Agarwal et al. *Boundary Value Problems* 2013, **2013**:184
http://www.boundaryvalueproblems.com/content/2013/1/184

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Positive solutions for a sixth-order boundary value problem with four parameters

Ravi P Agarwal[1*], B Kovacs[2] and D O'Regan[3]

*Correspondence:
Agarwal@tamuk.edu
[1]Department of Mathematics, Texas A&M University-Kingsville, 700 University Blvd., Kingsville, 78363-8202, USA
Full list of author information is available at the end of the article

Abstract
This paper investigates the existence and multiplicity of positive solutions of a sixth-order differential system with four variable parameters using a monotone iterative technique and an operator spectral theorem.
MSC: 34B15; 34B18

Keywords: positive solutions; variable parameters; fixed point theorem; operator spectral theorem

1 Introduction

It is well known that boundary value problems for ordinary differential equations can be used to describe a large number of physical, biological and chemical phenomena. In recent years, boundary value problems for sixth-order ordinary differential equations, which arise naturally, for example, in sandwich beam deflection under transverse shear have been studied extensively, see [1–4] and the references therein. The deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported can be described by a boundary value problem involving a sixth-order ordinary differential equation

$$\begin{aligned} &u^{(6)} + 2u^{(4)} + u'' = f(t,u), \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0. \end{aligned} \tag{1}$$

Liu and Li [5] studied the existence and nonexistence of positive solutions of the nonlinear fourth-order beam equation

$$\begin{aligned} &u^{(4)}(t) + \beta u''(t) - \alpha u(t) = \lambda f\bigl(t,u(t)\bigr), \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{2}$$

They showed that there exists a $\lambda^\star > 0$ such that the above boundary value problem has at least two, one, and no positive solutions for $0 < \lambda < \lambda^\star$, $\lambda = \lambda^\star$ and $\lambda > \lambda^\star$, respectively.

In this paper, we discuss the existence of positive solutions for the sixth-order boundary value problem

$$\begin{aligned} &-u^{(6)} + A(t)u^{(4)} + B(t)u'' + C(t)u = \bigl(D(t) + u\bigr)\varphi + \lambda f(t,u), \quad 0 < t < 1, \\ &-\varphi'' + \varkappa\varphi = \mu u, \quad 0 < t < 1, \end{aligned} \tag{3}$$

$$u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0,$$
$$\varphi(0) = \varphi(1) = 0.$$

For this, we shall assume the following conditions throughout

(H1) $f(t,u) : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is continuous;

(H2) $a,b,c \in R$, $a = \lambda_1 + \lambda_2 + \lambda_3 > -\pi^2$, $b = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 > 0$, $c = \lambda_1\lambda_2\lambda_3 < 0$ where $\lambda_1 \geq 0 \geq \lambda_2 \geq -\pi^2$, $0 \leq \lambda_3 < -\lambda_2$ and $\pi^6 + a\pi^4 - b\pi^2 + c > 0$, and $A,B,C,D \in C[0,1]$ with $a = \sup_{t\in[0,1]} A(t)$, $b = \inf_{t\in[0,1]} B(t)$ and $c = \sup_{t\in[0,1]} C(t)$.

Let $K = \max_{0\leq t\leq 1}[-A(t) + B(t) - C(t) - (-a + b - c)]$ and $\Gamma = \pi^6 + a\pi^4 - b\pi^2 + c$.

Assumption (H2) involves a three-parameter nonresonance condition.

More recently Li [6] studied the existence and multiplicity of positive solutions for a sixth-order boundary value problem with three variable coefficients. The main difference between our work and [6] is that we consider boundary value problem not only with three variable coefficients, but also with two positive parameters λ and μ, and the existence of the positive solution depends on these parameters. In this paper, we shall apply the monotone iterative technique [7] to boundary value problem (3) and then obtain several new existence and multiplicity results. In the special case, in [8] by using the fixed point theorem and the operator spectral theorem, we establish a theorem on the existence of positive solutions for the sixth-order boundary value problem (3) with $\lambda = 1$.

2 Preliminaries

Let $Y = C[0,1]$ and $Y_+ = \{u \in Y : u(t) \geq 0, t \in [0,1]\}$. It is well known that Y is a Banach space equipped with the norm $\|u\|_0 = \sup_{t\in[0,1]} |u(t)|$. Set $X = \{u \in C^4[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$. For given $\chi \geq 0$ and $\nu \geq 0$, we denote the norm $\|\cdot\|_{\chi,\nu}$ by

$$\|\cdot\|_{\chi,\nu} = \sup_{t\in[0,1]} \{ |u^{(4)}(t)| + \chi |u''(t)| + \nu |u(t)| \}, \quad u \in X.$$

We also need the space X, equipped with the norm

$$\|u\|_2 = \max\{ \|u\|_0, \|u''\|_0, \|u^{(4)}\|_0 \}.$$

In [8], it is shown that X is complete with the norm $\|\cdot\|_{\chi,\nu}$ and $\|u\|_2$, and moreover $\forall u \in X$, $\|u\|_0 \leq \|u''\|_0 \leq \|u^{(4)}\|_0$.

For $h \in Y$, consider the linear boundary value problem

$$\begin{aligned} &-u^{(6)} + au^{(4)} + bu'' + cu = h(t), \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0, \end{aligned} \tag{4}$$

where a, b, c satisfy the assumption

$$\pi^6 + a\pi^4 - b\pi^2 + c > 0, \tag{5}$$

and let $\Gamma = \pi^6 + a\pi^4 - b\pi^2 + c$. Inequality (5) follows immediately from the fact that $\Gamma = \pi^6 + a\pi^4 - b\pi^2 + c$ is the first eigenvalue of the problem $-u^{(6)} + au^{(4)} + bu'' + cu = \lambda u$, $u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0$, and $\phi_1(t) = \sin \pi t$ is the first eigenfunction, *i.e.*,

$\Gamma > 0$. Since the line $l_1 = \{(a,b,c) : \pi^6 + a\pi^4 - b\pi^2 + c = 0\}$ is the first eigenvalue line of the three-parameter boundary value problem $-u^{(6)} + au^{(4)} + bu'' + cu = 0$, $u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0$, if (a,b,c) lies in l_1, then by the Fredholm alternative, the existence of a solution of the boundary value problem (4) cannot be guaranteed.

Let $P(\lambda) = \lambda^2 + \beta\lambda - \alpha$, where $\beta < 2\pi^2$, $\alpha \geq 0$. It is easy to see that the equation $P(\lambda) = 0$ has two real roots $\lambda_1, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}$ with $\lambda_1 \geq 0 \geq \lambda_2 > -\pi^2$. Let λ_3 be a number such that $0 \leq \lambda_3 < -\lambda_2$. In this case, (4) satisfies the decomposition form

$$-u^{(6)} + au^{(4)} + bu'' + cu = \left(-\frac{d^2}{dt^2} + \lambda_1\right)\left(-\frac{d^2}{dt^2} + \lambda_2\right)\left(-\frac{d^2}{dt^2} + \lambda_3\right)u, \quad 0 < t < 1. \tag{6}$$

Suppose that $G_i(t,s)$ $(i = 1,2,3)$ is the Green's function associated with

$$-u'' + \lambda_i u = 0, \qquad u(0) = u(1) = 0. \tag{7}$$

We need the following lemmas.

