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Existence and multiplicity of solutions for a class of sublinear Schrödinger-Maxwell equations

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Abstract
In this paper I consider a class of sublinear Schrödinger-Maxwell equations, and new results about the existence and multiplicity of solutions are obtained by using the minimizing theorem and the dual fountain theorem respectively.

Keywords: Schrödinger-Maxwell equations; sublinear; minimizing theorem; dual fountain theorem

1 Introduction and main result

Consider the following semilinear Schrödinger-Maxwell equations:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } R^3, \\ -\Delta \phi = u^2, \qquad \lim_{|x|\to\infty} \phi(x) = 0, & \text{in } R^3. \end{cases} \tag{1}$$

Such a system, also known as the nonlinear Schrödinger-Poisson system, arises in an interesting physical context. Indeed, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger and the Maxwell equations (we refer to [1, 2] for more details on the physical aspects and on the qualitative properties of the solutions). In particular, if we are looking for electrostatic-type solutions, we just have to solve (1).

In recent years, system (1), with $V(x) \equiv 1$ or being radially symmetric, has been widely studied under various conditions on f; see, for example, [3–11]. Since (1) is set on R^3, it is well known that the Sobolev embedding $H^1(R^3) \hookrightarrow L^s(R^3)$ $(2 \le s \le 2^* = 6)$ is not compact, and then it is usually difficult to prove that a minimizing sequence or a sequence that satisfies the (PS) condition, briefly a Palais-Smale sequence, is strongly convergent if we seek solutions of (1) by variational methods. If $V(x)$ is radial (for example, $V(x) \equiv 1$), we can avoid the lack of compactness of Sobolev embedding by looking for solutions of (1) in the subspace of radial functions of $H^1(R^3)$, which is usually denoted by $H_r^1(R^3)$, since the embedding $H_r^1(R^3) \hookrightarrow L^s(R^3)$ $(2 < s < 6)$ is compact. Specially, Ruiz [11] dealt with (1) under the assumption that $V(x) \equiv 1$ and $f(u) = u^p$ $(1 < p < 5)$ and got some general existence, nonexistence and multiplicity results.

Moreover, in [12] the authors considered system (1) with periodic potential $V(x)$, and the existence of infinitely many geometrically distinct solutions was proved by the nonlinear superposition principle established in [13].

There are also some papers treating the case with nonradial potential $V(x)$. More precisely, Wang and Zhou [14] got the existence and nonexistence results of (1) when $f(u)$ is asymptotically linear at infinity. Chen and Tang [15] proved that (1) has infinitely many high energy solutions under the condition that $f(x,u)$ is superlinear at infinity in u by the fountain theorem. Soon after, Li, Su and Wei [16] improved their results.

Up to now, there have been few works concerning the case that $V(x)$ is nonradial potential and $f(x,u)$ is sublinear at infinity in u. Very recently, Sun [17] treated the above case based on the variant fountain theorem established in Zou [18].

Theorem 1.1 [17] *Assume that the following conditions hold*:

(V_1') $V \in C(R^3, R)$ *satisfies* $\inf_{x\in R^3} V(x) \geq a > 0$, *where* $a > 0$ *is a constant*. *For every* $M > 0$, $\operatorname{meas}\{x \in R^3 : v(x) \leq M\} < \infty$.

(H_1) $F(x,u) = a(x)|u|^r$, *where* $F(x,u) = \int_0^u f(x,y)\,dy$, $a : R^3 \to R^+$ *is a positive function such that* $a \in L^{\frac{2}{2-r}}(R^3)$ *and* $1 < r < 2$.

Then problem (1) *has infinitely many nontrivial solutions* $\{(u_k, \phi_k)\}$ *satisfying*

$$\frac{1}{2}\int_{R^3}\big(|\nabla u_k|^2 + V(x)u_k^2\big)\,dx - \frac{1}{4}\int_{R^3}|\nabla \phi_k|^2\,dx + \frac{1}{2}\int_{R^3}\phi_k u_k^2\,dx - \int_{R^3}F(x,u_k)\,dx \to 0^-$$

as $k \to \infty$.

In the present paper, based on the dual fountain theorem, we can prove the same result under a more generic condition, which generalizes the result in [17]. Our first result can be stated as follows.

Theorem 1.2 *Assume that* V *satisfies*

(V_1) $V \in C(R^3, R)$ *and* $\inf_{x\in R^3} V(x) > 0$;

and f *satisfies the following conditions*.

(W_1) *There exist constants* $\delta > 0$, $r_1 \in (1,2)$ *and a function* $a_1 \in L^{\frac{2}{2-r_1}}(R^3, [0,+\infty))$ *such that*

$$\big|f(x,u)\big| \leq a_1(x)|u|^{r_1-1}$$

for all $x \in R^3$ *and* $|u| \leq \delta$;

(W_2) *There exist constants* $M > 0$, $r_2 \in (1,2)$ *and a function* $a_2 \in L^{\frac{2}{2-r_2}}(R^3, [0,+\infty))$ *such that*

$$\big|f(x,u)\big| \leq a_2(x)|u|^{r_2-1}$$

for all $x \in R^3$ *and* $|u| \geq M$;

(W_3) *For every* $m > \delta$, *there exist a constant* $r_3 \in (1,2)$ *and a function* $b_m \in L^{\frac{2}{2-r_3}}(R^3, [0,+\infty))$ *such that*

$$\big|f(x,u)\big| \leq b_m(x)$$

for all $x \in R^3$ *and* $|u| \leq m$;

(W_4) *There exist constants* $r_4 \in (1,2)$, $\eta > 0$ *and* $\zeta > 0$ *such that*

$$F(x,u) \geq \eta |u|^{r_4}$$

for all $x \in \Omega$ *and* $|u| \leq \zeta$, *where* $\operatorname{meas}\{\Omega\} > 0$, $F(x,u) := \int_0^u f(x,y)\,dy$;

(W_5) $F(x,-u) = F(x,u)$ *for all* $x \in R^3$ *and* $u \in R$.

Then problem (1) *has infinitely many nontrivial solutions* $\{(u_k,\phi_k)\}$ *satisfying*

$$\frac{1}{2}\int_{R^3}\left(|\nabla u_k|^2 + V(x)u_k^2\right)dx - \frac{1}{4}\int_{R^3}|\nabla\phi_k|^2\,dx + \frac{1}{2}\int_{R^3}\phi_k u_k^2\,dx - \int_{R^3}F(x,u_k)\,dx \to 0^-$$

as $k \to \infty$.

By Theorem 1.2, we obtain the following corollary.

Corollary 1.3 *Assume that L satisfies* (V_1) *and W satisfies*

(W_6) $F(x,u) = a(x)|u|^r$, *where* $F(x,u) = \int_0^u f(x,y)\,dy$, $1 < r < 2$ *is a constant and* $a : R^3 \to R$ *is a function such that* $a \in L^{\frac{2}{2-r}}(R^3)$ *and* $a(x) > 0$ *for* $x \in \Omega$, *where* $\operatorname{meas}\{\Omega\} > 0$.

Then problem (1) *has infinitely many nontrivial solutions* $\{(u_k,\phi_k)\}$ *satisfying*

$$\frac{1}{2}\int_{R^3}\left(|\nabla u_k|^2 + V(x)u_k^2\right)dx - \frac{1}{4}\int_{R^3}|\nabla\phi_k|^2\,dx + \frac{1}{2}\int_{R^3}\phi_k u_k^2\,dx - \int_{R^3}F(x,u_k)\,dx \to 0^-$$

as $k \to \infty$.

Remark 1.4 In Theorem 1.2, infinitely many solutions for problem (1) are obtained under the symmetry condition (W_5) by using the dual fountain theorem. As a special case of Theorem 1.2, Corollary 1.3 generalizes and improves Theorem 1.1. To show this, it suffices to compare (V_1') and (V_1), (H_1) and (W_6). Firstly, it is clear that (V_1) is really weaker than (V_1'). Secondly, in (H_1) a is assumed to be positive, while in (W_6) we assume that a is indefinite.

Moreover, under all the conditions of Theorem 1.2 except (W_5) we obtain an existence result.

