_+$ *property, that is, any sequence* $\{u_n\}$ *weakly convergent to* u *with* $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$ *strongly converges to* u.

2 Constructing approximate solutions

Choose a function $\psi \in P \setminus \{0\}$. In this section, for such ψ and $\varepsilon > 0$, we consider the following elliptic equation:

$$\begin{cases} -\operatorname{div} A(x, \nabla u) = f(x, u, \nabla u) + \varepsilon\psi(x) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{$P;\varepsilon$}$$

In [7], the case $\psi \equiv 1$ in the above equation is considered.

Lemma 5 *Suppose* (f1) *or* (f2). *Then there exists* $\lambda_0 > 0$ *such that* $f(x,t,\xi)t + \lambda_0 t^p \geq 0$ *for every* $x \in \Omega$, $t \geq 0$ *and* $\xi \in \mathbb{R}^N$.

Proof From the growth condition of f_0 and (3), it follows that

$$f_0(x,t)t \geq -b_0\|m\|_\infty t^p - b_1' t^p \quad \text{for every } (x,t) \in \Omega \times [0,\infty)$$

holds, where b_1' is a positive constant independent of (x,t). Therefore, for $\lambda_0 \geq b_0\|m\|_\infty + b_1'$, we easily see that $f(x,t,\xi)t + \lambda_0 t^p \geq f_0(x,t)t + \lambda_0 t^p \geq 0$ for every $x \in \Omega$, $t \geq 0$ and $\xi \in \mathbb{R}^N$ holds. □

Proposition 6 *If* $u_\varepsilon \in W_0^{1,p}(\Omega)$ *is a non-negative solution of* $(P;\varepsilon)$ *for* $\varepsilon \geq 0$, *then* $u_\varepsilon \in L^\infty(\Omega)$. *Moreover, for any* $\varepsilon_0 > 0$, *there exists a positive constant* $D > 0$ *such that* $\|u_\varepsilon\|_\infty \leq D\max\{1, \|u_\varepsilon\|\}$ *holds for every* $\varepsilon \in [0, \varepsilon_0]$.

Proof Set $\overline{p}^* = Np/(N-p)$ if $N > p$, and in the case of $N \leq p$, $\overline{p}^* > p$ is an arbitrarily fixed constant. Let u_ε be a non-negative solution of $(P;\varepsilon)$ with $0 \leq \varepsilon \leq \varepsilon_0$ (some $\varepsilon_0 > 0$). For $r > 0$, choose a smooth increasing function $\eta(t)$ such that $\eta(t) = t^{r+1}$ if $0 \leq t \leq 1$, $\eta(t) = d_0 t$ if $t \geq d_1$ and $\eta'(t) \geq d_2 > 0$ if $1 \leq t \leq d_1$ for some $0 < d_2 < 1 < d_0, d_1$. Define $\xi_M(u) := M^{r+1}\eta(u/M)$ for $M > 1$.

If $u_\varepsilon \in L^{r+p}(\Omega)$, then by taking $\xi_M(u_\varepsilon)$ as a test function (note that η' is bounded), we have

$$\begin{aligned} &\frac{C_0}{p-1}\int_\Omega |\nabla u_\varepsilon|^p \xi_M'(u_\varepsilon)\, dx \\ &\quad \leq \int_\Omega A(x, \nabla u_\varepsilon)\nabla u_\varepsilon \xi_M'(u_\varepsilon)\, dx \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \bigl(f(x,u_\varepsilon,\nabla u_\varepsilon) + \varepsilon\psi\bigr)\xi_M(u_\varepsilon)\,dx \\
&\le b_1 \int_\Omega \bigl(1 + u_\varepsilon^{q-1} + \varepsilon_0\|\psi\|_\infty\bigr) M^{r+1}\eta(u_\varepsilon/M)\,dx + b_1\int_\Omega |\nabla u_\varepsilon|^{q-1}\xi_M(u_\varepsilon)\,dx \\
&\le d_0 d_1\bigl(2b_1 + \varepsilon_0\|\psi\|_\infty\bigr)\bigl(\|u_\varepsilon\|_{r+q}^{r+q} + \|u_\varepsilon\|_{r+1}^{r+1}\bigr) + b_1\int_\Omega |\nabla u_\varepsilon|^{q-1}\xi_M(u_\varepsilon)\,dx \qquad (8)
\end{aligned}
$$