Lemma 1 [5, 9] *Let $\omega_i = \sqrt{|\lambda_i|}$, then $G_i(t,s)$ $(i = 1,2,3)$ can be expressed as*

(i) *when $\lambda_i > 0$,*

$$G_i(t,s) = \left\{\begin{array}{l} \dfrac{\sinh \omega_i t \sinh \omega_i (1-s)}{\omega_i \sinh \omega_i}, \ 0 \leq t \leq s \leq 1, \\ \dfrac{\sinh \omega_i s \sinh \omega_i (1-t)}{\omega_i \sinh \omega_i}, \ 0 \leq s \leq t \leq 1 \end{array}\right\};$$

(ii) *when $\lambda_i = 0$,*

$$G_i(t,s) = \left\{\begin{array}{l} t(1-s), \ 0 \leq t \leq s \leq 1, \\ s(1-t), \ 0 \leq s \leq t \leq 1 \end{array}\right\};$$

(iii) *when $-\pi^2 < \lambda_i < 0$,*

$$G_i(t,s) = \left\{\begin{array}{l} \dfrac{\sin \omega_i t \sin \omega_i (1-s)}{\omega_i \sin \omega_i}, \ 0 \leq t \leq s \leq 1, \\ \dfrac{\sin \omega_i s \sin \omega_i (1-t)}{\omega_i \sin \omega_i}, \ 0 \leq s \leq t \leq 1 \end{array}\right\}.$$

Lemma 2 [5] *$G_i(t,s)$ $(i = 1,2,3)$ has the following properties*

(i) $G_i(t,s) > 0$, $\forall t,s \in (0,1)$;

(ii) $G_i(t,s) \leq C_i G_i(s,s)$, $\forall t,s \in [0,1]$;

(iii) $G_i(t,s) \geq \delta_i G_i(t,t) G_i(s,s)$, $\forall t,s \in [0,1]$,

where $C_i = 1$, $\delta_i = \frac{\omega_i}{\sinh \omega_i}$, if $\lambda_i > 0$; $C_i = 1$, $\delta_i = 1$, if $\lambda_i = 0$; $C_i = \frac{1}{\sin \omega_i}$, $\delta_i = \omega_i \sin \omega_i$, if $-\pi^2 < \lambda_i < 0$.

In what follows, we let $D_i = \max_{t \in [0,1]} \int_0^1 G_i(t,s)\,ds$.

Lemma 3 [10] *Let X be a Banach space, K a cone and Ω a bounded open subset of X. Let $\theta \in \Omega$ and $T : K \cap \bar{\Omega} \to K$ be condensing. Suppose that $Tx \neq \upsilon x$ for all $x \in K \cap \partial\Omega$ and $\upsilon \geq 1$. Then $i(T, K \cap \Omega, K) = 1$.*

Lemma 4 [10] *Let X be a Banach space, let K be a cone of X. Assume that $T : \bar{K}_r \to K$ (here $K_r = \{x \in K \mid \|x\| < r\}$, $r > 0$) is a compact map such that $Tx \neq x$ for all $x \in \partial K_r$. If $\|x\| \leq \|Tx\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.*

Now, since

$$\begin{aligned} -u^{(6)} + au^{(4)} + bu'' + cu &= \left(-\frac{d^2}{dt^2} + \lambda_1\right)\left(-\frac{d^2}{dt^2} + \lambda_2\right)\left(-\frac{d^2}{dt^2} + \lambda_3\right)u \\ &= \left(-\frac{d^2}{dt^2} + \lambda_2\right)\left(-\frac{d^2}{dt^2} + \lambda_1\right)\left(-\frac{d^2}{dt^2} + \lambda_3\right)u = h(t), \end{aligned} \tag{8}$$

the solution of boundary value problem (4) can be expressed as

$$u(t) = \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)h(\tau)\,d\tau\,ds\,dv, \quad t \in [0,1]. \tag{9}$$

Thus, for every given $h \in Y$, the boundary value problem (4) has a unique solution $u \in C^6[0,1]$, which is given by (9).

We now define a mapping $T : C[0,1] \to C[0,1]$ by

$$(Th)(t) = \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)h(\tau)\,d\tau\,ds\,dv, \quad t \in [0,1]. \tag{10}$$

Throughout this article, we shall denote $Th = u$ the unique solution of the linear boundary value problem (4).

Lemma 5 [8] *$T : Y \longrightarrow (X, \|\cdot\|_{\chi,\nu})$ is linear completely continuous, where $\chi = \lambda_1 + \lambda_3$, $\nu = \lambda_1\lambda_3$ and $\|T\| \leq D_2$. Moreover, $\forall h \in Y_+$, if $u = Th$, then $u \in X \cap Y_+$, and $u'' \leq 0$, $u^{(4)} \geq 0$.*

We list the following conditions for convenience

(H3) $f(t,u)$ is nondecreasing in u for $t \in [0,1]$;

(H4) $f(t,0) > \widehat{c} > 0$ for all $t \in [0,1]$;

(H5) $f_\infty = \lim_{u\to\infty} \frac{f(t,u)}{u} = \infty$ uniformly for $t \in [0,1]$;

(H6) $f(t,\rho u) \geq \rho^\alpha f(t,u)$ for $\rho \in (0,1)$ and $t \in [0,1]$, where $\alpha \in (0,1)$ is independent of ρ and u.

Suppose that $G(t,s)$ is the Green's function of the linear boundary value problem

$$-u'' + \varkappa u = 0, \qquad u(0) = u(1) = 0. \tag{11}$$

Then, the boundary value problem

$$-\varphi'' + \varkappa\varphi = \mu u, \qquad \varphi(0) = \varphi(1) = 0,$$

can be solved by using Green's function, namely,

$$\varphi(t) = \mu \int_0^1 G(t,s)u(s)\,ds, \quad 0 < t < 1, \tag{12}$$

where $\varkappa > -\pi^2$. Thus, inserting (12) into the first equation in (3), yields

$$\begin{aligned} &-u^{(6)} + A(t)u^{(4)} + B(t)u'' + C(t)u = \mu\big(D(t) + u(t)\big)\int_0^1 G(t,s)u(s)\,ds + \lambda f(t,u), \\ &u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0. \end{aligned} \tag{13}$$

Let us consider the boundary value problem

$$\begin{aligned} &-u^{(6)} + A(t)u^{(4)} + B(t)u'' + C(t)u = h(t), \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0. \end{aligned} \tag{14}$$

Now, we consider the existence of a positive solution of (14). The function $u \in C^6(0,1) \cap C^4[0,1]$ is a positive solution of (14), if $u \geq 0$, $t \in [0,1]$, and $u \not\equiv 0$.