Theorem 1.5 *Assume that L satisfies* (V_1) *and W satisfies* (W_1), (W_2), (W_3), (W_4). *Then problem* (1) *possesses a nontrivial solution.*

Remark 1.6 In Theorem 1.5 we obtain the existence of solutions for problem (1) under the assumption that $f(x,u)$ is indefinite and without any coercive assumptions respect to V such as (V_1'). There are functions V and f which satisfy Theorem 1.5, but do not satisfy the corresponding results in [2–16]. For example,

$$V(x) \equiv 1, \qquad f(x,u) = \tilde{a}(x)|u|^{\frac{3}{2}} \tag{2}$$

and

$$\tilde{a}(x) = \begin{cases} (-1)^n n^3(|x| - n) & \text{for } n \leq |x| \leq n + \frac{1}{n^2}, \\ 0 & \text{else}, \end{cases} \tag{3}$$

in which $n \geq 3$. It is clear that $\tilde{a} \in C(R^3, R)$ is indefinite. Denoting by π the area of the unit ball in R^3, we obtain

$$\begin{aligned}
\int_{R^3} \tilde{a}^4(x)\,dx &= \sum_{n=3}^{\infty} \left(\int_n^{n+\frac{1}{n^2}} n^{12} r^2 (r-n)^4\,dr + \int_{n+\frac{1}{n^2}}^{n+\frac{2}{n^2}} n^{12} r^2 \left(n + \frac{2}{n^2} - r \right)^4 dr \right) \pi \\
&= \pi \sum_{n=3}^{\infty} 2n^{12} \int_0^{\frac{1}{n^2}} r^6\,dx \\
&= \frac{2\pi}{7} \sum_{n=3}^{\infty} n^{-2} \\
&< \infty,
\end{aligned} \tag{4}$$

which means that $\tilde{a} \in L^{\frac{2}{2-\frac{3}{2}}}(R^3)$. So, (2) satisfies our results, but does not satisfy the results in [3–17].

2 Preliminary results

In order to establish our results via critical point theory, we firstly describe some properties of the space $H^1(R^3)$, on which the variational functional associated with problem (1) is defined. Define the function space

$$H^1(R^3) := \{ u \in L^2(R^3) : \nabla u \in (L^2(R^3))^3 \}$$

equipped with the norm

$$\|u\|_{H^1} := \left(\int_{R^3} (|\nabla u|^2 + u^2)\,dx \right)^{1/2}$$

and the function space

$$D^{1,2}(R^3) := \{ u \in L^{2^*} : \nabla u \in (L^2(R^3))^3 \}$$

with the norm

$$\|u\|_{D^{1,2}} = \left(\int_{R^3} |\nabla u|^2\,dx \right)^{1/2}.$$

Let

$$E := \left\{ u \in H^1(R^3) : \int_{R^3} V(x) u^2\,dx < +\infty \right\}$$

equipped with the inner product

$$(u, v) = \int_{R^3} (\nabla u \cdot \nabla v + V(x) uv)\,dx$$

and the corresponding norm

$$\|u\|^2 = (u, u).$$

Note that the following embeddings

$$E \hookrightarrow L^s(R^3), \quad 2 \le s \le 2^*, \qquad D^{1,2}(R^3) \hookrightarrow L^{2^*}(R^3)$$

are continuous, where $2^* = 6$ is the critical exponent for the Sobolev embeddings in dimension 3. Therefore, there exist constants C_p and C_* such that

$$\|u\|_{L^p} \le C_p \|u\|, \qquad \|u\|_{L^{2^*}} \le C_* \|u\|_{D^{1,2}} \tag{5}$$

for all $u \in E$. Here $L^p(R^3)$ $(2 \le p \le 2^*)$ denotes the Banach spaces of a function on R^3 with values in R under the norm

$$\|u\|_{L^p} = \left(\int_{R^3} |u(x)|^p \, dx \right)^{1/p}.$$

Let

$$L^r_a(R^3) := \left\{ u : R^3 \to R : \int_{R^3} a(x)|u|^r \, dx < +\infty \right\},$$

where $a(x) > 0$ for a.e. $x \in R^3$. Then $L^r_a(R^3)$ is a Banach space with the norm

$$\|u\|_{L^r_a} = \left(\int_{R^3} a(x)|u|^r \, dx \right)^{1/r}.$$

Lemma 2.1 *Suppose that assumption* (V_1) *holds. Then the embedding of* E *in* $L^r_a(R^3)$ *is compact, where* $r \in (1,2)$, $a \in L^{\frac{2}{2-r}}(R^3)$ *is positive for a.e.* $x \in R^3$.

Proof For any bounded set $K \subset E$, there exists a positive constant M_0 such that $\|u\| \le M_0$ for all $u \in K$. We claim that K is precompact in $L^r_a(R^3)$. In fact, since $a \in L^{\frac{2}{2-r}}(R^3)$, for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\left(\int_{|x| \ge T_\varepsilon} a(x)^{\frac{2}{2-r}} \, dx \right)^{(2-r)/2} < \varepsilon.$$

For any $u, v \in K$, applying the Hölder inequality for r such that $\frac{r}{2} + \frac{2-r}{2} = 1$ and the first inequality in (5), we have

$$\begin{aligned} \int_{|x| \ge T_\varepsilon} a(x)|u-v|^r \, dx &\le \left(\int_{|x| \ge T_\varepsilon} a(x)^{\frac{2}{2-r}} \, dx \right)^{(2-r)/2} \left(\int_{|x| \ge T_\varepsilon} |u-v|^2 \, dx \right)^{r/2} \\ &\le \|u-v\|^r_{L^2} \left(\int_{|x| \ge T_\varepsilon} a(x)^{\frac{2}{2-r}} \, dx \right)^{(2-r)/2} \\ &\le C_2^r \|u-v\|^r \varepsilon \\ &\le 2C_2^r M_0^r \varepsilon. \end{aligned} \tag{6}$$

Besides, since $E(B_{T_\varepsilon}(0)) \subset H^1(B_{T_\varepsilon}(0))$ is compactly embedded in $L^r_a(B_{T_\varepsilon}(0))$, where $B_{T_\varepsilon}(0) = \{x \in R^3 : |x| \le T_\varepsilon\}$, there are $u_1, u_2, \dots, u_m \in K$ such that for any $u \in K$,

$$\int_{|x| \le T_\varepsilon} a(x)|u-u_i|^r \, dx < \varepsilon. \tag{7}$$

Now it follows from (6) and (7) that K is precompact in $L_a^r(R^3)$. Obviously, we have E is compact embedded in $L_a^r(R^3)$, where $r \in (1,2)$, $a \in L^{\frac{2}{2-r}}(R^3)$ is positive for a.e. $x \in R^3$. □

Lemma 2.2 *Assume that assumptions* (V_1), (W_1), (W_2) *and* (W_3) *hold and* $u_n \rightharpoonup u$ *in* E. *Then*

$$f(x,u_n) \to f(x,u)$$

in $L^2(R^3)$.

Proof Assume that $u_n \rightharpoonup u$ in E. Then, by Lemma 2.1,

$$u_n \to u$$

in $L_a^r(R^3)$, where $r \in (1,2)$, $a \in L^{\frac{2}{2-r}}(R^3)$ is positive for a.e. $x \in R^3$. Passing to a subsequence if necessary, it can be assumed that

$$\sum_{n=1}^{\infty} \|u_n - u\|_{L_a^r} < +\infty.$$

It is clear that

$$h_k(x) := \sum_{n=1}^{k} \left|u_n(x) - u(x)\right| \in L_a^r\left(R^3\right) \tag{8}$$

and

$$\|h_g - h_l\|_{L_a^r} \le \sum_{n=l}^{g} \|u_n - u\|_{L_a^r} \tag{9}$$

for all $g > l \in N^+$. Since $\{u_n\}$ is a Cauchy sequence in $L_a^r(R^3)$, so by (9) we know that $\{h_k\}$ is also a Cauchy sequence in $L_a^r(R^3)$. Therefore, by the completeness of $L_a^r(R^3)$, there exists $h \in L_a^r(R^3)$ such that $h_k \to h$ in $L_a^r(R^3)$. Now we show that

$$h_k(x) \le h(x) \tag{10}$$

for all $k \in N^+$ and almost every $x \in R^3$. If not, there exist $k_0 \in N^+$ and $S \subset R^3$, with $\mathrm{meas}\{S\} > 0$, such that

$$h_{k_0}(x) > h(x)$$

for all $x \in S$. Then there exist a constant $c > 0$ and $S_0 \subset S$, with $\mathrm{meas}\{S_0\} > 0$, such that

$$h_{k_0}(x) \ge h(x) + c$$

for all $x \in S_0$. By the definition of h_k, we have

$$h_k(x) \geq h_{k_0}(x) \geq h(x) + c$$

for all $k \geq k_0$ and $x \in S_0$. Therefore, one has

$$\begin{aligned}\int_{R^3} a(x)|h_k - h|^r\,dx &\geq \int_{S_0} a(x)|h_k - h|^r\,dx\\ &\geq c^r \int_{S_0} a(x)\,dx.\end{aligned}$$