due to Remark 3(iii) and $M^{r+1}\eta(t/M) \le d_0 d_1 t^{r+1}$. Putting $\beta := p/(p-q+1) < p$, we see that $(\xi_M(u_\varepsilon))/(\xi_M'(u_\varepsilon))^{(q-1)/p} = u_\varepsilon^{r+1}/((r+1)u_\varepsilon^r)^{(q-1)/p} \le u_\varepsilon^{1+r/\beta}$ provided $0 < u_\varepsilon < M$ (note $r > 0$). Similarly, if $M \le u_\varepsilon \le d_1 M$, then $(\xi_M(u_\varepsilon))/(\xi_M'(u_\varepsilon))^{(q-1)/p} \le d_0 d_1 M^{r+1}/(d_2 M^r)^{(q-1)/p} = d_0 d_1 d_2^{(1-q)/p} M^{1+r/\beta} \le d_0 d_1 d_2^{(1-q)/p} u_\varepsilon^{1+r/\beta}$, and if $u_\varepsilon > d_1 M$, then $(\xi_M(u_\varepsilon))/(\xi_M'(u_\varepsilon))^{(q-1)/p} = d_0^{1/\beta} M^{r/\beta} u_\varepsilon \le d_0^{1/\beta} u_\varepsilon^{1+r/\beta}$ (note $d_1 > 1$). Thus, according to Young's inequality, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$
\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^{q-1}\xi_M(u_\varepsilon)\,dx &\le \delta\int_\Omega |\nabla u_\varepsilon|^p \xi_M'(u_\varepsilon)\,dx + C_\delta \int_{u_\varepsilon>0} \frac{(\xi_M(u_\varepsilon))^\beta}{(\xi_M'(u_\varepsilon))^{(q-1)\beta/p}}\,dx \\
&\le \delta\int_\Omega |\nabla u_\varepsilon|^p \xi_M'(u_\varepsilon)\,dx + C_\delta d_3 \int_\Omega u_\varepsilon^{r+\beta}\,dx, \qquad (9)
\end{aligned}
$$

where $\beta := p/(p-q+1) < p$ and $d_3 = \max\{d_0 d_1 d_2^{(1-q)/p}, d_0^{1/\beta}\}$ (>1). As a result, because of $r+p > r+q, r+\beta$, according to Hölder's inequality and the monotonicity of t^r with respect to r on $[1,\infty)$, taking a $0 < \delta < C_0/b_1(p-1)$ and setting $u_\varepsilon^M(x) := \min\{u_\varepsilon(x), M\}$, we obtain

$$
\begin{aligned}
b_4 (r')^p \max\bigl\{1, \|u_\varepsilon\|_{r+p}^{r+p}\bigr\} &\ge (r')^p \int_\Omega |\nabla u_\varepsilon|^p \xi_M'(u_\varepsilon)\,dx \ge (r')^p \int_\Omega \bigl|\nabla u_\varepsilon^M\bigr|^p \bigl(u_\varepsilon^M\bigr)^r\,dx \\
&= \bigl\|\bigl(u_\varepsilon^M\bigr)^{r'}\bigr\|^p \ge C_*\bigl\|\bigl(u_\varepsilon^M\bigr)^{r'}\bigr\|_{\overline{p}^*}^p = C_*\bigl\|u_\varepsilon^M\bigr\|_{\overline{p}^* r'}^{r+p} \qquad (10)
\end{aligned}
$$

provided $u_\varepsilon \in L^{r+p}(\Omega)$ by (8) and (9), where $r' = 1 + r/p$, C_* comes from the continuous embedding of $W_0^{1,p}(\Omega)$ into $L^{\overline{p}^*}(\Omega)$ and d_4 is a positive constant independent of u_ε, ε and r. Consequently, Moser's iteration process implies our conclusion. In fact, we define a sequence $\{r_m\}_m$ by $r_0 := \overline{p}^* - p$ and $r_{m+1} := \overline{p}^*(p + r_m)/p - p$. Then, we see that $u_\varepsilon \in L^{\overline{p}^*(p+r_m)/p}(\Omega) = L^{p+r_{m+1}}(\Omega)$ holds if $u_\varepsilon \in L^{p+r_m}(\Omega)$ by applying Fatou's lemma to (10) and letting $M \to \infty$. Here, we also see $r_{m+1} = \overline{p}^* r_m/p + \overline{p}^* - p \ge (\overline{p}^*/p)^{m+1} r_0 \to \infty$ as $m \to \infty$. Therefore, by the same argument as in Theorem C in [4], we can obtain $u_\varepsilon \in L^\infty(\Omega)$ and $\|u_\varepsilon\|_\infty \le D\max\{1, \|u_\varepsilon\|\}$ for some positive constant D independent of u_ε and ε. □

Lemma 7 *Suppose* (f1) *or* (f2). *If* $u_\varepsilon \in W_0^{1,p}(\Omega)$ *is a solution of* $(P;\varepsilon)$ *for* $\varepsilon > 0$, *then* $u_\varepsilon \in \operatorname{int} P$.

