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Blow-up criteria for 3D nematic liquid crystal models in a bounded domain

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Abstract

In this paper we prove some blow-up criteria for two 3D density-dependent nematic liquid crystal models in a bounded domain.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, and let ν be the unit outward normal vector on $\partial\Omega$. We consider the regularity criterion to the density-dependent incompressible nematic liquid crystal model as follows [1–4]:

$$\operatorname{div} u = 0, \tag{1.1}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.2}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla\pi - \Delta u = -\nabla \cdot (\nabla d \odot \nabla d), \tag{1.3}$$

$$\partial_t d + u \cdot \nabla d + \big(|d|^2 - 1\big)d - \Delta d = 0, \tag{1.4}$$

in $(0,\infty) \times \Omega$ with initial and boundary conditions

$$(\rho, u, d)(\cdot, 0) = (\rho_0, u_0, d_0) \quad \text{in } \Omega, \tag{1.5}$$

$$u = 0, \qquad \partial_\nu d = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where ρ denotes the density, u the velocity, π the pressure, and d represents the macroscopic molecular orientations, respectively. The symbol $\nabla d \odot \nabla d$ denotes a matrix whose (i,j)th entry is $\partial_i d\, \partial_j d$, and it is easy to find that $\nabla d \odot \nabla d = \nabla d^T \nabla d$.

When d is a given constant vector, then (1.1)-(1.3) represent the well-known density-dependent Navier-Stokes system, which has received many studies; see [5–7] and references therein.

When $\rho = 1$, Guillén-González *et al.* [8] proved the blow-up criterion

$$\int_0^T \Big(\|u(t)\|_{L^q}^{\frac{2q}{q-3}} + \|\nabla d(t)\|_{L^q}^{\frac{2q}{q-3}}\Big)\, dt < \infty \quad \text{with } 3 < q \le \infty \tag{1.7}$$

and $0 < T < \infty$.

It is easy to prove that the problem (1.1)-(1.6) has a unique local-in-time strong solution [6, 9], and thus we omit the details here. The aim of this paper is to consider the regularity criterion; we will prove the following theorem.

Theorem 1.1 *Let* $\rho_0 \in W^{1,q}(\Omega)$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $d_0 \in H^3(\Omega)$ *with* $3 < q \le 6$ *and* $\rho_0 \ge 0$, $\operatorname{div} u_0 = 0$ *in* Ω *and* $\partial_\nu d_0 = 0$ *on* $\partial\Omega$. *We also assume that the following compatibility condition holds true*: $\exists(\nabla\pi_0, g) \in L^2(\Omega)$ *such that*

$$\nabla\pi_0 - \Delta u_0 + \nabla \cdot (\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0} g \quad \text{in } \Omega.$$

Let (ρ, u, d) *be a local strong solution to the problem* (1.1)-(1.6). *If* u *satisfies*

$$\int_0^T \|u(t)\|_{L^q}^{\frac{2q}{q-3}} \, dt < \infty \quad \text{with } 3 < q \le \infty \tag{1.8}$$

and $0 < T < \infty$, *then the solution* (ρ, u, d) *can be extended beyond* $T > 0$.

Remark 1.1 When $\rho \equiv 1$, our result improves (1.7) to (1.8).

Remark 1.2 By similar calculations as those in [6], we can replace L^q-norm in (1.8) by L_w^q-norm, and thus we omit the details here.

Remark 1.3 When the space dimension $n = 2$, we can prove that the problem (1.1)-(1.6) has a unique global-in-time strong solution by the same method as that in [10], and thus we omit the details here.

Next we consider another liquid model: (1.1), (1.2), (1.3), (1.5), (1.6) and

$$\partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d, \tag{1.9}$$

with $|d| \equiv 1$ in $(0, \infty) \times \Omega$. Li and Wang [9] proved that the problem has a unique local strong solution. When $\Omega := \mathbb{R}^3$, Fan *et al.* [11] proved a regularity criterion. The aim of this paper is to study the regularity criterion of the problem in a bounded domain. We will prove the following theorem.