Let us rewrite equation (13) in the following form

$$\begin{aligned} -u^{(6)} + au^{(4)} + bu'' + cu = {} & -\big(A(t) - a\big)u^{(4)} - \big(B(t) - b\big)u'' - \big(C(t) - c\big)u \\ & + \mu\big(D(t) + u(t)\big)\int_0^1 G(t,s)u(s)\,ds + h(t). \end{aligned} \tag{15}$$

For any $u \in X$, let

$$Gu = -\big(A(t) - a\big)u^{(4)} - \big(B(t) - b\big)u'' - \big(C(t) - c\big)u + \mu D(t)\int_0^1 G(t,s)u(s)\,ds.$$

The operator $G : X \to Y$ is linear. By Lemmas 2 and 3 in [8], $\forall u \in X$, $t \in [0,1]$, we have

$$\begin{aligned} \big|(Gu)(t)\big| &\leq \big[-A(t) + B(t) - C(t) - (-a + b - c)\big]\|u\|_2 + \mu C d_1\|u\|_0 \\ &\leq (K + \mu C d_1)\|u\|_2 \leq (K + \mu C d_1)\|u\|_{\chi,\nu}, \end{aligned}$$

where $C = \max_{t\in[0,1]} D(t)$, $K = \max_{t\in[0,1]}[-A(t) + B(t) - C(t) - (-a + b - c)]$, $d_1 = \max_{t\in[0,1]}\int_0^1 G(t,s)\,ds$, $\chi = \lambda_1 + \lambda_3 \geq 0$, $\nu = \lambda_1\lambda_3 \geq 0$. Hence $\|Gu\|_0 \leq (K + \mu C d_1)\|u\|_{\chi,\nu}$, and so $\|G\| \leq (K + \mu C d_1)$. Also $u \in C^4[0,1] \cap C^6(0,1)$ is a solution of (13) if $u \in X$ satisfies $u = T(Gu + h_1)$, where $h_1(t) = \mu u(t)\int_0^1 G(t,s)u(s)\,ds + h(t)$, *i.e.*,

$$u \in X, \quad (I - TG)u = Th_1. \tag{16}$$

The operator $I - TG$ maps X into X. From $\|T\| \leq D_2$ together with $\|G\| \leq (K + \mu C d_1)$ and the condition $D_2(K + \mu C d_1) < 1$, and applying the operator spectral theorem, we find that $(I - TG)^{-1}$ exists and is bounded. Let $\mu \in (0, \frac{1-D_2K}{D_2Cd_1})$, where $1 - D_2K > 0$, then the condition $D_2(K + \mu C d_1) < 1$ is fulfilled. Let $L = D_2(K + \mu C d_1)$, and let $\mu^{**} = \frac{1-D_2K}{D_2Cd_1}$.

Let $H = (I - TG)^{-1}T$. Then (16) is equivalent to $u = Hh_1$. By the Neumann expansion formula, H can be expressed by

$$H = \big(I + TG + \cdots + (TG)^n + \cdots\big)T = T + (TG)T + \cdots + (TG)^nT + \cdots. \tag{17}$$

The complete continuity of T with the continuity of $(I - TG)^{-1}$ guarantees that the operator $H : Y \to X$ is completely continuous.

Now $\forall h \in Y_+$, let $u = Th$, then $u \in X \cap Y_+$, and $u'' \leq 0$, $u^{(4)} \geq 0$. Thus, we have

$$\begin{aligned}(Gu)(t) = &-\big(A(t)-a\big)u^{(4)} - \big(B(t)-b\big)u'' - \big(C(t)-c\big)u \\ &+ \mu D(t)\int_0^1 G(t,s)u(s)\,ds \geq 0, \quad t \in [0,1].\end{aligned}$$

Hence

$$\forall h \in Y_+, \quad (GTh)(t) \geq 0, \quad t \in [0,1], \tag{18}$$

and so, $(TG)(Th)(t) = T(GTh)(t) \geq 0$, $t \in [0,1]$.

It is easy to see [11] that the following inequalities hold: $\forall h \in Y_+$,

$$\frac{1}{1-L}(Th)(t) \geq (Hh)(t) \geq (Th)(t), \quad t \in [0,1], \tag{19}$$

and, moreover,

$$\big\|(Hh)\big\|_0 \leq \frac{1}{1-L}\big\|(Th)\big\|_0. \tag{20}$$

Lemma 6 [8] *$H : Y \longrightarrow (X, \|\cdot\|_{\varkappa,\nu})$ is completely continuous, where $\chi = \lambda_1 + \lambda_3$, $\nu = \lambda_1\lambda_3$ and $\forall h \in Y_+$, $\frac{1}{1-L}(Th)(t) \geq (Hh)(t) \geq (Th)(t)$, $t \in [0,1]$, and, moreover, $\|Th\|_0 \geq (1-L)\|Hh\|_0$.*

For any $u \in Y_+$, define $Fu = \mu u(t)\int_0^1 G(t,s)u(s)\,ds + \lambda f(t,u)$. From (H1), we have that $F : Y_+ \to Y_+$ is continuous. It is easy to see that $u \in C^4[0,1] \cap C^6(0,1)$, being a positive solution of (13), is equivalent to $u \in Y_+$, being a nonzero solution of

$$u = HFu. \tag{21}$$

Let us introduce the following notations

$$\begin{aligned}T_{\lambda,\mu}u(t) := TFu(t) = &\int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau) \\ &\times \left(\mu u(\tau)\int_0^1 G(\tau,s)u(s)\,ds + \lambda f\big(\tau,u(\tau)\big)\right)d\tau\,ds\,dv,\\ Q_{\lambda,\mu}u := HFu = &\, TFu + (TG)TFu + (TG)^2TFu + \cdots + (TG)^nTFu + \cdots \\ = &\, T_{\lambda,\mu}u + (TG)T_{\lambda,\mu}u + (TG)^2T_{\lambda,\mu}u + \cdots + (TG)^nT_{\lambda,\mu}u + \cdots,\end{aligned}$$

i.e., $Q_{\lambda,\mu}u = HFu$. Obviously, $Q_{\lambda,\mu} : Y_+ \to Y_+$ is completely continuous. We next show that the operator $Q_{\lambda,\mu}$ has a nonzero fixed point in Y_+.

Let $P = \{u \in Y_+ : u(t) \geq \delta_1(1-L)g_1(t)\|u(t)\|_0, t \in [\frac{1}{4},\frac{3}{4}]\}$, where $g_1(t) = \frac{1}{C_1}G_1(t,t)$. It is easy to see that P is a cone in Y, and now, we show $Q_{\lambda,\mu}(P) \subset P$.