Proof Taking $-(u_\varepsilon)_-$ as a test function in $(P;\varepsilon)$, we have

$$
\frac{C_0}{p-1}\bigl\|\nabla(u_\varepsilon)_-\bigr\|_p^p \le \int_\Omega A(x,\nabla u_\varepsilon)\bigl(-\nabla(u_\varepsilon)_-\bigr)\,dx = -\varepsilon\int_\Omega \psi(u_\varepsilon)_-\,dx \le 0
$$

because of $f(x,t,\xi) = 0$ if $t \le 0$ and by Remark 3(iii). Hence, $u_\varepsilon \ge 0$ follows. Because Proposition 6 guarantees that $u_\varepsilon \in L^\infty(\Omega)$, we have $u_\varepsilon \in C_0^{1,\alpha}(\overline{\Omega})$ (for some $0 < \alpha < 1$) by the regularity result in [11]. Note that $u_\varepsilon \not\equiv 0$ because of $\varepsilon > 0$ and $\psi \not\equiv 0$. In addition, Lemma 5 implies the existence of $\lambda_0 > 0$ such that $-\operatorname{div} A(x,\nabla u_\varepsilon) + \lambda_0 u_\varepsilon^{p-1} \ge 0$ in the distribution sense.

Therefore, according to Theorem A and Theorem B in [4], $u_\varepsilon > 0$ in Ω and $\partial u_\varepsilon/\partial\nu < 0$ on $\partial\Omega$, namely, $u_\varepsilon \in \operatorname{int} P$. □

The following result can be shown by the same argument as in [9, Theorem 3.1].

Proposition 8 *Suppose* (f1) *or* (f2). *Then, for every* $\varepsilon > 0$, $(P;\varepsilon)$ *has a positive solution* $u_\varepsilon \in \operatorname{int} P$.

Proof Fix any $\varepsilon > 0$ and let $\{e_1,\dots,e_m,\dots\}$ be a Schauder basis of $W_0^{1,p}(\Omega)$ (refer to [12] for the existence). For each $m \in \mathbb{N}$, we define the m-dimensional subspace V_m of $W_0^{1,p}(\Omega)$ by $V_m := \text{lin.sp.}\{e_1,\dots,e_m\}$. Moreover, set a linear isomorphism $T_m\colon \mathbb{R}^m \to V_m$ by $T_m(\xi_1,\dots,\xi_m) := \sum_{i=1}^m \xi_i e_i \in V_m$, and let $T_m^* : V_m^* \to (\mathbb{R}^m)^*$ be a dual map of T_m. By identifying $\mathbb{R}^m$ and $(\mathbb{R}^m)^*$, we may consider that T_m^* maps from V_m^* to $\mathbb{R}^m$. Define maps A_m and B_m from V_m to V_m^* as follows:

$$\langle A_m(u), v\rangle := \int_\Omega A(x,\nabla u)\nabla v\,dx \quad\text{and}\quad \langle B_m(u), v\rangle := \int_\Omega f(x,u,\nabla u)v\,dx + \varepsilon\int_\Omega \psi v\,dx$$

for u, $v \in V_m$. We claim that for every $m \in \mathbb{N}$, there exists $u_m \in V_m$ such that $A_m(u_m) - B_m(u_m) = 0$ in V_m^*. Indeed, by the growth condition of f, Remark 3(iii) and Hölder's inequality, we easily have

$$\begin{aligned}&\langle A_m(u) - B_m(u), u\rangle\\ &\quad\geq \frac{C_0}{p-1}\|u\|^p - b_1\bigl(\|u\|_1 + \|u\|_q^q + \|\nabla u\|_p^{q-1}\|u\|_\beta\bigr) - \varepsilon\|\psi\|_\infty\|u\|_1 \qquad (11)\end{aligned}$$