Theorem 1.2 *Let the initial data satisfy the same conditions in Theorem* 1.1 *and* $|d_0| \equiv 1$ *in* Ω. *Let* (ρ, u, d) *be a local strong solution to the problem* (1.1)-(1.3), (1.5), (1.6) *and* (1.9). *If* u *and* ∇d *satisfy*

$$\int_0^T \left(\|u(t)\|_{L^q}^{\frac{2q}{q-3}} + \|\nabla d\|_{L^q}^{\frac{2q}{q-3}} \right) dt < \infty \quad \text{with } 3 < q \le \infty \tag{1.10}$$

and $0 < T < \infty$, *then the solution* (ρ, u, d) *can be extended beyond* $T > 0$.

2 Proof of Theorem 1.1

We only need to establish *a priori* estimates.

Below we shall use the notation

$$\int = \int_\Omega .$$

First, thanks to the maximum principle, it follows from (1.1) and (1.2) that

$$0 \le \rho \le \|\rho_0\|_{L^\infty} < \infty. \tag{2.1}$$

Testing (1.3) by u and using (1.1) and (1.2), we see that

$$\frac{1}{2}\frac{d}{dt}\int \rho u^2\,dx + \int |\nabla u|^2\,dx = -\int (u\cdot\nabla)d\cdot\Delta d\,dx. \tag{2.2}$$

Testing (1.4) by $-\Delta d + (|d|^2-1)d$ and using (1.1), we find that

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{1}{2}|\nabla d|^2 + \frac{1}{4}\big(|d|^2-1\big)^2\right)dx + \int\big(-\Delta d + \big(|d|^2-1\big)d\big)^2\,dx\\ &\quad = \int (u\cdot\nabla)d\cdot\Delta d\,dx.\end{aligned} \tag{2.3}$$

Summing up (2.2) and (2.3), we have the well-known energy inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\rho u^2 + |\nabla d|^2 + \frac{1}{2}\big(|d|^2-1\big)^2\right)dx + \int\left(|\nabla u|^2 + \big(-\Delta d + \big(|d|^2-1\big)d\big)^2\right)dx\\ &\quad \le 0.\end{aligned} \tag{2.4}$$

Next, we prove the following estimate:

$$\|d\|_{L^\infty(0,T;L^\infty)} \le \max\big(1, \|d_0\|_{L^\infty}\big). \tag{2.5}$$

Without loss of generality, we assume that $1 \le \|d_0\|_{L^\infty}$. Multiplying (1.4) by $2d$, we get

$$\partial_t\phi + u\cdot\nabla\phi - \Delta\phi + 2|d|^2\phi = -2|d|^2\big(\|d_0\|_{L^\infty}^2 - 1\big) - 2|\nabla d|^2 \le 0 \tag{2.6}$$

with $\phi := |d|^2 - \|d_0\|_{L^\infty}^2$ and $\phi(\cdot,0) = |d_0|^2 - \|d_0\|_{L^\infty}^2 \le 0$ and $\partial_\nu\phi = 0$ on $\partial\Omega\times(0,\infty)$. Then (2.5) follows from (2.6) by the maximum principle.

In the following calculations, we use the following Gauss-Green formula [12]:

$$\begin{aligned}-\int_\Omega \Delta f\cdot f|f|^{p-2}\,dx &= \frac{1}{2}\int_\Omega |f|^{p-2}|\nabla f|^2\,dx + 4\frac{p-2}{p^2}\int_\Omega \big|\nabla|f|^{p/2}\big|^2\,dx\\ &\quad - \int_{\partial\Omega}|f|^{p-2}(f\cdot\nabla)f\cdot\nu\,dS - \int_{\partial\Omega}|f|^{p-2}(\operatorname{curl} f\times\nu)\cdot f\,dS\end{aligned} \tag{2.7}$$

and the following estimate [13, 14]:

$$\|f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|f\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \tag{2.8}$$

with $1 < p < \infty$.