Lemma 7 *$Q_{\lambda,\mu}(P) \subset P$ and $Q_{\lambda,\mu} : P \to P$ is completely continuous.*

Proof It is clear that $Q_{\lambda,\mu}: P \to P$ is completely continuous. Now $\forall u \in P$, let $h_1 = Fu$, then $h_1 \in Y_+$. Using Lemma 6, *i.e.*, $(Q_{\lambda,\mu}u)(t) = (HFu)(t) \geq (TFu)(t)$, $t \in [0,1]$ and by Lemma 2, for all $u \in P$, we have

$$(TFu)(t) \leq C_1 \int_0^1 \int_0^1 \int_0^1 G_1(\nu,\nu)G_2(\nu,s)G_3(s,\tau)(Fu)(\tau)\,d\tau\,ds\,d\nu, \quad \forall t \in [0,1].$$

Thus,

$$\int_0^1 \int_0^1 \int_0^1 G_1(\nu,\nu)G_2(\nu,s)G_3(s,\tau)(Fu)(\tau)\,d\tau\,ds\,d\nu \geq \frac{1}{C_1}\|TFu\|_0. \tag{22}$$

On the other hand, by Lemma 6 and (22), we have

$$\begin{aligned}(TFu)(t) &\geq \delta_1 G_1(t,t)\int_0^1 \int_0^1 \int_0^1 G_1(\nu,\nu)G_2(\nu,s)G_3(s,\tau)(Fu)(\tau)\,d\tau\,ds\,d\nu \\ &\geq \delta_1 G_1(t,t)\frac{1}{C_1}\|TFu\|_0 \geq \delta_1 G_1(t,t)\frac{1}{C_1}(1-L)\|Qu\|_0, \quad \forall t \in [0,1].\end{aligned}$$

Thus, $Q_{\lambda,\mu}(P) \subset P$. □

3 Main results

Lemma 8 *Let $f(t,u)$ be nondecreasing in u for $t \in [0,1]$ and $f(t,0) > \widehat{c} > 0$ for all $t \in [0,1]$, where $\widehat{c}$ is a constant and $L < 1$. Then there exists $\lambda^{*} > 0$ and $\mu^{*} > 0$ such that the operator $Q_{\lambda,\mu}$ has a fixed point $u^\star$ at (λ^{*},μ^{*}) with $u^\star \in P\backslash\{\theta\}$.*

Proof Set $\widehat{u}_1(t) = (Q_{\lambda^{*},\mu^{*}}u_\star)(t)$, where

$$u_\star(t) = \frac{2}{1-L}\int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)\,ds\,d\tau\,d\nu.$$

It is easy to see that $u_\star(t) \in P$. Let $\lambda^{*} = M_{fu}^{-1}$ and $\mu^{*} = \min(N_{fu}^{-1};\mu^{**})$, where $M_{fu} = \max_{t\in[0,1]} f(t,u_\star(t))$ and $N_{fu} = \max_{t\in[0,1]} u_\star(t)\int_0^1 G(t,s)u_\star(s)\,ds$, respectively. Then $M_{fu} > 0$ and $N_{fu} > 0$, and from Lemma 6, we obtain

$$\begin{aligned}\widehat{u}_0(t) = u_\star(t) &= \frac{2}{1-L}\int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)\,ds\,d\tau\,d\nu \\ &\geq \frac{1}{1-L}\int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s) \\ &\quad \times \left(\lambda^{*} f\big(s,u_\star(s)\big) + \mu^{*} u_\star(t)\int_0^1 G(t,s)u_\star(s)\,ds\right) ds\,d\tau\,d\nu \\ &= \frac{1}{1-L}(T_{\lambda^{*},\mu^{*}}u_\star)(t) \geq (Q_{\lambda^{*},\mu^{*}}u_\star)(t) = \widehat{u}_1(t).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\widehat{u}_n(t) &= (Q_{\lambda^{*},\mu^{*}}\widehat{u}_{n-1})(t) \\ &= \big(T_{\lambda^{*},\mu^{*}}\widehat{u}_{n-1} + (TG)T_{\lambda^{*},\mu^{*}}\widehat{u}_{n-1} + (TG)^2 T_{\lambda^{*},\mu^{*}}\widehat{u}_{n-1} + \cdots\end{aligned}$$

$$
\begin{aligned}
&\quad + (TG)^n T_{\lambda^*,\mu^*}\widehat{u}_{n-1} + \cdots\big) \\
&\geq \big(T_{\lambda^*,\mu^*}\widehat{u}_{n-2} + (TG)T_{\lambda^*,\mu^*}\widehat{u}_{n-2} + (TG)^2 T_{\lambda^*,\mu^*}\widehat{u}_{n-2} + \cdots \\
&\quad + (TG)^n T_{\lambda^*,\mu^*}\widehat{u}_{n-2} + \cdots\big) = \widehat{u}_{n-1}(t).
\end{aligned}
$$

Indeed, for $h_1, h_2 \in Y_+$, let $h_1(t) \geq h_2(t)$, then from (10), we have $u_1(t) = Th_1 \geq Th_2 = u_2(t)$. Using equation (4) and (6), we obtain

$$
-u'' + \lambda_2 u = \int_0^1 \int_0^1 G_1(t,\nu)G_3(\nu,\tau)h(\tau)\,d\tau\,d\nu, \quad t \in [0,1] \tag{23}
$$

and

$$
u^{(4)} - (\lambda_2 + \lambda_3)u'' + \lambda_2\lambda_3 u = \int_0^1 G_1(t,\nu)h(\nu)\,d\nu, \quad t \in [0,1]. \tag{24}
$$

Then by (23), we have for $t \in [0,1]$

$$
u_1''(t) - u_2''(t) = \lambda_2\big(u_1(t) - u_2(t)\big) - \int_0^1 \int_0^1 G_1(t,\nu)G_3(\nu,\tau)\big(h_1(t) - h_2(t)\big)\,d\tau\,d\nu \leq 0,
$$

because $\lambda_2 < 0$, and finally, from (24), we have

$$
\begin{aligned}
u_1^{(4)}(t) - u_2^{(4)}(t) &= (\lambda_2 + \lambda_3)\big(u_1''(t) - u_2''(t)\big) - \lambda_2\lambda_3\big(u_1(t) - u_2(t)\big) \\
&\quad + \int_0^1 \int_0^1 G_1(t,\nu)\big(h_1(t) - h_2(t)\big)\,d\tau\,d\nu \geq 0, \quad t \in [0,1]
\end{aligned}
$$

because $\lambda_2 + \lambda_3 \leq 0$ and $\lambda_2\lambda_3 \leq 0$. From the equation

$$
\begin{aligned}
&(Gu)(t) = -\big(A(t) - a\big)u^{(4)} - \big(B(t) - b\big)u'' - \big(C(t) - c\big)u + \mu D(t)\int_0^1 G(t,s)u(s)\,ds \geq 0, \\
&\quad t \in [0,1]
\end{aligned}
$$