for every $u \in V_m$, where $\beta = p/(p-q+1) < p$. This implies that $A_m - B_m$ is coercive on V_m by $q < p$. Set a homotopy $H_m(t,y) := ty + (1-t)T_m^*(A_m(T_m(y)) - B_m(T_m(y)))$ for $t \in [0,1]$ and $y \in \mathbb{R}^m$. By recalling that $A_m - B_m$ is coercive on V_m, we see that there exists an $R > 0$ such that $(H_m(t,y),y) > 0$ for every $t \in [0,1]$ and $|y| \geq R$ because $\|\cdot\|$ and the norm of $\mathbb{R}^m$ are equivalent on V_m. Therefore, we have

$$\begin{aligned}1 &= \deg\bigl(I_m, B_R(0), 0\bigr) = \deg\bigl(H_m(1,\cdot), B_R(0), 0\bigr)\\ &= \deg\bigl(H_m(0,\cdot), B_R(0), 0\bigr) = \deg\bigl(T_m^* \circ (A_m - B_m)\circ T_m, B_R(0), 0\bigr),\end{aligned}$$

where I_m is the identity map on $\mathbb{R}^m$, $B_R(0) := \{y \in \mathbb{R}^m; |y| < R\}$ and $\deg(g,B,0)$ denotes the degree on $\mathbb{R}^m$ for a continuous map $g : B \to \mathbb{R}^m$ (*cf.* [13]). Hence, this yields the existence of $y_m \in \mathbb{R}^m$ such that $(T_m^* \circ (A_m - B_m)\circ T_m)(y_m) = 0$, and so the desired u_m is obtained by setting $u_m = T_m(y_m) \in V_m$ since T_m^* is injective.

Because (11) with $u = u_m \in W_0^{1,p}(\Omega)$ leads to the boundedness of $\|u_m\|$ by $q < p$, we may assume, by choosing a subsequence, that u_m converges to some u_0 weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Let P_m be a natural projection onto V_m, that is, $P_m u = \sum_{i=1}^m \xi_i e_i$ for $u = \sum_{i=1}^\infty \xi_i e_i$. Since $u_m, P_m u_0 \in V_m$ and $A_m(u_m) - B_m(u_m) = 0$ in V_m^*, by noting that $A_m = A$ on V_m for a map A defined in Proposition 4, we obtain

$$\begin{aligned}&\langle A(u_m), u_m - u_0\rangle + \langle A(u_m), u_0 - P_m u_0\rangle\\ &\quad= \langle A_m(u_m), u_m - P_m u_0\rangle = \langle B_m(u_m), u_m - P_m u_0\rangle\end{aligned}$$

$$= \int_\Omega \bigl(f(x, u_m, \nabla u_m) + \varepsilon\psi\bigr)(u_m - u_0)\,dx$$
$$+ \int_\Omega \bigl(f(x, u_m, \nabla u_m) + \varepsilon\psi\bigr)(u_0 - P_m u_0)\,dx \to 0$$

as $m \to \infty$, where we use the boundedness of $\|u_m\|$, the growth condition of f and $u_m \to u_0$ in $L^p(\Omega)$. In addition, since $\|A(u_m)\|_{W_0^{1,p}(\Omega)^*}$ is bounded, by the boundedness of $\|u_m\|$, we see that $\langle A(u_m), u_0 - P_m u_0\rangle \to 0$ as $m \to \infty$, whence $\langle A(u_m), u_m - u_0\rangle \to 0$ as $m \to \infty$ holds. As a result, it follows from the $(S)_+$ property of A that $u_m \to u_0$ in $W_0^{1,p}(\Omega)$ as $m \to \infty$.

Finally, we shall prove that u_0 is a solution of $(P;\varepsilon)$. Fix any $l \in \mathbb{N}$ and $\varphi \in V_l$. For each $m \geq l$, by letting $m \to \infty$ in $\langle A_m(u_m), \varphi\rangle = \langle B_m(u_m), \varphi\rangle$, we have

$$\int_\Omega A(x, \nabla u_0)\nabla\varphi\,dx = \int_\Omega f(x, u_0, \nabla u_0)\varphi\,dx + \varepsilon\int_\Omega \psi\varphi\,dx. \tag{12}$$

Since l is arbitrary, (12) holds for every $\varphi \in \bigcup_{l\geq 1} V_l$. Moreover, the density of $\bigcup_{l\geq 1} V_l$ in $W_0^{1,p}(\Omega)$ guarantees that (12) holds for every $\varphi \in W_0^{1,p}(\Omega)$. This means that u_0 is a solution of $(P;\varepsilon)$. Consequently, our conclusion $u_0 \in \operatorname{int} P$ follows from Lemma 7. □

3 Proof of theorems

Lemma 9 *Let $\varphi, u \in \operatorname{int} P$. Then*

$$\int_\Omega A(x, \nabla u)\nabla\left(\frac{\varphi^p}{u^{p-1}}\right)dx \leq A_p\|\nabla\varphi\|_p^p$$

holds, where A_p is the positive constant defined by (2).