Taking ∇ to $(1.4)_i$, we deduce that

$$\partial_t\nabla d_i + (u\cdot\nabla)\nabla d_i + \nabla\big(\big(|d|^2-1\big)d_i\big) - \Delta\nabla d_i = \sum_j \nabla u_j\,\partial_j d_i.$$

Testing the above equation by $|\nabla d_i|^{p-2}\nabla d_i$ ($2 \le p \le 6$), using (1.1), (2.7), (2.8), (2.5) and summing over i, we derive

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla d|^p\,dx + \frac{1}{2}\int_\Omega |\nabla d|^{p-2}\big|\nabla^2 d\big|^2\,dx + 4\frac{p-2}{p^2}\int_\Omega \big|\nabla|\nabla d|^{p/2}\big|^2\,dx \\
&\quad = -\sum_i \int_{\partial\Omega} |\nabla d_i|^{p-2}(\nabla d_i\cdot\nabla)\nu\cdot\nabla d_i\,dS + \sum_{i,j}\int_\Omega \big[\nabla u_j \partial_j d_i |\nabla d_i|^{p-2}\big]\cdot\nabla d_i\,dx \\
&\qquad - \sum_i \int_\Omega \nabla\big(\big(|d|^2-1\big)d_i\big)\cdot|\nabla d_i|^{p-2}\nabla d_i\,dx \\
&\quad \le C\int_{\partial\Omega}|\nabla d|^p\,dS - \sum_{i,j}\int_\Omega u_j\nabla\cdot\big(\partial_j d_i|\nabla d_i|^{p-2}\nabla d_i\big)\,dx + C\int_\Omega |\nabla d|^p\,dx \\
&\quad \le C\int_{\partial\Omega}|\nabla d|^p\,dS + C\int_\Omega |u||\nabla d|^{p/2}\cdot\big|\nabla|\nabla d|^{p/2}\big|\,dx + C\int_\Omega|\nabla d|^p\,dx \\
&\qquad + \int_\Omega |u||\nabla d|^{\frac{p}{2}}\cdot|\nabla d|^{\frac{p}{2}-1}\big|\nabla^2 d\big|\,dx \\
&\quad \le C\int_{\partial\Omega} w^2\,dS + C\int_\Omega |u|w|\nabla w|\,dx + C\int_\Omega w^2\,dx + \int_\Omega |u|w|\nabla d|^{\frac{p}{2}-1}\big|\nabla^2 d\big|\,dx \\
&\qquad \big(w := |\nabla d|^{p/2}\big) \\
&\quad \le C\|w\|_{L^2}\|w\|_{H^1} + C\|u\|_{L^q}\|w\|_{L^{\frac{2q}{q-2}}}\|\nabla w\|_{L^2} + C\|w\|_{L^2}^2 \\
&\qquad + C\|u\|_{L^q}\|\nabla w\|_{L^{\frac{2q}{q-2}}}\big\||\nabla d|^{\frac{p}{2}-1}\big|\nabla^2 d\big|\big\|_{L^2} \\
&\quad \le 2\frac{p-2}{p^2}|\nabla w|_{L^2}^2 + \frac{1}{4}\big\||\nabla d|^{\frac{p}{2}-1}\big|\nabla^2 d\big|\big\|_{L^2}^2 + C\|w\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{2q}{q-3}}\|w\|_{L^2}^2,
\end{aligned}$$

which gives

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega w^2\,dx + C\int_\Omega |\nabla w|^2\,dx + C\int_\Omega |\nabla d|^{p-2}\big|\nabla^2 d\big|^2\,dx \\
&\quad \le C\|w\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{2q}{q-3}}\|w\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\|\nabla d\|_{L^\infty(0,T;L^p)} \le C \quad \text{with } 2 \le p \le 6. \tag{2.9}$$

Testing (1.3) by u_t, using (1.1), (1.2), (2.1) and (2.9), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx + \int \rho|u_t|^2\,dx \\
&\quad = -\int \rho u\cdot\nabla u\cdot u_t\,dx + \frac{d}{dt}\int \nabla d\odot\nabla d : \nabla u\,dx - 2\int \nabla d_t\odot\nabla d : \nabla u\,dx \\
&\quad \le \|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^q}\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\sqrt{\rho}u_t\|_{L^2} \\
&\qquad + \frac{d}{dt}\int \nabla d\odot\nabla d : \nabla u\,dx + 2\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^6}\|\nabla u\|_{L^3}
\end{aligned}$$

$$\begin{aligned}
&\le C\|u\|_{L^q}\|\nabla u\|_{L^2}^{1-\frac{3}{q}}\|u\|_{H^2}^{\frac{3}{q}}\|\sqrt{\rho}u_t\|_{L^2} \\
&\quad + \frac{d}{dt}\int \nabla d \odot \nabla d : \nabla u\,dx + C\|\nabla d_t\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}.
\end{aligned} \tag{2.10}$$