we have

$$
\begin{aligned}
&(Gu_1)(t) - (Gu_2)(t) \\
&\quad = -\big(A(t) - a\big)\big(u_1^{(4)}(t) - u_2^{(4)}(t)\big) - \big(B(t) - b\big)\big(u_1''(t) - u_2''(t)\big) \\
&\qquad - \big(C(t) - c\big)\big(u_1(t) - u_2(t)\big) + \mu D(t)\int_0^1 G(t,s)\big(u_1(t) - u_2(t)\big)\,ds \geq 0, \\
&\qquad t \in [0,1]
\end{aligned} \tag{25}
$$

i.e., $(Gu_1)(t) \geq (Gu_2)(t)$ for all $t \in [0,1]$. Finally, if $h_1 = Fu_1$ and $h_2 = Fu_2$, then

$$
\begin{aligned}
(Hh_1)(t) &= T(h_1) + (TG)(Th_1) + (TG)^2(Th_1) + \cdots + (TG)^n(Th_1) + \cdots \\
&\geq (Hh_2)(t) = T(h_2) + (TG)(Th_2) + (TG)^2(Th_2) + \cdots + (TG)^n(Th_2) + \cdots
\end{aligned}
$$

i.e.,

$$\begin{aligned}
&Q_{\lambda^*,\mu^*}u_1 \\
&\quad = (HFu_1)(t) = T(Fu_1) + (TG)(TFu_1) + (TG)^2(TFu_1) + \cdots + (TG)^n(TFu_1) + \cdots \\
&\quad \geq (HFu_2)(t) = T(Fu_2) + (TG)(TFu_2) + (TG)^2(TFu_2) + \cdots + (TG)^n(TFu_2) + \cdots \\
&\quad = Q_{\lambda^*,\mu^*}u_2,
\end{aligned} \tag{26}$$

and from (26), it follows that for $u_1, u_2 \in Y_+$, if $u_1(t) \geq u_2(t)$ then, we have

$$Q_{\lambda^*,\mu^*}u_1 \geq Q_{\lambda^*,\mu^*}u_2. \tag{27}$$

Set $\widehat{u}_0(t) = u_\star(t)$ and $\widehat{u}_n(t) = (Q_{\lambda^\star,\mu^\star}\widehat{u}_{n-1})(t)$, $n = 1, 2, \ldots$, $t \in [0,1]$. Then

$$\widehat{u}_0(t) = u_\star(t) \geq \widehat{u}_1(t) \geq \cdots \geq \widehat{u}_n(t) \geq \cdots \geq L_1 G_1(t,t),$$

where

$$L_1 = \lambda^* \delta_1 \delta_2 \delta_3 \widehat{c} C_{23} C_{12} C_3. \qquad \square$$

Indeed, by Lemma 6, we have

$$\begin{aligned}
\widehat{u}_n(t) &= Q_{\lambda^*,\mu^*}\widehat{u}_{n-1} = (HF)(\widehat{u}_{n-1}) \geq (TF)(\widehat{u}_{n-1}) \\
&\geq \lambda^* \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) f\big(s, \widehat{u}_{n-1}(s)\big)\, ds\, d\tau\, d\nu \\
&\geq \lambda^* \widehat{c} \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s)\, ds\, d\tau\, d\nu \\
&\geq \lambda^* \widehat{c} \delta_1 \delta_2 \delta_3 C_{12} C_{23} C_3 G_1(t,t).
\end{aligned}$$

Now, $f(t,u)$ nondecreasing in u for $t \in [0,1]$, Lemma 2, and the Lebesgue convergence theorem guarantee that $\{u_n\}_{n=0}^\infty = \{Q_{\lambda^*,\mu^*}\widehat{u}_0\}_{n=0}^\infty$ decreases to a fixed point $u^\star \in P \backslash \{\theta\}$ of the operator Q_{λ^*,μ^*}.

Lemma 9 *Suppose that* (H3)-(H5) *hold, and* $L < 1$. *Set*

$$S_{\lambda,\mu} = \big\{ u \in P : Q_{\lambda,\mu} u = u, (\lambda,\mu) \in A \big\},$$

where $A \subset [a,\infty) \times [b,\infty)$ *for some constants* $a > 0$, $b > 0$. *Then there exists a constant* C_A *such that* $\|u\|_0 < C_A$ *for all* $u \in S_{\lambda,\mu}$.

Proof Suppose, to the contrary, that there exists a sequence $\{u_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \|u_n\|_0 = +\infty$, where $u_n \in P$ is a fixed point of the operator $Q_{\lambda,\mu}$ at $(\lambda_n, \mu_n) \in A$ $(n = 1, 2, \ldots)$. Then

$$u_n(t) \geq k \|u_n\|_0 \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

where $k = \frac{\delta_1}{C_1}(1 - L) \min_{t \in [\frac{1}{4},\frac{3}{4}]} G_1(t,t)$.

Choose $J_1 > 0$, so that

$$J_1 a\delta_1\delta_2\delta_3 C_{12}C_{23}m_1m_2k > 2,$$

and $l_1 > 0$ such that

$$f(t,u) \geq J_1 u \quad \text{for } u > l_1 \text{ and } t \in \left[\frac{1}{4}, \frac{3}{4}\right],$$

and N_0, so that $\|u_{N_0}\| > \frac{l_1}{k}$. Now,

$$\begin{aligned}
(Q_{\lambda_{N_0},\mu_{N_0}} u_{N_0})\left(\frac{1}{2}\right) &\geq (TFu_{N_0})\left(\frac{1}{2}\right) \\
&\geq \lambda_{N_0} \int_0^1 \int_0^1 \int_0^1 G_1\left(\frac{1}{2}, v\right) G_2(v,\tau) G_3(\tau,s) f\bigl(s, u_{N_0}(s)\bigr)\,ds\,d\tau\,dv \\
&\geq \lambda_{N_0}\delta_1\delta_2\delta_3 C_{12}C_{23} G_1\left(\frac{1}{2}, \frac{1}{2}\right) \int_0^1 G_3(s,s) f\bigl(s, u_{N_0}(s)\bigr)\,ds \\
&\geq \lambda_{N_0}\delta_1\delta_2\delta_3 C_{12}C_{23} G_1\left(\frac{1}{2}, \frac{1}{2}\right) \int_{\frac{1}{4}}^{\frac{3}{4}} G_3(s,s) f\bigl(s, u_{N_0}(s)\bigr)\,ds \\
&\geq \frac{1}{2} a\delta_1\delta_2\delta_3 C_{12}C_{23} m_1m_2 J_1 u_{N_0}(t) \\
&\geq \frac{1}{2} a\delta_1\delta_2\delta_3 C_{12}C_{23} m_1m_2 J_1 k \|u_{N_0}\|_0 > \|u_{N_0}\|_0,
\end{aligned}$$

and so,

$$\|u_{N_0}\|_0 = \|Q_{\lambda_{N_0},\mu_{N_0}} u_{N_0}\|_0 \geq \bigl\|(TF)u_{N_0}\bigr\|_0 \geq (TFu_{N_0})\left(\frac{1}{2}\right) > \|u_{N_0}\|_0,$$

which is a contradiction. □

Lemma 10 *Suppose that* $L < 1$, (H3) *and* (H4) *hold and that the operator* $Q_{\lambda,\mu}$ *has a positive fixed point in* P *at* $\widehat{\lambda} > 0$ *and* $\widehat{\mu} > 0$. *Then for every* $(\lambda_\star, \mu_\star) \in (0, \widehat{\lambda}) \times (0, \widehat{\mu})$ *there exists a function* $u_\star \in P\backslash\{\theta\}$ *such that* $Q_{\lambda_\star,\mu_\star} u_\star = u_\star$.