Letting $k \to \infty$, we get

$$0 \geq c^r \int_{S_0} a(x)\,dx,$$

which contradicts the fact that $a(x) > 0$ for a.e. $x \in R^3$. Now we have proved (10). It follows from (W_2) that there exists $M > 0$ such that

$$\bigl|f(x,u)\bigr| \leq a_2(x)|u|^{r_2-1} \tag{11}$$

for all $x \in R^3$ and $|u| \geq M$. By (W_1), there exists $\delta > 0$ such that

$$\bigl|f(x,u)\bigr| \leq a_1(x)|u|^{r_1-1} \tag{12}$$

for all $x \in R^3$ and $|u| \leq \delta$, which together with (W_3) shows there exists $b_M \in L^{\frac{2}{2-r_3}}(R^3)$ such that

$$\bigl|f(x,u)\bigr| \leq a_1(x)|u|^{r_1-1} + \frac{b_M(x)}{\delta^{r_3-1}}|u|^{r_3-1} \tag{13}$$

for all $x \in R^3$ and $|u| \leq M$. Combining (11) and (13), we have

$$\bigl|f(x,u)\bigr| \leq a_1(x)|u|^{r_1-1} + a_2(x)|u|^{r_2-1} + \frac{b_M}{\delta^{r_3-1}}|u|^{r_3-1} \tag{14}$$

for all $x \in R^3$ and $u \in R$. Hence, by (10) one has

$$\begin{aligned}\bigl|f(x,u_n) - f(x,u)\bigr| &\leq a_1(x)\bigl(|u_n|^{r_1-1} + |u|^{r_1-1}\bigr) + a_2(x)\bigl(|u_n|^{r_2-1} + |u|^{r_2-1}\bigr)\\ &\quad + \frac{b_M(x)}{\delta^{r_3-1}}\bigl(|u_n|^{r_3-1} + |u|^{r_3-1}\bigr)\\ &\leq a_1(x)\bigl(|u_n - u|^{r_1-1} + 2\bigl|u(x)\bigr|^{r_1-1}\bigr) + a_2(x)\bigl(|u_n - u|^{r_2-1} + 2|u|^{r_2-1}\bigr)\\ &\quad + \frac{b_M(x)}{\delta^{r_3-1}}\bigl(|u_n - u|^{r_3-1} + 2|u|^{r_3-1}\bigr)\\ &\leq a_1(x)\bigl(|h|^{r_1-1} + 2|u|^{r_1-1}\bigr) + a_2(x)\bigl(|h|^{r_2-1} + 2|u|^{r_2-1}\bigr)\\ &\quad + \frac{b_M(x)}{\delta^{r_3-1}}\bigl(|h|^{r_3-1} + 2|u|^{r_3-1}\bigr)\end{aligned}$$

for all $n \in N$ and $x \in R^3$. It follows that

$$\begin{aligned}\left|f(x,u_n)-f(x,u)\right|^2 dx &\le 6a_1^2(x)\left(|h|^{2(r_1-1)}+4|u|^{2(r_1-1)}\right)dx \\ &\quad + 6a_2^2(x)\left(|h|^{2(r_2-1)}+4|u|^{2(r_2-1)}\right)dx \\ &\quad + \frac{6b_M^2(x)}{\delta^{2(r_3-1)}}\left(|h|^{2(r_3-1)}+4|u|^{2(r_3-1)}\right)dx \\ &=: \varrho(x) \end{aligned} \tag{15}$$

for all $n \in N$. By the Hölder inequality, we have

$$\begin{aligned}\int_{R^3} a_1^2(x)|h|^{2(r_1-1)}\,dx &\le \left(\int_{R^3} a_1(x)^{\frac{2}{2-r_1}}\,dx\right)^{\frac{2-r_1}{r_1}}\left(\int_{R^3} a_1(x)|h|^{r_1}\,dx\right)^{\frac{2(r_1-1)}{r_1}} \\ &= \|a_1\|^{\frac{2}{r_1}}_{L^{\frac{2}{2-r_1}}}\|h\|^{2(r_1-1)}_{L^{r_1}_{a_1}} \\ &< \infty. \end{aligned} \tag{16}$$

Similarly, we can prove

$$\begin{aligned} &\int_{R^3} a_1^2(x)|u|^{2(r_1-1)}\,dx < \infty, \qquad \int_{R^3} a_2^2(x)|h|^{2(r_2-1)}\,dx < \infty, \\ &\int_{R^3} a_2^2(x)|u|^{2(r_2-1)}\,dx < \infty, \end{aligned} \tag{17}$$

also

$$\int_{R^3} b_M^2(x)|h|^{2(r_3-1)}\,dx < \infty, \qquad \int_{R^3} b_M^2(x)|u|^{2(r_3-1)}\,dx < \infty. \tag{18}$$

It follows from (15), (16), (17) and (18) that

$$\varrho \in L^1\left(R^3\right),$$

which together with Lebesgue's convergence theorem shows

$$\int_{R^3}\left|f(x,u_n)-f(x,u)\right|^2 dx \to 0 \tag{19}$$

as $n \to \infty$. Now we have proved the lemma. □

In the proof of Theorem 1.2, the following lemma is needed.

Lemma 2.3 *Assume that $G \subset R^3$ is an open set. Then, for any closed set $H \subset G$, there exists a function $\varphi \in C_0^\infty(R^3)$ such that $\varphi(x) = 0$ for all $x \in R^3 \setminus G$, $\varphi(x) = 1$ for all $x \in H$ and $0 \le \phi(x) \le 1$ for all $x \in G \setminus H$.*

Proof Letting

$$\tilde{\alpha}(x) = \begin{cases} e^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

then $\tilde{\alpha} \in C_0^\infty(R^3)$ and $\operatorname{supp}\tilde{\alpha} = B_1(0)$. For any given $\varepsilon > 0$, defining α and α_ε as follows,

$$\alpha(x) = \frac{\tilde{\alpha}(x)}{\int_{R^3} \tilde{\alpha}(x)\,dx}, \qquad \alpha_\varepsilon(x) = \frac{1}{\varepsilon^3}\alpha\left(\frac{x}{\varepsilon}\right),$$

one has $\alpha_\varepsilon \in C_0^\infty(R^3)$, $\operatorname{supp}\alpha_\varepsilon = \{x : |x| \leq \varepsilon\}$ and $\int_{R^3} \alpha_\varepsilon(x)\,dx = 1$. Denoting

$$d_0 = \inf_{x\in H, y\in\partial G} d(x,y)$$

and

$$G_\theta := \{x \in G, d(x,\partial G) \geq \theta\},$$

it is clear that $d_0 > 0$ and $H \subset G_{d_0}$. Lastly, we define

$$\psi(x) = \begin{cases} 1, & x \in G_{\frac{d_0}{2}}, \\ 0, & x \in R \setminus G_{\frac{d_0}{2}} \end{cases}$$

and

$$\varphi(x) = \int_{R^3} \psi(x-y)\alpha_{\frac{d_0}{4}}(y)\,dy,$$

then $\varphi(x) = 1$ for all $x \in H$ and $\varphi(x) = 0$ for all $x \in G_{\frac{d_0}{4}}$. Moreover, by the definition of α_ε, we have $\varphi \in C_0^\infty(R^3)$ and $0 \leq \varphi(x) \leq 1$. □

Since E is a Hilbert space, then there exists a basis $\{v_n\} \subset X$ such that $X = \overline{\bigoplus_{j\geq 1} X_j}$, where $X_j = \operatorname{span}\{v_j\}$. Letting $Y_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j\geq k} X_j}$, now we show the following lemma, which will be used in the proof of Theorem 1.2.