By the H^2-regularity theory of the Stokes system, it follows from (1.3) that

$$\begin{aligned}
\|u\|_{H^2} &\le C\|\nabla d^T\cdot\Delta d\|_{L^2} + C\|\rho u_t + \rho u\cdot\nabla u\|_{L^2} \\
&\le C\|\nabla d\|_{L^6}\|\Delta d\|_{L^3} + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^q}\|\nabla u\|_{L^{\frac{2q}{q-2}}} \\
&\le C\|\Delta d\|_{L^3} + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^q}\|\nabla u\|_{L^2}^{1-\frac{3}{q}}\|u\|_{H^2}^{\frac{3}{q}} \\
&\le \frac{1}{2}\|u\|_{H^2} + C\|\Delta d\|_{L^3} + C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^q}^{\frac{q}{q-3}}\|\nabla u\|_{L^2},
\end{aligned}$$

which yields

$$\|u\|_{H^2} \le C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^q}^{\frac{q}{q-3}}\|\nabla u\|_{L^2} + C\|\Delta d\|_{L^3}. \tag{2.11}$$

Testing (1.4) by $-\Delta d_t$, using (2.5) and (2.9), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2\,dx + \int |\nabla d_t|^2\,dx \\
&\quad = \int \big((|d|^2-1)d + u\cdot\nabla d\big)\Delta d_t\,dx \\
&\quad \le \int \big|\big[\nabla\big(|d|^2 d - d\big) + \nabla(u\cdot\nabla d)\big]\nabla d_t\big|\,dx \\
&\quad \le C\big(1 + \|u\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}} + \|\nabla u\|_{L^3}\|\nabla d\|_{L^6}\big)\|\nabla d_t\|_{L^2} \\
&\quad \le C\big(1 + \|u\|_{L^q}\|\Delta d\|_{L^2}^{1-\frac{3}{q}}\|d\|_{H^3}^{\frac{3}{q}} + \|\nabla u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}\big)\|\nabla d_t\|_{L^2}.
\end{aligned} \tag{2.12}$$

On the other hand, by the H^3-regularity theory of the elliptic equation, from (1.4), (2.5) and (2.9) we infer that

$$\begin{aligned}
\|d\|_{H^3} &\le C\big(\|d\|_{L^2} + \|\nabla\Delta d\|_{L^2}\big) \\
&\le C\big(1 + \big\|\nabla\big(\partial_t d + u\cdot\nabla d + |d|^2 d - d\big)\big\|_{L^2}\big) \\
&\le C\big(1 + \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla d\|_{L^6} + \|u\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}}\big) \\
&\le C\big(1 + \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} + \|u\|_{L^q}\|\Delta d\|_{L^2}^{1-\frac{3}{q}}\|d\|_{H^3}^{\frac{3}{q}}\big),
\end{aligned}$$

which gives

$$\|d\|_{H^3} \le C\big(1 + \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} + \|u\|_{L^q}^{\frac{q}{q-3}}\|\Delta d\|_{L^2}\big). \tag{2.13}$$

Combining (2.11) and (2.13), we have

$$\begin{aligned}
\|u\|_{H^2} + \|d\|_{H^3} &\le C + C\|\sqrt{\rho}u_t\|_{L^2} + C\|\nabla d_t\|_{L^2} + C\|\nabla u\|_{L^2} \\
&\quad + C\|u\|_{L^q}^{\frac{q}{q-3}}\|\nabla u\|_{L^2} + C\|\Delta d\|_{L^2} + C\|u\|_{L^q}^{\frac{q}{q-3}}\|\Delta d\|_{L^2}.
\end{aligned} \tag{2.14}$$