Proof Because of $\varphi, u \in \operatorname{int} P$, there exist $\delta_1 > \delta_2 > 0$ such that $\delta_1 u \geq \varphi \geq \delta_2 u$ in $\overline{\Omega}$. Thus, $\delta_1 \geq \varphi/u \geq \delta_2$ and $1/\delta_2 \geq u/\varphi \geq 1/\delta_1$ in Ω. Hence, $u/\varphi, \varphi/u \in L^\infty(\Omega)$ hold. Therefore, we have

$$\begin{aligned}
A(x, \nabla u)\nabla\left(\frac{\varphi^p}{u^{p-1}}\right) &= p\left(\frac{\varphi}{u}\right)^{p-1} A(x, \nabla u)\nabla\varphi - (p-1)\left(\frac{\varphi}{u}\right)^p A(x, \nabla u)\nabla u \\
&\leq \frac{pC_1}{p-1}\left(\frac{\varphi}{u}\right)^{p-1}|\nabla u|^{p-1}|\nabla\varphi| - C_0\left(\frac{\varphi}{u}\right)^p|\nabla u|^p \\
&= \left\{\left(\frac{pC_0}{p-1}\right)^{1/p}\frac{\varphi}{u}|\nabla u|\right\}^{p-1}\left(\frac{p}{p-1}\right)^{1/p} C_1 C_0^{(1-p)/p}|\nabla\varphi| \\
&\quad - C_0\left(\frac{\varphi}{u}\right)^p|\nabla u|^p \leq A_p|\nabla\varphi|^p
\end{aligned} \tag{13}$$

in Ω by (ii) and (iii) in Remark 3 and Young's inequality. □

Lemma 10 *Assume that $a_0 \in C(\overline{\Omega}, [0,\infty))$ and let $\varphi, u \in \operatorname{int} P$. Then*

$$\int_\Omega a_0(x)|\nabla\varphi|^{p-2}\nabla\varphi\nabla\left(\frac{\varphi^p - u^p}{\varphi^{p-1}}\right)dx - \int_\Omega a_0(x)|\nabla u|^{p-2}\nabla u\nabla\left(\frac{\varphi^p - u^p}{u^{p-1}}\right)dx \geq 0$$

holds.

Proof First, we note that $u/\varphi, \varphi/u \in L^\infty(\Omega)$ hold by the same reason as in Lemma 9. Applying Young's inequality to the second term of the right-hand side in (14) (refer to (13) with $C_0 = C_1 = p-1$), we obtain

$$\begin{aligned} a_0(x)|\nabla\varphi|^{p-2}\nabla\varphi\nabla\left(\frac{\varphi^p - u^p}{\varphi^{p-1}}\right) & \\ \geq a_0(x)\left(|\nabla\varphi|^p - p\left(\frac{u}{\varphi}\right)^{p-1}|\nabla\varphi|^{p-1}|\nabla u| + (p-1)\left(\frac{u}{\varphi}\right)^p|\nabla\varphi|^p\right) & \qquad (14) \\ \geq a_0(x)\left(|\nabla\varphi|^p - |\nabla u|^p\right) & \qquad (15) \end{aligned}$$

in Ω. Similarly, we also have

$$a_0(x)|\nabla u|^{p-2}\nabla u\nabla\left(\frac{\varphi^p - u^p}{u^{p-1}}\right) \leq a_0(x)\left(|\nabla\varphi|^p - |\nabla u|^p\right) \quad \text{in } \Omega. \qquad (16)$$

The conclusion follows from (15) and (16). □

Under (f1) or (f2), we denote a solution $u_\varepsilon \in \operatorname{int} P$ of $(P;\varepsilon)$ for each $\varepsilon > 0$ obtained by Proposition 8.