Lemma 2.4 *Suppose $r \in (1,2)$ and $a \in L^{\frac{2}{2-r}}(R^3)$, then we have*

$$\beta_k(a,r) := \sup_{u\in Z_k, \|u\|=1} \|u\|_{L_a^r} \to 0$$

as $k \to \infty$.

Proof It is clear that $0 < \beta_{k+1}(a,r) \leq \beta_k(a,r)$, so there exists $\beta(a,r) \geq 0$ such that

$$\beta_k(a,r) \to \beta(a,r) \tag{20}$$

as $k \to \infty$. By the definition of $\beta_k(a,r)$, there exists $u_k \in Z_k$ with $\|u_k\| = 1$ such that

$$\|u_k\|_{L_a^r} > \frac{\beta_k(a,r)}{2}. \tag{21}$$

Since $\{u_k\}_{k\in N}$ is bounded, then there exists $u \in E$ such that

$$u_k \rightharpoonup u$$

as $k \to \infty$. Now, since $\{v_j\}$ is a basis of E, it follows that for all $j \in N$,

$$\begin{aligned} 0 &= (u_k, v_j) \quad \forall k > j \\ &\to (u, v_j) \end{aligned}$$

as $k \to \infty$, which shows that $u = 0$. By Lemma 2.1 we have

$$u_k \to 0$$

in $L_a^r(R^3)$ for all $r \in (1,2)$ and $a \in L^{\frac{2}{2-r}}(R^3)$, which together with (20) and (21) implies that $\beta(a,r) = 0$ for all $r \in (1,2)$ and $a \in L^{\frac{2}{2-r}}(R^3)$. □

We obtain the existence of a solution for problem (1) by using the following standard minimizing argument.

Lemma 2.5 [19] *Let E be a real Banach space and $\Phi \in C^1(E,R)$ satisfying the (PS) condition. If Φ is bounded from below,*

$$c := \inf_E \Phi$$

is a critical value of Φ.

In order to prove the multiplicity of solutions, we will use the dual fountain theorem. Firstly, we introduce the definition of the $(PS)_c^*$ condition.

Definition 2.6 Let $\Phi \in C^1(E,R)$ and $c \in R$. The function Φ satisfies the $(PS)_c^*$ condition if any sequence $\{u_{n_j}\} \in E$, such that

$$\Phi(u_{n_j}) \to c, \qquad \Phi|'_{Y_{n_j}}(u_{n_j}) \to 0 \quad \text{as } n_j \to \infty,$$

contains a subsequence converging to a critical point of Φ.

Now we show the following dual fountain theorem.

Lemma 2.7 [20] *If $\Phi(-u) = \Phi(u)$ and for every $k \geq k_0$, there exists $\rho_k > \gamma_k > 0$ such that*

(i) $a_k := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi(u) \geq 0$,

(ii) $b_k := \max_{u \in Y_k, \|u\| = \gamma_k} \Phi(u) < 0$,

(iii) $d_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi(u) \to 0$ *as* $k \to \infty$.

Moreover, if $\Phi \in C^1(X,R)$ satisfies the $(PS)_c^*$ condition for all $c \in [d_{k_0}, 0)$, then Φ has a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \to 0^-$ as $k \to \infty$.

3 Proof of theorems

Define the functional $I : E \times D^{1,2}(R^3) \to R$ by

$$I(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{R^3} |\nabla\phi|^2\,dx + \frac{1}{2}\int_{R^3} \phi u^2\,dx - \int_{R^3} F(x,u)\,dx. \tag{22}$$

It is easy to know that I exhibits a strong indefiniteness, namely it is unbounded both from below and from above on an infinitely dimensional subspace. This indefiniteness can be removed using the reduction method described in [1], by which we are led to study a variable functional that does not present such a strong indefinite nature.

Now we recall this method. For any $u \in E$, consider the linear functional $T_u : D^{1,2}(R^3) \to R$ defined as

$$T_u(v) = \int_{R^3} u^2 v\,dx.$$

By the Hölder inequality and using the second inequality in (5), we have

$$\begin{aligned}\int_{R^3} u^2 v\,dx &\le \|u^2\|_{L^{6/5}} \|v\|_{L^6} \\ &\le \|u\|_{L^{12/5}} \|v\|_{L^6} \\ &\le C_{12/5} C_* \|u\|^2 \|v\|_{D^{1,2}}.\end{aligned}$$

So, T_u is continuous on $D^{1,2}(R^3)$. Set

$$\mu(u,v) = \int_{R^3} \nabla u \cdot \nabla v\,dx$$

for all $u, v \in D^{1,2}(R^3)$. Obviously, $\mu(u,v)$ is bilinear, bounded and coercive. Hence, the Lax-Milgram theorem implies that for every $u \in E$, there exists a unique $\phi_u \in D^{1,2}(R^3)$ such that

$$T_u(v) = \mu(\phi_u, v)$$

for any $v \in D^{1,2}(R^3)$, that is,

$$\int_{R^3} u^2 v\,dx = \int_{R^3} \nabla\phi_u \cdot \nabla v\,dx$$

for any $v \in D^{1,2}(R^3)$. Using integration by parts, we get

$$\int_{R^3} \nabla\phi_u \cdot \nabla v\,dx = -\int_{R^3} v\Delta\phi_u\,dx$$

for any $v \in D^{1,2}(R^3)$, therefore

$$-\Delta\phi_u = u^2 \tag{23}$$

in a weak sense. We can write an integral expression for ϕ_u in the form

$$\phi_u = \frac{1}{4\pi}\int_{R^3} \frac{u^2(y)}{|x-y|}\,dy$$

for any $u \in C_0^\infty(R^3)$ (see [21], Theorem 1); by density it can be extended for any $u \in E$ (see Lemma 2.1 of [22]). Clearly, $\phi_u \ge 0$ and $\phi_{-u} = \phi_u$ for all $u \in E$.

It follows from (23) that

$$\int_{R^3} \phi_u u^2\,dx = \int_{R^3} \phi_u(-\Delta\phi_u)\,dx = \int_{R^3} |\nabla\phi_u|^2\,dx, \tag{24}$$

and by the Hölder inequality, we have

$$\begin{aligned}\|\phi_u\|^2_{D^{1,2}} &= \int_{R^3} \phi_u u^2\,dx \\ &\le \left(\int_{R^3} \phi_u^6\,dx\right)^{1/6} \left(\int_{R^3} |u|^{\frac{12}{5}}\right)^{5/6} \\ &= C_*\|\phi_u\|_{D^{1,2}}\|u\|^2_{L^{12/5}},\end{aligned}$$

and it follows that

$$\|\phi_u\|_{D^{1,2}} \le C_*\|u\|^2_{L^{12/5}}. \tag{25}$$

Hence,

$$\int_{R^3} \phi_u u^2\,dx \le {C_*}^2\|u\|^4_{L^{12/5}} \le {C_*}^2 C^4_{12/5}\|u\|^4 := C\|u\|^4. \tag{26}$$

So, we can consider the functional $\Phi : E \to R$ defined by $\Phi(u) = I(u,\phi_u)$. By (24), the reduced functional takes the form

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{R^3} \phi_u u^2\,dx - \int_{R^3} F(x,u)\,dx. \tag{27}$$

By (12), we have

$$\left|F(x,u)\right| \le \frac{a_1(x)}{r_1}|u|^{r_1} \tag{28}$$

for all $x \in R^3$ and $|u| \le \delta$, where $r_1 \in (1,2)$ and $a_1 \in L^{\frac{2}{2-r_1}}(R^3)$. Let $u \in E$, then $u \in C^0(R^3)$, the space of continuous function u on R^3, such that $u(x) \to 0$ as $|x| \to \infty$. Therefore there exists $T_1 > 0$ such that

$$\left|u(x)\right| \le \delta \tag{29}$$

for all $|x| > T_1$. Hence, one has

$$\begin{aligned}\int_{|x|>T_1} \left|F(x,u)\right|dx &\le \int_{|x|>T_1} \frac{a_1(x)}{r_1}\left|u(x)\right|^{r_1}dx \\ &\le \frac{1}{r_1}\left(\int_{|x|\ge T_1} a_1(x)^{\frac{2}{2-r_1}}\,dx\right)^{(2-r_1)/2}\left(\int_{|x|\ge T_1}\left|u(x)\right|^2 dx\right)^{r_1/2} \\ &\le \frac{1}{r_1}\left(\int_{|x|\ge T_1} a_1(x)^{\frac{2}{2-r_1}}\,dx\right)^{(2-r_1)/2}\|u\|^{r_1}_{L^2} \\ &\le \frac{1}{r_1}C_2^{r_1}\|u\|^{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}} \\ &< \infty,\end{aligned}$$

which together with (26) shows that Φ is well defined. Furthermore, it is well known that Φ is a C^1 functional with derivative given by

$$\langle \Phi'(u), v\rangle = \int_{R^3} \big[(\nabla u \cdot \nabla v) + V(x)uv + \phi_u uv - f(x,u)v\big]\,dx.$$

It can be proved that $(u,\phi) \in E \times D^{1,2}(R^3)$ is a solution of problem (1) if and only if $u \in E$ is a critical point of the functional Φ and $\phi = \phi_u$; see, for instance, [1].