Putting (2.14) into (2.10) and (2.12) and summing up, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(|\nabla u|^2+|\Delta d|^2\right)dx+\int\left(\rho|u_t|^2+|\nabla d_t|^2\right)dx-\frac{d}{dt}\int\nabla d\odot\nabla d:\nabla u\,dx\\
&\quad\le\frac{1}{4}\int\rho|u_t|^2\,dx+\frac{1}{4}\int|\nabla d_t|^2\,dx+C+C\|\nabla u\|_{L^2}^2\\
&\qquad+C\|u\|_{L^q}^{\frac{2q}{q-3}}\|\nabla u\|_{L^2}^2+C\|\Delta d\|_{L^2}^2+C\|u\|_{L^q}^{\frac{2q}{q-3}}\|\Delta d\|_{L^2}^2,
\end{aligned}$$

which leads to

$$\|u\|_{L^\infty(0,T;H^1)}\le C,\qquad\|\sqrt{\rho}u_t\|_{L^2(0,T;L^2)}\le C,\tag{2.15}$$

$$\|d\|_{L^\infty(0,T;H^2)}+\|d_t\|_{L^2(0,T;H^1)}\le C.\tag{2.16}$$

It follows from (2.14), (2.15) and (2.16) that

$$\|u\|_{L^2(0,T;H^2)}+\|d\|_{L^2(0,T;H^3)}\le C.\tag{2.17}$$

Taking ∂_t to (1.3), testing by u_t, using (1.1), (1.2) and (2.15), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2\,dx+\int|\nabla u_t|^2\,dx\\
&\quad\le\left|\int\rho u\cdot\nabla\left(u_t^2+u\cdot\nabla u\cdot u_t\right)dx\right|+\left|\int\rho u_t\cdot\nabla u\cdot u_t\,dx\right|\\
&\qquad+2\left|\int\nabla d_t\odot\nabla d:\nabla u_t\,dx\right|\\
&\quad\le C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}+C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\qquad+C\|u\|_{L^6}^2\|\Delta u\|_{L^2}\|u_t\|_{L^6}+C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^6}^2\\
&\qquad+C\|\sqrt{\rho}u_t\|_{L^2}\|u_t\|_{L^6}\|\nabla u\|_{L^3}+C\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\|\nabla u_t\|_{L^2}\\
&\quad\le\frac{1}{4}\|\nabla u_t\|_{L^2}^2+C\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u\|_{H^2}^2+C\|u\|_{H^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2\\
&\qquad+C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2.
\end{aligned}\tag{2.18}$$

Taking ∂_t to (1.4), testing by $-\Delta d_t$, using (2.5), (2.15), (2.16) and (2.17), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla d_t|^2\,dx+\int|\Delta d_t|^2\,dx\\
&\quad=\int\left(u\cdot\nabla d+|d|^2d-d\right)_t\cdot\Delta d_t\,dx\\
&\quad\le\int\left[u_t\cdot\nabla d+u\cdot\nabla d_t+\left(|d|^2d-d\right)_t\right]\Delta d_t\,dx\\
&\quad=-\int\nabla(u_t\cdot\nabla d)\cdot\nabla d_t\,dx+\int\left[u\cdot\nabla d_t+\left(|d|^2d-d\right)_t\right]\Delta d_t\,dx\\
&\quad\le\left(\|\nabla u_t\|_{L^2}\|\nabla d\|_{L^\infty}+\|u_t\|_{L^6}\|\Delta d\|_{L^3}\right)\|\nabla d_t\|_{L^2}\\
&\qquad+\|u\|_{L^6}\|\nabla d_t\|_{L^3}\|\Delta d_t\|_{L^2}+C\|\nabla d_t\|_{L^2}^2
\end{aligned}$$

$$\le \frac{1}{4}\|\nabla u_t\|_{L^2}^2 + \frac{1}{4}\|\Delta d_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2 + C\|\Delta d\|_{L^3}^2\|\nabla d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2. \tag{2.19}$$

Combining (2.18) and (2.19), we have

$$\|\sqrt{\rho}u_t\|_{L^\infty(0,T;L^2)} + \|u_t\|_{L^2(0,T;H^1)} \le C, \tag{2.20}$$

$$\|d_t\|_{L^\infty(0,T;H^1)} + \|d_t\|_{L^2(0,T;H^2)} \le C. \tag{2.21}$$

It follows from (1.4), (2.21) and (2.16) that

$$\|d\|_{L^\infty(0,T;H^2)} \le C. \tag{2.22}$$

It follows from (2.14), (2.15), (2.20) and (2.21) that

$$\|u\|_{L^\infty(0,T;H^2)} + \|d\|_{L^\infty(0,T;H^3)} \le C. \tag{2.23}$$

It follows from (1.3), (2.20) and (2.23) that

$$\|u\|_{L^2(0,T;W^{2,6})} \le C, \tag{2.24}$$

from which it follows that

$$\|\nabla u\|_{L^2(0,T;L^\infty)} \le C. \tag{2.25}$$

Applying ∇ to (1.2), testing by $|\nabla\rho|^{q-2}\nabla\rho$ $(2 \le q \le 6)$ and using (2.25), we have