Proof Let $\widehat{u}(t)$ be a fixed point of the operator $Q_{\lambda,\mu}$ at $(\widehat{\lambda}, \widehat{\mu})$. Then

$$\widehat{u}(t) = Q_{\widehat{\lambda},\widehat{\mu}} \widehat{u}(t) \geq Q_{\lambda_\star,\mu_\star} \widehat{u}(t),$$

where $0 < \lambda_\star < \widehat{\lambda}$, $0 < \mu_\star < \widehat{\mu}$. Hence

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^1 G_1(t,v) G_2(v,\tau) G_3(\tau,s) \left(\widehat{\lambda} f\bigl(s, \widehat{u}(s)\bigr) + \widehat{\mu}\widehat{u}(s) \int_0^1 G(s,p)\widehat{u}(p)\,dp\right) ds\,d\tau\,dv \\
&\quad\geq \int_0^1 \int_0^1 \int_0^1 G_1(t,v) G_2(v,\tau) G_3(\tau,s) \\
&\qquad\times \left(\lambda_\star f\bigl(s, \widehat{u}(s)\bigr) + \mu_\star \widehat{u}(s) \int_0^1 G(s,p)\widehat{u}(p)\,dp\right) ds\,d\tau\,dv.
\end{aligned}$$

Set

$$(T_{\lambda_\star,\mu_\star}u)(t) = \int_0^1\int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s) \\ \times\left(\lambda_\star f\bigl(s,u(s)\bigr) + \mu_\star u(s)\int_0^1 G(s,p)u(p)\,dp\right)ds\,d\tau\,d\nu$$

and

$$(Q_{\lambda_\star,\mu_\star}u)(t) = T_{\lambda_\star,\mu_\star}u + (TG)T_{\lambda_\star,\mu_\star}u + (TG)^2T_{\lambda_\star,\mu_\star}u + \cdots + (TG)^nT_{\lambda_\star,\mu_\star}u + \cdots$$

$u_0(t) = \widehat{u}(t)$ and $u_n(t) = Q_{\lambda_\star,\mu_\star}u_{n-1}$. Then

$$\begin{aligned} u_0(t) = \widehat{u}(t) &= T_{\widehat{\lambda},\widehat{\mu}}\widehat{u} + (TG)T_{\widehat{\lambda},\widehat{\mu}}\widehat{u} + (TG)^2T_{\widehat{\lambda},\widehat{\mu}}\widehat{u} + \cdots + (TG)^nT_{\widehat{\lambda},\widehat{\mu}}\widehat{u} + \cdots \\ &\geq T_{\lambda_\star,\mu_\star}\widehat{u} + (TG)T_{\lambda_\star,\mu_\star}\widehat{u} + (TG)^2T_{\lambda_\star,\mu_\star}\widehat{u} + \cdots + (TG)^nT_{\lambda_\star,\mu_\star}\widehat{u} + \cdots = u_1(t) \end{aligned}$$

and

$$\begin{aligned} u_n(t) = Q_{\lambda_\star,\mu_\star}u_{n-1} &= T_{\lambda_\star,\mu_\star}u_{n-1} + (TG)T_{\lambda_\star,\mu_\star}u_{n-1} + (TG)^2T_{\lambda_\star,\mu_\star}u_{n-1} + \cdots \\ &\quad + (TG)^nT_{\lambda_\star,\mu_\star}u_{n-1} + \cdots \\ &\geq T_{\lambda_\star,\mu_\star}u_{n-2} + (TG)T_{\lambda_\star,\mu_\star}u_{n-2} + (TG)^2T_{\lambda_\star,\mu_\star}u_{n-2} + \cdots \\ &\quad + (TG)^nT_{\lambda_\star,\mu_\star}u_{n-2} + \cdots = u_{n-1}(t) \end{aligned}$$

because $f(t,u)$ is nondecreasing in u for $t \in [0,1]$ and $T_{\lambda_\star,\mu_\star}u$ is also nondecreasing in u. Thus

$$u_0(t) \geq u_1(t) \geq \cdots \geq u_n(t) \geq u_{n+1}(t) \geq \cdots \geq L_2G_1(t,t), \tag{28}$$

where

$$L_2 = \lambda_{\divideontimes}\widehat{c}\delta_1\delta_2\delta_3C_{12}C_{23}C_3.$$ □

Indeed, by Lemma 6, we have

$$\begin{aligned} u_n(t) &= Q_{\lambda_\star,\mu_\star}u_{n-1} = (HF)(u_{n-1}) \geq T_{\lambda_\star,\mu_\star}(u_{n-1}) \\ &\geq \lambda_\star\int_0^1\int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,u_{n-1}(s)\bigr)\,ds\,d\tau\,d\nu \\ &\geq \lambda_\star\widehat{c}\int_0^1\int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)\,ds\,d\tau\,d\nu \geq \lambda_\star\widehat{c}\delta_1\delta_2\delta_3C_{12}C_{23}C_3G_1(t,t). \end{aligned}$$

Lemma 2 implies that $\{Q^n_{\lambda^\star}u\}_{n=1}^\infty$ decreases to a fixed point $u_\star \in P\backslash\{\theta\}$.

Lemma 11 *Suppose that* $L < 1$, (H3)-(H5) *hold*. *Let*

$$\Lambda = \bigl\{\lambda > 0, \mu > 0 : Q_{\lambda,\mu} \textit{ have at least one fixed point at } (\lambda,\mu) \textit{ in } P\bigr\}.$$

Then Λ *is bounded*.