Lemma 3.1 *Under conditions* (V_1), (W_1), (W_2), (W_3), Φ *satisfies the* $(PS)_c^*$ *condition.*

Proof Assume that $\{u_{n_j}\} \subset E$ is a sequence such that

$$\Phi(u_{n_j}) \to c, \qquad \Phi|'_{Y_{n_j}}(u_{n_j}) \to 0 \quad \text{as } n_j \to \infty.$$

Then there exists $\sigma > 0$ such that

$$\big|\Phi(u_{n_j})\big| \le \sigma, \qquad \big\|\Phi|'_{Y_{n_j}}(u_{n_j})\big\|_E^* \le \sigma$$

for all $n_j \in N$.

Firstly, we show that $\{u_{n_j}\}$ is bounded. By (14), we have

$$\big|F(x,u)\big| \le \frac{a_1(x)}{r_1}|u|^{r_1} + \frac{a_2(x)}{r_2}|u|^{r_2} + \frac{b_M(x)}{r_3\delta^{r_3-1}}|u|^{r_3} \tag{30}$$

for all $u \in R$ and $x \in R^3$, which together with $\int_{R^3} \phi_{u_{n_j}} u_{n_j}^2\,dx \ge 0$ implies

$$\begin{aligned}
\|u_{n_j}\|^2 &= 2\Phi(u_{n_j}) - \frac{1}{2}\int_{R^3} \phi_{u_{n_j}} u_{n_j}^2\,dx + 2\int_{R^3} F(x,u_{n_j})\,dx \\
&\le 2\sigma + \frac{2}{r_1}\int_{R^3} a_1(x)|u_{n_j}|^{r_1}\,dx + \frac{2}{r_2}\int_{R^3} a_2(x)|u_{n_j}|^{r_2}\,dx \\
&\quad + \frac{2}{r_3\delta^{r_3-1}}\int_{R^3} b_M(x)|u_{n_j}|^{r_3}\,dx \\
&\le 2\sigma + \frac{2}{r_1}\left(\int_{R^3} a_1(x)^{\frac{2}{2-r_1}}\,dx\right)^{(2-r_1)/2}\left(\int_{R^3} |u_{n_j}|^2\,dx\right)^{r_1/2} \\
&\quad + \frac{2}{r_2}\left(\int_{R^3} a_2(x)^{\frac{2}{2-r_2}}\,dx\right)^{(2-r_2)/2}\left(\int_{R^3} |u_{n_j}|^2\,dx\right)^{r_2/2} \\
&\quad + \frac{2}{r_3\delta^{r_3-1}}\left(\int_{R^3} b_M(x)^{\frac{2}{2-r_3}}\,dx\right)^{(2-r_3)/2}\left(\int_{R^3} |u_{n_j}|^2\,dx\right)^{r_3/2} \\
&\le 2\sigma + \frac{2}{r_1}C_2^{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}\|u_{n_j}\|^{r_1} + \frac{2}{r_2}C_2^{r_2}\|a_2\|_{L^{\frac{2}{2-r_2}}}\|u_{n_j}\|^{r_2} \\
&\quad + \frac{2}{r_3\delta^{r_3-1}}C_2^{r_3}\|b_M\|_{L^{\frac{2}{2-r_3}}}\|u_{n_j}\|^{r_3}.
\end{aligned} \tag{31}$$

Noting that $r_i < 2$ for all $i = 1,2,3$, so $\|u_{n_j}\|$ is bounded.

By the fact that $\{u_{n_j}\}$ is bounded in E, there exists $u \in E$ and a constant $d > 0$ such that

$$\sup_{n_j \in N}\|u_{n_j}\| \le d, \qquad \|u\| \le d \tag{32}$$

and

$$u_{n_j} \rightharpoonup u$$

in E as $n_j \to \infty$. It is obvious that

$$\langle \Phi'(u_{n_j}) - \Phi'(u), u \rangle \to 0 \tag{33}$$

and

$$\phi_u u(u_{n_j} - u) \to 0 \tag{34}$$

as $n_j \to \infty$. On the other hand, by (V_1), (32) and Lemma 2.2, one has

$$\begin{aligned} \left| \int_{R^3} \big(f(x,u_{n_j}) - f(x,u)\big) u_{n_j}\, dx \right| &\le \big\| f(x,u_{n_j}) - f(x,u) \big\|_{L^2} \|u_{n_j}\|_{L^2} \\ &\le C_2 \big\| f(x,u_{n_j}) - f(x,u) \big\|_{L^2} \|u_{n_j}\| \\ &\le C_2 d \big\| f(x,u_{n_j}) - f(x,u) \big\|_{L^2} \\ &\to 0 \end{aligned} \tag{35}$$

as $n_j \to \infty$, which implies

$$\langle \Phi'(u_{n_j}) - \Phi'(u), u_{n_j} \rangle \to 0 \tag{36}$$

as $n_j \to \infty$. Summing up (33) and (36), we have

$$\langle \Phi'(u_{n_j}) - \Phi'(u), u_{n_j} - u \rangle \to 0 \tag{37}$$

as $n_j \to \infty$. By the Hölder inequality and (25), one gets

$$\begin{aligned} \int_{R^3} \phi_{u_{n_j}} u_{n_j} (u_{n_j} - u)\, dx &\le \|\phi_{u_{n_j}} u_{n_j}\|_{L^2} \|u_{n_j} - u\|_{L^2} \\ &\le \|\phi_{u_{n_j}}\|_{L^6} \|u_{n_j}\|_{L^3} \|u_{n_j} - u\|_{L^2} \\ &\le C_* \|\phi_{u_{n_j}}\|_{D^{1,2}} \|u_{n_j}\|_{L^3} \|u_{n_j} - u\|_{L^2} \\ &\le {C_*}^2 \|u_{n_j}\|^2_{L^{12/5}} \|u_{n_j}\|_{L^3} \|u_{n_j} - u\|_{L^2} \\ &\le {C_*}^2 C^2_{12/5} C_3 C_2 \|u_{n_j}\|^3 \|u_{n_j} - u\| \\ &\le 2{C_*}^2 C^2_{12/5} C_3 C_2 d^4 \\ &< \infty. \end{aligned}$$

Then by Lebesgue's convergence theorem, we have

$$\int_{R^3} \phi_{u_{n_j}} u_{n_j} (u_{n_j} - u)\, dx \to 0$$

as $n_j \to \infty$, which together with (34) implies

$$\int_{R^3} (\phi_{u_{n_j}} u_{n_j} - \phi_u u)(u_{n_j} - u)\,dx \to 0 \tag{38}$$

as $n_j \to \infty$. By Lemma 2.2 and (32), we get

$$\begin{aligned}\left|\int_{R^3} \bigl(f(x,u_{n_j}) - f(x,u)\bigr)(u_{n_j} - u)\,dx\right| &\le \bigl\|f(x,u_{n_j}) - f\bigl(x,u(x)\bigr)\bigr\|_{L^2} \|u_{n_j} - u\|_{L^2} \\ &\le C_2 \bigl\|f(x,u_{n_j}) - f(x,u)\bigr\|_{L^2} \|u_{n_j} - u\| \\ &\le 2C_2 d \bigl\|f(x,u_{n_j}) - f(x,u)\bigr\|_{L^2} \\ &\to 0\end{aligned}$$

as $n_j \to \infty$. Moreover, an easy computation shows that

$$\begin{aligned}\bigl\langle \Phi'(u_{n_j}) - \Phi'(u), u_{n_j} - u\bigr\rangle = \|u_{n_j} - u\|^2 &+ \int_{R^3} (\phi_{u_{n_j}} u_{n_j} - \phi_u u)(u_{n_j} - u)\,dx \\ &- \int_{R^3} \bigl(f(x,u_{n_j}) - f(x,u)\bigr)(u_{n_j} - u)\,dx.\end{aligned}$$