Lemma 11 *Assume* (f1) *or* (f2). *Let* $I := (0,1]$. *Then* $\{u_\varepsilon\}_{\varepsilon\in I}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Proof Taking u_ε as a test function in $(P;\varepsilon)$, we have

$$\begin{aligned} \frac{C_0}{p-1}\|\nabla u_\varepsilon\|_p^p &\leq \int_\Omega A(x,\nabla u_\varepsilon)\nabla u_\varepsilon\,dx = \int_\Omega f(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\,dx + \varepsilon\int_\Omega \psi u_\varepsilon\,dx \\ &\leq b_1\left(\|u_\varepsilon\|_1 + \|u_\varepsilon\|_q^q + \|\nabla u_\varepsilon\|_p^{q-1}\|u_\varepsilon\|_\beta\right) + \|\psi\|_\infty\|u_\varepsilon\|_1 \\ &\leq b_1'\left(\|u_\varepsilon\| + \|u_\varepsilon\|^q\right) \end{aligned}$$

by Remark 3(iii), the growth condition of f, Hölder's inequality and the continuity of the embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, where $\beta = p/(p-q+1)$ $(< p)$ and b_1' is a positive constant independent of u_ε. Because of $q < p$, this yields the boundedness of $\|u_\varepsilon\|$ $(= \|\nabla u_\varepsilon\|_p)$. □

Lemma 12 *Assume* (f1) *or* (f2). *Then* $|\nabla u_\varepsilon|/u_\varepsilon \in L^p(\Omega)$ *and* $\||\nabla u_\varepsilon|/u_\varepsilon\|_p^p \leq \lambda_0|\Omega|/C_0$ *hold for every* $\varepsilon > 0$, *where* $|\Omega|$ *denotes the Lebesgue measure of* Ω, *and where* C_0 *and* λ_0 *are positive constants as in* (A) *and Lemma* 5, *respectively.*

Proof Fix any $\varepsilon > 0$ and choose any $\rho > 0$. By taking $(u_\varepsilon + \rho)^{1-p}$ as a test function, we obtain

$$\begin{aligned} (1-p)\int_\Omega \frac{A(x,\nabla u_\varepsilon)\nabla u_\varepsilon}{(u_\varepsilon+\rho)^p}\,dx &= \int_\Omega \frac{f(x,u_\varepsilon,\nabla u_\varepsilon) + \varepsilon\psi}{(u_\varepsilon+\rho)^{p-1}}\,dx \geq -\lambda_0\int_\Omega \frac{u_\varepsilon^{p-1}}{(u_\varepsilon+\rho)^{p-1}}\,dx \\ &\geq -\lambda_0|\Omega|, \end{aligned} \qquad (17)$$

by Lemma 5 and $\varepsilon\psi \geq 0$. On the other hand, by Remark 3(iii) and $1-p < 0$, we have

$$(1-p)\int_\Omega \frac{A(x,\nabla u_\varepsilon)\nabla u_\varepsilon}{(u_\varepsilon+\rho)^p}\,dx \leq -C_0\int_\Omega \frac{|\nabla u_\varepsilon|^p}{(u_\varepsilon+\rho)^p}\,dx. \qquad (18)$$

Therefore, (17) and (18) imply the inequality $\int_\Omega |\nabla u_\varepsilon|^p/(u_\varepsilon+\rho)^p\,dx \le \lambda_0|\Omega|/C_0$ for every $\rho>0$. As a result, by letting $\rho\to 0+$, our conclusion is shown. □

Lemma 13 *Assume* (f2) *and* (AH0). *Let* $\varphi\in\operatorname{int}P$. *If* $u_\varepsilon\to 0$ *in* $C_0^1(\overline{\Omega})$ *as* $\varepsilon\to 0+$, *then*

$$\lim_{\varepsilon\to 0+}\left|\int_\Omega \widetilde{a}_0\big(x,|\nabla u_\varepsilon|\big)\nabla u_\varepsilon\nabla\left(\frac{\varphi^p-u_\varepsilon^p}{u_\varepsilon^{p-1}}\right)dx\right| = 0$$

holds, where $\widetilde{a}_0$ *is a continuous function as in* (AH0).

Proof Note that $u_\varepsilon/\varphi, \varphi/u_\varepsilon\in L^\infty(\Omega)$ hold (as in the proof of Lemma 9). Because we easily see that $|\int_\Omega \widetilde{a}_0(x,|\nabla u|)|\nabla u|^2\,dx|\le C\|\nabla u\|_p^p$ for every $u\in W_0^{1,p}(\Omega)$ with some $C>0$ independent of u (see (6)), it is sufficient to show $|\int_\Omega \widetilde{a}_0(x,|\nabla u_\varepsilon|)\nabla u_\varepsilon\nabla(\varphi^p/u_\varepsilon^{p-1})\,dx|\to 0$ as $\varepsilon\to 0+$. Here, we fix any $\delta>0$. By the property of $\widetilde{a}_0$ (see (6)) and because we are assuming that $u_\varepsilon\to 0$ in $C_0^1(\overline{\Omega})$ as $\varepsilon\to 0+$, we have $|\widetilde{a}_0(x,|\nabla u_\varepsilon|)|\le\delta|\nabla u_\varepsilon|^{p-2}$ for every $x\in\Omega$ provided sufficiently small $\varepsilon>0$. Therefore, for such sufficiently small $\varepsilon>0$, we obtain