Proof Suppose, to the contrary, that there exists a fixed point sequence $\{u_n\}_{n=0}^{\infty} \subset P$ of $Q_{\lambda,\mu}$ at (λ_n, μ_n) such that $\lim_{n\to\infty} \lambda_n = \infty$ and $0 < \mu_n < \mu^{**}$. Then there are two cases to be considered: (i) there exists a subsequence $\{u_{n_i}\}_{n=0}^{\infty}$ such that $\lim_{i\to\infty} \|u_{n_i}\|_0 = \infty$, which is impossible by Lemma 9, so we only consider the next case: (ii) there exists a constant $H > 0$ such that $\|u_n\|_0 \le H$, $n = 0, 1, 2, 3, \ldots$. In view of (H3) and (H4), we can choose $l_0 > 0$ such that $f(t, 0) > l_0 H$, and further, $f(t, u_n) > l_0 H$ for $t \in [0, 1]$. We know that

$$u_n = Q_{\lambda_n,\mu_n} u_n \ge T_{\lambda_n,\mu_n} u_n.$$

Let $v_n(t) = T_{\lambda_n,\mu_n} u_n$, *i.e.*, $u_n(t) \ge v_n(t)$. Then it follows that

$$-v_n^{(6)} + a v_n^{(4)} + b v_n'' + c v_n = \lambda_n f(t, u_n) + \mu_n u_n(t) \int_0^1 G(t,p) u_n(p)\, dp, \quad 0 < t < 1. \tag{29}$$

Multiplying (29) by $\sin \pi t$ and integrating over $[0, 1]$, and then using integration by parts on the left side of (29), we have

$$\Gamma \int_0^1 v_n(t) \sin \pi t\, dt = \lambda_n \int_0^1 f(t, u_n) \sin \pi t\, dt + \mu_n \int_0^1 u_n(t) \sin \pi t \int_0^1 G(t,p) u_n(p)\, dp\, dt.$$

Next, assume that (ii) holds. Then

$$\begin{aligned}\Gamma \int_0^1 u_n(t) \sin \pi t\, dt &\ge \Gamma \int_0^1 v_n(t) \sin \pi t\, dt \\ &= \lambda_n \int_0^1 f(t, u_n) \sin \pi t\, dt + \mu_n \int_0^1 u_n(t) \sin \pi t \int_0^1 G(t,p) u_n(p)\, dp\, dt\end{aligned}$$

and

$$\begin{aligned}\Gamma H \int_0^1 \sin \pi t\, dt \ge \Gamma \|u_n\|_0 \int_0^1 \sin \pi t\, dt \ge \Gamma \int_0^1 u_n(t) \sin \pi t\, dt &\ge \Gamma \int_0^1 v_n(t) \sin \pi t\, dt \\ = \lambda_n \int_0^1 f(t, u_n) \sin \pi t\, dt + \mu_n \int_0^1 u_n(t) \sin \pi t &\int_0^1 G(t,p) u_n(p)\, dp\, dt \\ \ge \lambda_n l_0 H \int_0^1 \sin \pi t\, dt&\end{aligned}$$

lead to $\Gamma \ge \lambda_n l_0$, which is a contradiction. The proof is complete. □

Lemma 12 *Suppose that* $L < 1$, (H3)-(H4) *hold. Let*

$$\Lambda_\mu = \{\lambda > 0 : (\lambda, \mu) \in \Lambda \text{ and } \mu \text{ is fixed}\},$$

and let $\widetilde{\lambda}_\mu = \sup \Lambda_\mu$. *Then* $\Lambda_\mu = (0, \widetilde{\lambda}_\mu]$, *where* Λ *is defined in Lemma* 11.

Proof By Lemma 10, it follows that $(0, \widetilde{\lambda}) \times (0, \mu) \subset \Lambda$. We only need to prove $(\widetilde{\lambda}_\mu, \mu) \in \Lambda$. We may choose a distinct nondecreasing sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \Lambda$ such that $\lim_{n\to\infty} \lambda_n = \widetilde{\lambda}_\mu$. Set $u_n \in P$ as a fixed point of $Q_{\lambda,\mu}$ at (λ_n, μ), $n = 1, 2, \ldots$, *i.e.*, $u_n = Q_{\lambda_n,\mu} u_n$. By Lemma 9,

$\{u_n\}_{n=1,}^{\infty}$ is uniformly bounded, so it has a subsequence, denoted by $\{u_{n_k}\}_{k=1}^{\infty}$, converging to $\widetilde{u} \in P$. Note that

$$\begin{aligned} u_n &= T_{\lambda_n,\mu} u_n + (TG) T_{\lambda_n,\mu} u_n + (TG)^2 T_{\lambda_n,\mu} u_n + \cdots + (TG)^n T_{\lambda_n,\mu} u_n + \cdots \\ &= Q_{\lambda_n,\mu} u_n. \end{aligned} \tag{30}$$

Taking the limit as $n \to \infty$ on both sides of (30), and using the Lebesgue convergence theorem, we have

$$\widetilde{u} = T_{\widetilde{\lambda},\mu}\widetilde{u} + (TG)T_{\widetilde{\lambda},\mu}\widetilde{u}_n + (TG)^2 T_{\widetilde{\lambda},\mu}\widetilde{u} + \cdots + (TG)^n T_{\widetilde{\lambda},\mu}\widetilde{u} + \cdots$$

which shows that $Q_{\lambda,\mu}$ has a positive fixed point $\widetilde{u}$ at $(\widetilde{\lambda}, \mu)$. □

Theorem 1 *Suppose that* (H3)-(H5) *hold, and* $L < 1$. *For fixed* $\mu^\star \in (0, \mu^{\star\star})$, *then there exists at* $\lambda^\star > 0$ *such that* (3) *has at least two, one and has no positive solutions for* $0 < \lambda < \lambda^\star$, $\lambda = \lambda^\star$ *for* $\lambda > \lambda^\star$, *respectively.*

Proof Suppose that (H3) and (H4) hold. Then there exists $\lambda^\star > 0$ and $\mu^\star > 0$ such that $Q_{\lambda,\mu}$ has a fixed point $u_{\lambda^\star,\mu^\star} \in P\backslash\{\theta\}$ at $\lambda = \lambda^\star$ and $\mu = \mu^\star$. In view of Lemma 12, $Q_{\lambda,\mu}$ also has a fixed point $u_{\underline{\lambda},\underline{\mu}} < u_{\lambda^\star,\mu^\star}$, $u_{\underline{\lambda},\underline{\mu}} \in P\backslash\{\theta\}$, and $0 < \underline{\lambda} < \lambda^\star$, $0 < \underline{\mu} < \mu^\star$, $\mu^\star \in (0, \mu^{\star\star})$. For $0 < \underline{\lambda} < \lambda^\star\backslash$, there exists $\delta_0 > 0$ such that

$$f(t, u_{\lambda^\star,\mu^\star} + \delta) - f(t, u_{\lambda^\star,\mu^\star}) \le f(t,0)\left(\frac{\lambda^\star}{\underline{\lambda}} - 1\right)$$

for $t \in [0,1]$, $0 < \delta \le \delta_0$. In this case, it is easy to see that

$$\begin{aligned} T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}, + \delta) &= \underline{\lambda} \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s) f\big(s, u_{\lambda^\star,\mu^\star}(s) + \delta\big)\, ds\, d\tau\, dv \\ &\quad + \underline{\mu} \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s)\big(u_{\lambda^\star,\mu^\star}(s) + \delta\big) \\ &\quad \times \int_0^1 G(s,p)\big(u_{\lambda^\star,\mu^\star}(p) + \delta\big)\, dp\, ds\, d\tau\, dv \\ &\le \lambda^\star \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s) f\big(s, u_{\lambda^\star}(s)\big)\, ds\, d\tau\, dv \\ &\quad + \mu^\star \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s) u_{\lambda^\star,\mu^\star}(s) \\ &\quad \times \int_0^1 G(s,p) u_{\lambda^\star,\mu^\star}(p)\, dp\, ds\, d\tau\, dv = T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star}. \end{aligned}$$