Consequently, $\|u_{n_j} - u\| \to 0$ as $n_j \to \infty$. Φ satisfies the $(PS)_c^*$ condition. □

Remark 3.2 Under conditions (V_1), (W_1), (W_2), (W_3), Φ satisfies the (PS) condition. Assume that $\{u_n\} \subset E$ is a sequence such that $I(u_n)$ is bounded and

$$I'(u_n) \to 0$$

as $n \to \infty$. Then there exists $\sigma > 0$ such that

$$\bigl|I(u_n)\bigr| \le \sigma, \qquad \bigl\|I'(u_n)\bigr\|_E^* \le \sigma$$

for all $n \in N$. The rest of the proof is the same as that of Lemma 3.1.

Proof of Theorem 1.2 For any $k \in N$, we take k disjoint open sets $\{\Omega_i | i = 1, \dots, k\}$ such that

$$\bigcup_{i=1}^k \Omega_i \subset \Omega.$$

For any $\varepsilon > 0$ and Ω_i, there exist a closed set H_i and an open set G_i such that $H_i \subset \Omega_i \subset G_i$ and

$$\operatorname{meas}\{G_i \setminus \Omega_i\} < \varepsilon, \qquad \operatorname{meas}\{\Omega_i \setminus H_i\} < \varepsilon.$$

For every G_i $(i = 1, \dots, k)$, by Lemma 2.3 there exists $\varphi_i \in C_0^\infty(G_i, R)$ such that $\varphi_i|_{H_i} = 1$ and $0 \le \varphi_i \le 1$. Letting $v_i = \frac{\varphi_i}{\|\varphi_i\|}$, can be extended to be a basis $\{v_n\} \subset X$. Therefore $X = \overline{\bigoplus_{j\ge 1} X_j}$, where $X_j = \operatorname{span}\{v_j\}$. Now we define $Y_k := \bigoplus_{j=1}^k X_j$, $Z_k := \overline{\bigoplus_{j\ge k} X_j}$.

By Lemma 3.1, $\Phi \in C^1(E,R)$ satisfies the $(PS)_c^*$ condition and $\Phi(u) = \Phi(-u)$. Hence, to prove Theorem 1.2, we should just show that Φ has the geometric property (i), (ii) and (iii) in Lemma 2.7.

(i) By Lemma 2.4

$$\beta_k(a,r) = \sup_{u\in Z_k,\|u\|=1} \|u\|_{L_a^r} \to 0$$

as $k \to \infty$ for $r \in (1,2)$ and $a \in L^{\frac{2}{2-r}}(R^3)$. In view of (30) and the fact that $\int_{R^3} \phi_u u^2\,dx \geq 0$, we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{R^3} \phi_u u^2\,dx - \int_{R^3} F(x,u)\,dx \\
&\geq \frac{1}{2}|u|^2 - \int_{R^3} F(x,u)\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{2}{r_1}\int_{R^3} a_1(x)|u|^{r_1}\,dx - \frac{2}{r_2}\int_{R^3} a_2(x)|u|^{r_2}\,dx \\
&\quad - \frac{2}{r_3\delta^{r_3-1}}\int_{R^3} b_M(x)|u|^{r_3}\,dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{2\|u\|_{L_{a_1}^{r_1}}^{r_1}}{r_1} - \frac{2\|u\|_{L_{a_2}^{r_2}}^{r_2}}{r_2} - \frac{2\|u\|_{L_{a_3}^{r_3}}^{r_3}}{r_3\delta^{r_3-1}} \\
&\geq \frac{1}{2}\|u\|^2 - \frac{2\beta_k(a_1,r_1)^{r_1}}{r_1}\|u\|^{r_1} - \frac{2\beta_k(a_2,r_2)^{r_2}}{r_2}\|u\|^{r_2} - \frac{2\beta_k(b_M,r_3)^{r_3}}{r_3\delta^{r_3-1}}\|u\|^{r_3}.
\end{aligned} \tag{39}$$

Let $r := \min\{r_1,r_2,r_3\}$, $\beta_k := \max\{\beta_k(a_1,r_1),\beta_k(a_2,r_2),\beta_k(b_M,r_3)\}$, $C' := \max\{\frac{2}{r_1},\frac{2}{r_2},\frac{2}{r_3\delta^{r_3-1}}\}$, then $\beta_k \to 0$ as $k \to \infty$. Hence, we have

$$\Phi(u) \geq \frac{1}{2}\|u\|^2 - 3C'\beta_k^r\|u\|^r \tag{40}$$

when $\|u\| \leq 1$ and $\beta_k \leq 1$. Now we can choose $\rho_k = (12\beta_k^r C')^{1/(2-r)}$, then $\rho_k \to 0$ as $k \to \infty$. When k is large enough, we have $\rho_k \leq 1$, $\beta_k \leq 1$, which together with (40) shows

$$a_k := \inf_{u\in Z_k,\|u\|=\rho_k} \Phi(u) \geq \frac{1}{4}\rho_k^2 > 0.$$

(ii) For any $u \in Y_k$, there exists $\lambda_i = 1,2,\ldots,k$ such that

$$u = \sum_{i=1}^k \lambda_i v_i.$$

Then we have

$$\begin{aligned}
\|u\|_{L^{r_4}}^{r_4} &= \int_{R^3} \left|u(x)\right|^{r_4}\,dx \\
&= \sum_{i=1}^k |\lambda_i|^{r_4}\int_{\Omega_i} \left|v_i(x)\right|^{r_4}\,dx + \sum_{i=1}^k |\lambda_i|^{r_4}\int_{G_i\setminus\Omega_i} \left|v_i(x)\right|^{r_4}\,dx \\
&= \sum_{i=1}^k |\lambda_i|^{r_4}\int_{\Omega_i} \left|v_i(x)\right|^{r_4}\,dx + \sum_{i=1}^k |\lambda_i|^{r_4}\int_{G_i\setminus\Omega_i} \frac{|\varphi_i(x)|^{r_4}}{\|\varphi_i\|^{r_4}}\,dx
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^{k} |\lambda_i|^{r_4} \int_{\Omega_i} \left|v_i(x)\right|^{r_4} dx + \sum_{i=1}^{k} \frac{|\lambda_i|^{r_4}}{\|\varphi_i\|^{r_4}} \operatorname{meas}\{G_i \setminus \Omega_i\} \\
&\leq \sum_{i=1}^{k} |\lambda_i|^{r_4} \int_{\Omega_i} \left|v_i(x)\right|^{r_4} + \sum_{i=1}^{k} \frac{|\lambda_i|^{r_4}}{\|\varphi_i\|^{r_4}} \varepsilon
\end{aligned} \tag{41}$$

and also

$$\begin{aligned}
\|u\|^2 &= \int_{R^3} \left[|\nabla u|^2 + V(x)u^2\right] dx \\
&= \sum_{i=1}^{k} \lambda_i^2 \int_{G_i} \left[|\nabla v_i|^2 + V(x)v_i^2\right] dx \\
&= \sum_{i=1}^{k} \lambda_i^2 \|v_i\|^2 \\
&= \sum_{i=1}^{k} \lambda_i^2.
\end{aligned} \tag{42}$$