$$\begin{aligned}
&\left|\int_\Omega \widetilde{a}_0\big(x,|\nabla u_\varepsilon|\big)\nabla u_\varepsilon\nabla\left(\frac{\varphi^p}{u_\varepsilon^{p-1}}\right)dx\right|\\
&\quad\le p\int_\Omega\frac{|\widetilde{a}_0(x,|\nabla u_\varepsilon|)||\nabla u_\varepsilon||\nabla\varphi|\varphi^{p-1}}{u_\varepsilon^{p-1}}\,dx+(p-1)\int_\Omega\frac{|\widetilde{a}_0(x,|\nabla u_\varepsilon|)||\nabla u_\varepsilon|^2\varphi^p}{u_\varepsilon^p}\,dx\\
&\quad\le\delta\|\varphi\|^p_{C_0^1(\overline{\Omega})}\left\{p\int_\Omega\left(\frac{|\nabla u_\varepsilon|}{u_\varepsilon}\right)^{p-1}dx+(p-1)\int_\Omega\left(\frac{|\nabla u_\varepsilon|}{u_\varepsilon}\right)^p dx\right\}\\
&\quad\le\delta\|\varphi\|^p_{C_0^1(\overline{\Omega})}|\Omega|\big(p(\lambda_0/C_0)^{1-1/p}+(p-1)(\lambda_0/C_0)\big)
\end{aligned}$$

because of $|\nabla u_\varepsilon|/u_\varepsilon\in L^p(\Omega)$ by Lemma 12. Since $\delta>0$ is arbitrary, our conclusion is shown. □

3.1 Proof of main results

Proof of Theorems

Let $\varepsilon\in(0,1]$. Due to Proposition 6 and Lemma 11, we have $\|u_\varepsilon\|_\infty\le M$ for some $M>0$ independent of $\varepsilon\in(0,1]$. Hence, there exist $M'>0$ and $0<\alpha<1$ such that $u_\varepsilon\in C_0^{1,\alpha}(\overline{\Omega})$ and $\|u_\varepsilon\|_{C_0^{1,\alpha}(\overline{\Omega})}\le M'$ for every $\varepsilon\in(0,1]$ by the regularity result in [11]. Because the embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ is compact and by $u_\varepsilon\in\operatorname{int}P$, there exists a sequence $\{\varepsilon_n\}$ and $u_0\in P$ such that $\varepsilon_n\to 0+$ and $u_n:=u_{\varepsilon_n}\to u_0$ in $C_0^1(\overline{\Omega})$ as $n\to\infty$. If $u_0\not\equiv 0$ occurs, then $u_0\in\operatorname{int}P$ by the same reason as in Lemma 7, and hence our conclusion is proved. Now, we shall prove $u_0\not\equiv 0$ by contradiction for each theorem. So, we suppose that $u_0=0$, whence $u_n\to 0$ in $C_0^1(\overline{\Omega})$ as $n\to\infty$.

Proof of Theorem 1 Let $\varphi\in\operatorname{int}P$ be an eigenfunction corresponding to the first positive eigenvalue $\mu_1(m)$ (*cf.* [14, 15], it is well known that we can obtain φ as the minimizer of (4)), namely, φ is a positive solution of $-\Delta_p u=\mu_1(m)m(x)|u|^{p-2}u$ in Ω and $u=0$ on $\partial\Omega$. Since p-Laplacian is $(p-1)$-homogeneous, we may assume that φ satisfies $\int_\Omega m(x)\varphi^p\,dx=1$, and hence $\|\nabla\varphi\|_p^p=\mu_1(m)\int_\Omega m(x)\varphi^p\,dx=\mu_1(m)$ holds by taking φ as a test function. Choose $\rho>0$ satisfying $b_0-A_p\mu_1(m)>\rho\|\varphi\|_p^p$ (note that $b_0-A_p\mu_1(m)>0$ as in (f1)). Due to (f1), there exists a $\delta>0$ such that $f_0(x,t)\ge(b_0m(x)-\rho)t^{p-1}$ for every $0\le t\le\delta$ and $x\in\Omega$. Since we are assuming $u_n \to 0$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$, $\|u_n\|_\infty \le \delta$ occurs for sufficiently large n. Then, for such sufficiently large n, according to Lemma 9, (1) and $\psi \ge 0$, we obtain