$$\frac{d}{dt}\|\nabla\rho\|_{L^q}^q \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}^q,$$

which implies

$$\|\nabla\rho\|_{L^\infty(0,T;L^q)} \le C, \tag{2.26}$$

and therefore

$$\begin{aligned}\|\rho_t\|_{L^\infty(0,T;L^q)} &= \|u\nabla\rho\|_{L^\infty(0,T;L^q)} \\ &\le \|u\|_{L^\infty(0,T;L^\infty)}\|\nabla\rho\|_{L^\infty(0,T;L^q)} \\ &\le C.\end{aligned} \tag{2.27}$$

This completes the proof.

3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. We only need to establish *a priori* estimates.

First, we still have (2.1) and (2.2).

Next, we easily infer that

$$|d| \equiv 1 \quad \text{in } (0,\infty)\times\Omega. \tag{3.1}$$

Testing (1.9) by $-\Delta d - |\nabla d|^2 d$ and using (1.1) and (3.1), we find that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2\,dx + \int \bigl|\Delta d + |\nabla d|^2 d\bigr|^2\,dx = \int (u\cdot\nabla)d\cdot\Delta d\,dx. \tag{3.2}$$

Summing up (2.2) and (3.2), we have the well-known energy inequality

$$\frac{1}{2}\frac{d}{dt}\int \bigl(\rho u^2 + |\nabla d|^2\bigr)\,dx + \int \bigl(|\nabla u|^2 + \bigl|\Delta d + |\nabla d|^2 d\bigr|^2\bigr)\,dx \le 0. \tag{3.3}$$

Taking ∇ to $(1.9)_i$, testing by $|\nabla d_i|^{p-2}\nabla d_i$ $(2\le p\le 6)$, using (1.1), (2.7), (2.8) and (3.1), similarly to (2.9), we deduce that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega |\nabla d|^p\,dx + \frac{1}{2}\int_\Omega |\nabla d|^{p-2}\bigl|\nabla^2 d\bigr|^2\,dx + 4\frac{p-2}{p^2}\int_\Omega \bigl|\nabla|\nabla d|^{p/2}\bigr|^2\,dx \\
&\quad = -\sum_i \int_{\partial\Omega} |\nabla d_i|^{p-2}(\nabla d_i\cdot\nabla)\nu\cdot\nabla d_i\,dS + \sum_{i,j}\int_\Omega \nabla u_j \partial_j d_i |\nabla d_i|^{p-2}\nabla d_i\,dx \\
&\qquad + \int_\Omega \nabla\bigl(|\nabla d|^2 d\bigr)\cdot|\nabla d|^{p-2}\nabla d\,dx \\
&\quad \le \frac{p-2}{p^2}\int_\Omega |\nabla w|^2\,dx + C\|w\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{2q}{q-3}}\|w\|_{L^2}^2 \\
&\qquad + \int_\Omega |\nabla d|^2 w^2\,dx + C\int_\Omega |\nabla d|\bigl|\nabla^2 d\bigr||\nabla d|^{\frac{p}{2}-1}|\nabla d|^{\frac{p}{2}}\,dx \quad \bigl(w := |\nabla d|^{p/2}\bigr) \\
&\quad \le \frac{p-2}{p^2}\int_\Omega |\nabla w|^2\,dx + C\|w\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{2q}{q-3}}\|w\|_{L^2}^2 \\
&\qquad + \|\nabla d\|_{L^q}^2\|w\|_{L^{\frac{2q}{q-2}}}^2 + \frac{1}{4}\int_\Omega |\nabla d|^{p-2}\bigl|\nabla^2 d\bigr|^2\,dx \\
&\quad \le 2\frac{p-2}{p^2}\int_\Omega |\nabla w|^2\,dx + C\|w\|_{L^2}^2 + C\|u\|_{L^q}^{\frac{2q}{q-3}}\|w\|_{L^2}^2 \\
&\qquad + C\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}\|w\|_{L^2}^2 + \frac{1}{4}\int_\Omega |\nabla d|^{p-2}\bigl|\nabla^2 d\bigr|^2\,dx,
\end{aligned}$$

which yields

$$\|\nabla d\|_{L^\infty(0,T;L^p)} + \int_0^T\int |\nabla d|^2\bigl|\nabla^2 d\bigr|^2\,dx\,dt \le C. \tag{3.4}$$

We still have (2.10) and (2.11).