Since all the norms of a finite dimensional space are equivalent, there is a constant $\tilde{C}$ such that

$$\tilde{C}\|u\| \leq \|u\|_{L^{r_4}}$$

for all $u \in Y_k$. By (30), one has

$$F(x, \lambda_i v_i) \geq -\frac{a_1(x)}{r_1}|\lambda_i v_i|^{r_1} - \frac{a_2(x)}{r_2}|\lambda_i v_i|^{r_2} - \frac{b_M(x)}{r_3 \delta^{r_3-1}}|\lambda_i v_i|^{r_3}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{i=1}^{k} \int_{G_i \setminus \Omega_i} F(x, \lambda_i v_i)\, dx \\
&\quad \geq -\sum_{i=1}^{k} \int_{G_i \setminus \Omega_i} \frac{|\lambda_i|^{r_1}}{r_1} a_1(x)|v_i|^{r_1} dx - \sum_{i=1}^{k} \int_{G_i \setminus \Omega_i} \frac{|\lambda_i|^{r_2}}{r_2} a_2(x)|v_i|^{r_2} dx \\
&\qquad - \sum_{i=1}^{k} \int_{G_i \setminus \Omega_i} \frac{|\lambda_i|^{r_3}}{r_3 \delta^{r_3-1}} b_M(x)|v_i|^{r_3} dx \\
&\quad \geq -\sum_{i=1}^{k} \frac{|\lambda_i|^{r_1}}{r_1} \|a_1\|_{L^{\frac{2}{2-r_1}}} \left(\int_{G_i \setminus \Omega_i} |v_i|^2 dx\right)^{r_1/2} \\
&\qquad - \sum_{i=1}^{k} \frac{|\lambda_i|^{r_2}}{r_2} \|a_2\|_{L^{\frac{2}{2-r_2}}} \left(\int_{G_i \setminus \Omega_i} |v_i|^2 dx\right)^{r_2/2} \\
&\qquad - \sum_{i=1}^{k} \frac{|\lambda_i|^{r_3}}{r_3 \delta^{r_3-1}} \|b_M\|_{L^{\frac{2}{2-r_3}}} \left(\int_{G_i \setminus \Omega_i} |v_i|^2 dx\right)^{r_3/2} \\
&\quad \geq -\sum_{i=1}^{k} \frac{|\lambda_i|^{r_1}}{r_1} \|a_1\|_{L^{\frac{2}{2-r_1}}} \left(\int_{G_i \setminus \Omega_i} \frac{|\varphi_i|^2}{\|\varphi_i\|^2} dx\right)^{r_1/2}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{k} \frac{|\lambda_i|^{r_2}}{r_2} \|a_2\|_{L^{\frac{2}{2-r_2}}} \left(\int_{G_i \setminus \Omega_i} \frac{|\varphi_i|^2}{\|\varphi_i\|^2} \, dx \right)^{r_2/2} \\
&- \sum_{i=1}^{k} \frac{|\lambda_i|^{r_3}}{r_3 \delta^{r_3-1}} \|b_M\|_{L^{\frac{2}{2-r_3}}} \left(\int_{G_i \setminus \Omega_i} \frac{|\varphi_i|^2}{\|\varphi_i\|^2} \, dx \right)^{r_3/2} \\
= {} & -\frac{1}{r_1} \|a_1\|_{L^{\frac{2}{2-r_1}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_1}}{\|\varphi_i\|^{r_1}} \left(\mathrm{meas}\{G_i \setminus \Omega_i\} \right)^{r_1/2} \\
&- \frac{1}{r_2} \|a_2\|_{L^{\frac{2}{2-r_2}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_2}}{\|\varphi_i\|^{r_2}} \left(\mathrm{meas}\{G_i \setminus \Omega_i\} \right)^{r_2/2} \\
&- \frac{1}{r_3 \delta^{r_3-1}} \|b_M\|_{L^{\frac{2}{2-r_3}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_3}}{\|\varphi_i\|^{r_3}} \left(\mathrm{meas}\{G_i \setminus \Omega_i\} \right)^{r_3/2} \\
\geq {} & -\frac{1}{r_1} \|a_1\|_{L^{\frac{2}{2-r_1}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_1}}{\|\varphi_i\|^{r_1}} \varepsilon^{r_1/2} - \frac{1}{r_2} \|a_2\|_{L^{\frac{2}{2-r_2}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_2}}{\|\varphi_i\|^{r_2}} \varepsilon^{r_2/2} \\
&- \frac{1}{r_3 \delta^{r_3-1}} \|b_M\|_{L^{\frac{2}{2-r_3}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_3}}{\|\varphi_i\|^{r_3}} \varepsilon^{r_3/2}. \qquad (43)
\end{aligned}
$$

For any $u \in Y_k$ with $\|u\| = \sum_{i=1}^{k} \lambda_i^2 = \gamma_k$, we can choose γ_k small enough such that $|\lambda_i \nu_i(x)| < \zeta$ for all $x \in R^3$ and $i = 1, \dots, k$, which together with (W_4) implies

$$
F(x, \lambda_i \nu_i) \geq \eta |\lambda_i \nu_i|^{r_4} \qquad (44)
$$

for all $x \in \Omega_i$ and $i = 1, \dots, k$. Combining (24), (41), (42), (43) and (44), we have

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{R^3} \phi_u u^2 \, dx - \int_{R^3} F(x, u) \, dx \\
&= \frac{1}{2} \|u\|^2 + \frac{C}{4} \|u\|^4 - \sum_{i=1}^{k} \int_{G_i} F(x, \lambda_i \nu_i) \, dx \\
&\leq \frac{1}{2} \|u\|^2 - \sum_{i=1}^{k} \left[\int_{G_i \setminus \Omega_i} F(x, \lambda_i \nu_i) \, dx + \int_{\Omega_i} F(x, \lambda_i \nu_i) \, dx \right] \\
&\leq \frac{1}{2} \|u\|^2 + \frac{C}{4} \|u\|^4 + \frac{1}{r_1} \|a_1\|_{L^{\frac{2}{2-r_1}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_1}}{\|\varphi_i\|^{r_1}} \varepsilon^{r_1/2} \\
&\quad + \frac{1}{r_2} \|a_2\|_{L^{\frac{2}{2-r_2}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_2}}{\|\varphi_i\|^{r_2}} \varepsilon^{r_2/2} + \frac{1}{r_3 \delta^{r_3-1}} \|b_M\|_{L^{\frac{2}{2-r_3}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_3}}{\|\varphi_i\|^{r_3}} \varepsilon^{r_3/2} \\
&\quad - \eta \sum_{i=1}^{k} |\lambda_i|^{r_4} \int_{\Omega_i} |\nu_i|^{r_4} \, dx \\
&= \frac{1}{2} \|u\|^2 + \frac{C}{4} \|u\|^4 + \frac{1}{r_1} \|a_1\|_{L^{\frac{2}{2-r_1}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_1}}{\|\varphi_i\|^{r_1}} \varepsilon^{r_1/2} \\
&\quad + \frac{1}{r_2} \|a_2\|_{L^{\frac{2}{2-r_2}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_2}}{\|\varphi_i\|^{r_2}} \varepsilon^{r_2/2} + \frac{1}{r_3 \delta^{r_3-1}} \|b_M\|_{L^{\frac{2}{2-r_3}}} \sum_{i=1}^{k} \frac{|\lambda_i|^{r_3}}{\|\varphi_i\|^{r_3}} \varepsilon^{r_3/2}
\end{aligned}
$$