$$\begin{aligned} A_p\mu_1(m) = A_p\|\nabla\varphi\|_p^p &\ge \int_\Omega A(x,\nabla u_n)\nabla\left(\frac{\varphi^p}{u_n^{p-1}}\right)dx = \int_\Omega \frac{f(x,u_n,\nabla u_n)+\varepsilon\psi}{u_n^{p-1}}\varphi^p\,dx \\ &\ge \int_\Omega \frac{f_0(x,u_n)}{u_n^{p-1}}\varphi^p\,dx \ge b_0\int_\Omega m(x)\varphi^p\,dx - \rho\|\varphi\|_p^p = b_0 - \rho\|\varphi\|_p^p > A_p\mu_1(m). \end{aligned}$$

This is a contradiction. □

Proof of Theorem 2 Since $\infty > \sup_{x\in\Omega} a_0(x) \ge \inf_{x\in\Omega} a_0(x) > 0$ holds, by the standard argument as in the p-Laplacian, we see that $\lambda_1(m) > 0$ and it is the first positive eigenvalue of $-\operatorname{div}(a_0(x)|\nabla u|^{p-2}\nabla u) = \lambda m(x)|u|^{p-2}u$ in Ω and $u = 0$ on $\partial\Omega$. Therefore, by the well-known argument, there exists a positive eigenfunction $\varphi_1 \in \operatorname{int} P$ corresponding to $\lambda_1(m)$ (we can obtain φ_1 as the minimizer of (7)). Hence, by taking φ_1 as a test function, we have $0 < \int_\Omega a_0(x)|\nabla\varphi_1|^p\,dx = \lambda_1(m)\int_\Omega m(x)\varphi_1^p\,dx$. Thus, $\int_\Omega m(x)\varphi_1^p\,dx > 0$ follows. Because $u_n \in \operatorname{int} P$ is a solution of $(P;\varepsilon_n)$ and $\varphi_1 \in \operatorname{int} P$ is an eigenfunction corresponding to $\lambda_1(m)$, according to Lemma 11 and Lemma 13 (note $A(x,y) = a_0|y|^{p-2}y + \widetilde{a}_0(x,|y|)y$ as in (AH0)), we obtain

$$\begin{aligned} 0 &\le \int_\Omega a_0(x)|\nabla\varphi_1|^{p-2}\nabla\varphi_1\nabla\left(\frac{\varphi_1^p - u_n^p}{\varphi_1^{p-1}}\right)dx - \int_\Omega a_0(x)|\nabla u_n|^{p-2}\nabla u_n\nabla\left(\frac{\varphi_1^p - u_n^p}{u_n^{p-1}}\right)dx \\ &\le \lambda_1(m)\int_\Omega m\big(\varphi_1^p - u_n^p\big)\,dx - \int_\Omega \frac{f_0(x,u_n)}{u_n^{p-1}}\varphi_1^p\,dx \\ &\quad + \int_\Omega \widetilde{a}_0\big(x,|\nabla u_n|\big)\nabla u_n\nabla\left(\frac{\varphi_1^p - u_n^p}{u_n^{p-1}}\right)dx + \int_\Omega f(x,u_n,\nabla u_n)u_n\,dx + \varepsilon_n\int_\Omega \psi u_n\,dx \\ &= -\int_\Omega\left(\frac{f_0(x,u_n)}{u_n^{p-1}} - b_0 m(x)\right)\varphi_1^p\,dx - \big(b_0 - \lambda_1(m)\big)\int m(x)\varphi_1^p\,dx + o(1) \end{aligned} \tag{19}$$

as $n \to \infty$ since we are assuming $u_n \to 0$ in $C_0^1(\overline{\Omega})$, where we use the facts that $\psi \ge 0$ and $\varphi_1 > 0$ in Ω. Furthermore, by Fatou's lemma and (3), we have

$$\liminf_{n\to\infty}\int_\Omega\left(\frac{f_0(x,u_n)}{u_n^{p-1}} - b_0 m(x)\right)\varphi_1^p\,dx \ge 0.$$

As a result, by taking a limit superior with respect to n in (19), we have $0 \le -(b_0 - \lambda_1(m))\int m(x)\varphi_1^p\,dx < 0$. This is a contradiction. □

Competing interests
The author declares that she has no competing interests.

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