Similarly to (2.12), testing (1.9) by $-\Delta d_t$, using (3.1) and (3.4), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2\,dx + \int |\nabla d_t|^2\,dx \\
&\quad = \int \bigl(u\cdot\nabla d - |\nabla d|^2 d\bigr)\Delta d_t\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \nabla\big(|\nabla d|^2 d - u\cdot\nabla d\big)\cdot\nabla d_t\,dx \\
&\le C\big(1 + \|u\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}} + \|\nabla u\|_{L^3}\|\nabla d\|_{L^6}\big)\|\nabla d_t\|_{L^2} \\
&\quad + C\big(\|\nabla d\|_{L^6}^3 + \big\|\nabla|\nabla d|^2\big\|_{L^2}\big)\|\nabla d_t\|_{L^2} \\
&\le C\big(1 + \|u\|_{L^q}\|\Delta d\|_{L^2}^{1-\frac{3}{q}}\|d\|_{H^3}^{\frac{3}{q}} + \|\nabla u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2} + \big\|\nabla|\nabla d|^2\big\|_{L^2}\big)\|\nabla d_t\|_{L^2}. \qquad (3.5)
\end{aligned}
$$

Similarly to (2.13), we have

$$
\begin{aligned}
\|d\|_{H^3} &\le C\Big(1 + \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} + \|u\|_{L^q}^{\frac{q}{q-3}}\|\Delta d\|_{L^2} + \big\|\nabla\big(|\nabla d|^2 d\big)\big\|_{L^2}\Big) \\
&\le C\Big(1 + \|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^3} + \|u\|_{L^q}^{\frac{q}{q-3}}\|\Delta d\|_{L^2} + \big\|\nabla|\nabla d|^2\big\|_{L^2}\Big). \qquad (3.6)
\end{aligned}
$$

Combining (2.11) and (3.6), we have

$$
\|u\|_{H^2} + \|d\|_{H^3} \le \text{ the right hand side of (2.14)} + C\big\|\nabla|\nabla d|^2\big\|_{L^2}. \qquad (3.7)
$$

Putting (3.7) into (3.5) and (2.10) and using the Gronwall inequality, we still have (2.15), (2.16), (2.17) and (2.18).

Similarly to (2.19), applying ∂_t to (1.9), testing by $-\Delta d_t$ and using (3.4), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla d_t|^2\,dx + \int|\Delta d_t|^2\,dx \\
&\quad = \int\big(u\cdot\nabla d - |\nabla d|^2 d\big)_t\cdot\Delta d_t\,dx \\
&\quad = \int\big(u_t\cdot\nabla d + u\cdot\nabla d_t - |\nabla d|^2 d_t - d\,\partial_t|\nabla d|^2\big)\Delta d_t\,dx \\
&\quad = -\int\nabla(u_t\cdot\nabla d)\cdot\nabla d_t\,dx + \int\big(u\cdot\nabla d_t - |\nabla d|^2 d_t - d\,\partial_t|\nabla d|^2\big)\Delta d_t\,dx \\
&\quad \le \frac{1}{4}\|\nabla u_t\|_{L^2}^2 + \frac{1}{4}\|\Delta d_t\|_{L^2}^2 + C\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2 \\
&\qquad + C\|\Delta d\|_{L^3}^2\|\nabla d_t\|_{L^2}^2 + C\|\nabla d_t\|_{L^2}^2. \qquad (3.8)
\end{aligned}
$$

Combining (2.18) and (3.8) and using the Gronwall inequality, we still obtain (2.20) and (2.21).

By similar calculations as those in (2.22)-(2.27), we still arrive at (2.22)-(2.27).

This completes the proof.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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