$$
\begin{aligned}
&-\eta\left(\|u\|_{L^{r_4}}^{r_4}-\sum_{i=1}^{k}\frac{|\lambda_i|^{r_4}}{\|\varphi_i\|^{r_4}}\varepsilon\right)\\
&\le \frac{1}{2}\|u\|^2+\frac{C}{4}\|u\|^4-\eta\tilde{C}^{r_4}\|u\|^{r_4}+\frac{1}{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_1}}{\|\varphi_i\|^{r_1}}\varepsilon^{r_1/2}\\
&\quad+\frac{1}{r_2}\|a_2\|_{L^{\frac{2}{2-r_2}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_2}}{\|\varphi_i\|^{r_2}}\varepsilon^{r_2/2}+\frac{1}{r_3\delta^{r_3-1}}\|b_M\|_{L^{\frac{2}{2-r_3}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_3}}{\|\varphi_i\|^{r_3}}\varepsilon^{r_3/2}\\
&\quad+\eta\sum_{i=1}^{k}\frac{|\lambda_i|^{r_4}}{\|\varphi_i\|^{r_4}}\varepsilon\\
&=\frac{1}{2}\sum_{i=1}^{k}\lambda_i^2+\frac{C}{4}\left(\sum_{i=1}^{k}\lambda_i^2\right)^2-\eta\tilde{C}^{r_4}\left(\sum_{i=1}^{k}\lambda_i^2\right)^{r_4/2}+\frac{1}{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_1}}{\|\varphi_i\|^{r_1}}\varepsilon^{r_1/2}\\
&\quad+\frac{1}{r_2}\|a_2\|_{L^{\frac{2}{2-r_2}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_2}}{\|\varphi_i\|^{r_2}}\varepsilon^{r_2/2}+\frac{1}{r_3\delta^{r_3-1}}\|b_M\|_{L^{\frac{2}{2-r_3}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_3}}{\|\varphi_i\|^{r_3}}\varepsilon^{r_3/2}\\
&\quad+\eta\sum_{i=1}^{k}\frac{|\lambda_i|^{r_4}}{\|\varphi_i\|^{r_4}}\varepsilon\\
&=\frac{1}{2}\gamma_k^2+\frac{C}{4}\gamma_k^4-\eta(\tilde{C}\gamma_k)^{r_4}+\frac{1}{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_1}}{\|\varphi_i\|^{r_1}}\varepsilon^{r_1/2}\\
&\quad+\frac{1}{r_2}\|a_2\|_{L^{\frac{2}{2-r_2}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_2}}{\|\varphi_i\|^{r_2}}\varepsilon^{r_2/2}+\frac{1}{r_3\delta^{r_3-1}}\|b_M\|_{L^{\frac{2}{2-r_3}}}\sum_{i=1}^{k}\frac{|\lambda_i|^{r_3}}{\|\varphi_i\|^{r_3}}\varepsilon^{r_3/2}\\
&\quad+\eta\sum_{i=1}^{k}\frac{|\lambda_i|^{r_4}}{\|\varphi_i\|^{r_4}}\varepsilon\\
&\le \gamma_k^2+\frac{C}{4}\gamma_k^4-\eta(\tilde{C}\gamma_k)^{r_4}
\end{aligned}
$$

for all $u\in Y_k$ with $\|u\|=\gamma_k$, when ε and γ_k are both small enough. Since $r_4<2$, we can choose $\gamma_k<\rho_k$ small enough such that

$$b_k:=\max_{u\in Y_k,\|u\|=\gamma_k}\Phi(u)<0.$$

(iii) By (40), for any $u\in Z_k$ with $\|u\|=\rho_k$, we have

$$\Phi(u)\ge -3C'\beta_k^r\|u\|^r.$$

Therefore

$$0\ge \inf_{u\in Z_k,\|u\|\le\rho_k}\Phi(u)\ge -3C'\beta_k^r\rho_k^r.$$

Since $\beta_k,\rho_k\to 0$ as $k\to\infty$, we have

$$d_k:=\inf_{u\in Z_k,\|u\|\le\rho_k}\Phi(u)\to 0$$

as $k\to\infty$.

Hence, by Lemma 2.7, we obtain that problem (1) has infinitely many solutions $\{(u_k, \phi_k)\}$ satisfying

$$\frac{1}{2}\int_{R^3}\left(|\nabla u_k|^2 + V(x)u_k^2\right)dx - \frac{1}{4}\int_{R^3}|\nabla\phi_k|^2\,dx + \frac{1}{2}\int_{R^3}\phi_k u_k^2\,dx - \int_{R^3}F(x,u_k)\,dx \to 0^-$$

as $k \to \infty$. □

Proof of Theorem 1.5 Similar to (31), there exist constants $k_i > 0$, $i = 1, 2, 3$, such that

$$\Phi(u) \geq \frac{1}{2}\|u\|^2 - \sum_{i=1}^{3} k_i\|u\|^{r_i} \tag{45}$$

for all $u \in E$. Since $1 < r_i < 2$, it follows from (45) that the functional Φ is bounded from below. By Lemma 2.5 and Remark 3.2, Φ possesses a critical point u satisfying

$$\Phi(u) = \inf_E \Phi, \qquad \Phi'(u) = 0.$$

It remains to show that u is nontrivial. For every $\varepsilon > 0$, there exist an open set G and a closed set H such that $H \subset \Omega \subset G$ and

$$\operatorname{meas}\{G \setminus \Omega\} < \varepsilon, \qquad \operatorname{meas}\{\Omega \setminus H\} < \varepsilon.$$

By Lemma 2.3, there exists a function $\varphi \in C_0^\infty(R^3)$ such that $0 \leq \varphi(x) \leq 1$ and $\varphi|_H(x) = 1$, $\varphi|_{R\setminus G}(x) = 0$, then $\varphi \in E$. Choosing $0 < \lambda < \min\{\delta, \zeta\}$, then $|\lambda\varphi(x)| < \delta$ for all $x \in R^3$, which together with (28) shows

$$F\bigl(x, \lambda\varphi(x)\bigr) \geq -\frac{a_1(x)}{r_1}\bigl|\lambda\varphi(x)\bigr|^{r_1}$$

for all $x \in R^3$. Therefore, one has

$$\begin{aligned}
\int_{G\setminus H} F(x,\lambda\varphi)\,dx &\geq -\int_{G\setminus H}\frac{\lambda^{r_1}}{r_1}a_1(x)\varphi^{r_1}\,dx \\
&\geq -\frac{\lambda^{r_1}}{r_1}\left(\int_{G\setminus H} a_1(x)^{\frac{2}{2-r_1}}\,dx\right)^{(2-r_1)/2}\left(\int_{G\setminus H}\varphi^2\,dx\right)^{r_1/2} \\
&\geq -\frac{\lambda^{r_1}}{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}\left(\int_{G\setminus H} 1\,dx\right)^{r_1/2} \\
&\geq -\frac{\lambda^{r_1}}{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}\bigl(\operatorname{meas}\{G\setminus H\}\bigr)^{r_1/2} \\
&\geq -\frac{\lambda^{r_1}}{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}(2\varepsilon)^{r_1/2}.
\end{aligned} \tag{46}$$

In view of $\lambda < \zeta$, we have $|\lambda\varphi(x)| < \zeta$ for all $x \in R^3$, which together with (W_4) implies

$$F(x,\lambda\varphi) \geq \eta|\lambda\varphi|^{r_4} \tag{47}$$

for all $x \in \Omega$. It follows from (24), (46), (47) that

$$\begin{aligned}
\Phi(\lambda\varphi) &= \frac{\lambda^2}{2}\|\varphi\|^2 + \frac{1}{4}\int_{R^3}\phi_{\lambda\varphi}(\lambda\varphi)^2\,dx - \int_{R^3}F(x,\lambda\varphi)\,dx \\
&\le \frac{\lambda^2}{2}\|\varphi\|^2 + C\lambda^4\|\varphi\|^4 - \int_{R^3}F(x,\lambda\varphi)\,dx \\
&\le \frac{\lambda^2}{2}\|\varphi\|^2 + C\lambda^4\|\varphi\|^4 - \int_{G}F(x,\lambda\varphi)\,dx \\
&= \frac{\lambda^2}{2}\|\varphi\|^2 + C\lambda^4\|\varphi\|^4 - \left[\int_{H}F(x,\lambda\varphi)\,dx + \int_{G\setminus H}F(x,\lambda\varphi)\,dx\right] \\
&\le \frac{\lambda^2}{2}\|\varphi\|^2 + C\lambda^4\|\varphi\|^4 - \lambda^{r_4}\int_{H}\eta|\varphi|^{r_4}\,dx + \frac{\lambda^{r_1}}{r_1}\|a_1\|_{L^{\frac{2}{2-r_1}}}(2\varepsilon)^{r_1/2} \\
&\le \lambda^2\|\varphi\|^2 + C\lambda^4\|\varphi\|^4 - \lambda^{r_4}\eta\,\mathrm{meas}\{H\} \\
&< 0
\end{aligned}$$

when ε and λ are both small enough. Since $\Phi(0) = 0$, then $u \neq 0$. Hence, (u, ϕ_u) is a nontrivial solution of problem (1). □

Competing interests

The author declares that she has no competing interests.

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