Indeed, we have

$$\begin{aligned} &\underline{\lambda} \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s) f\big(s, u_{\lambda^\star}(s) + \delta\big)\, ds\, d\tau\, dv \\ &\quad - \lambda^\star \int_0^1 \int_0^1 \int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s) f\big(s, u_{\lambda^\star}(s)\big)\, ds\, d\tau\, dv \end{aligned}$$

$$
\begin{aligned}
&= \underline{\lambda} \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) \bigl\{ f\bigl(s, u_{\lambda^\star}(s)+\delta\bigr) - f\bigl(s, u_{\lambda^\star}(s)\bigr) \bigr\}\, ds\, d\tau\, d\nu \\
&\quad - \bigl(\lambda^\star - \underline{\lambda}\bigr) \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) f\bigl(s, u_{\lambda^\star}(s)\bigr)\, ds\, d\tau\, d\nu \\
&\le \bigl(\lambda^\star - \underline{\lambda}\bigr) \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) f(s,0)\, ds\, d\tau\, d\nu \\
&\quad - \bigl(\lambda^\star - \underline{\lambda}\bigr) \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) f\bigl(s, u_{\lambda^\star}(s)\bigr)\, ds\, d\tau\, d\nu \\
&= \bigl(\lambda^\star - \underline{\lambda}\bigr) \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) \bigl\{ f(s,0) - f\bigl(s, u_{\lambda^\star}(s)\bigr) \bigr\}\, ds\, d\tau\, d\nu \le 0.
\end{aligned}
$$

Similarly, it is easy to see that

$$
\begin{aligned}
&\underline{\mu} \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) \bigl(u_{\lambda^\star,\mu^\star}(s)+\delta\bigr) \int_0^1 G(s,p) \bigl(u_{\lambda^\star,\mu^\star}(p)+\delta\bigr)\, dp\, ds\, d\tau\, d\nu \\
&\quad - \mu^\star \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) u_{\lambda^\star,\mu^\star}(s) \int_0^1 G(s,p) u_{\lambda^\star,\mu^\star}(p)\, dp\, ds\, d\tau\, d\nu \le 0.
\end{aligned}
$$

Moreover, from (25), it follows that for $T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) \le T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star}$ we have

$$
G\bigl(T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta)\bigr) \le G(T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star}).
$$

Finally, we have

$$
(TG) T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) \le (TG) T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star}.
$$

By induction, it is easy to see that

$$
(TG)^n T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) \le (TG)^n T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star}, \quad n = 1, 2, \ldots. \tag{31}
$$

Hence, using (31), we have

$$
\begin{aligned}
Q_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) &= T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) + (TG) T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) \\
&\quad + (TG)^2 T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) + \cdots + (TG)^n T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) + \cdots \\
&\le T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star} + (TG) T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star} + (TG)^2 T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star} + \cdots \\
&\quad + (TG)^n T_{\lambda^\star,\mu^\star} u_{\lambda^\star,\mu^\star} + \cdots \\
&= Q_{\lambda^\star,\mu^\star}(u_{\lambda^\star,\mu^\star})
\end{aligned}
$$

i.e.,

$$
Q_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) - Q_{\lambda^\star,\mu^\star}(u_{\lambda^\star,\mu^\star}) \le 0,
$$

so that

$$
Q_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star}+\delta) \le Q_{\lambda^\star,\mu^\star}(u_{\lambda^\star,\mu^\star}) = u_{\lambda^\star,\mu^\star} < u_{\lambda^\star,\mu^\star}+\delta.
$$

Set $D_{u_{\lambda^\star,\mu^\star}} = \{u \in C[0,1] : -\delta < u(t) < u_{\lambda^\star,\mu^\star} + \delta\}$. Then $Q_{\underline{\lambda},\underline{\mu}} : P \cap D_{u_{\lambda^\star,\mu^\star}} \to P$ is completely continuous. Furthermore, $Q_{\underline{\lambda},\underline{\mu}} u \neq \upsilon u$ for $\upsilon \geq 1$ and $u \in P \cap \partial D_{u_{\lambda^\star,\mu^\star}}$. Indeed set $u \in P \cap \partial D_{u_{\lambda^\star,\mu^\star}}$. Then there exists $t_0 \in [0,1]$ such that $u(t_0) = \|u\|_0 = \|u_{\lambda^\star,\mu^\star} + \delta\|_0$ and

$$\begin{aligned}
(Q_{\underline{\lambda},\underline{\mu}} u)(t_0) &= \bigl(T_{\underline{\lambda},\underline{\mu}}(u) + (TG)T_{\underline{\lambda},\underline{\mu}}(u) + (TG)^2 T_{\underline{\lambda},\underline{\mu}}(u) + \cdots + (TG)^n T_{\underline{\lambda},\underline{\mu}}(u) + \cdots\bigr)(t_0) \\
&\leq \bigl(T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star} + \delta) + (TG)T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star} + \delta) + (TG)^2 T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star} + \delta) + \cdots \\
&\quad + (TG)^n T_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star} + \delta) + \cdots\bigr)(t_0) = Q_{\underline{\lambda},\underline{\mu}}(u_{\lambda^\star,\mu^\star} + \delta)(t_0) \\
&< u_{\lambda^\star,\mu^\star}(t_0) + \delta = u(t_0) \leq \upsilon u(t_0), \quad \upsilon \geq 1.
\end{aligned}$$

By Lemma 3, $i(Q_{\underline{\lambda},\underline{\mu}}, P \cap \partial D_{u_{\lambda^\star,\mu^\star}}, P) = 1$.

Let k be such that

$$u(t) \geq k\|u\|_0 \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

We know that $\lim_{u\to\infty} \frac{f(t,u)}{u} = \infty$ uniformly for $t \in [0,1]$, so we may choose $J_3 > 0$, so that

$$\underline{\lambda} J_3 \delta_1 \delta_2 \delta_3 C_{12} C_{23} m_1 C_3 k > 2,$$

$l_3 > \|u_{\lambda^\star,\mu^\star} + \delta\|_0 > 0$, so that

$$f(t,u) \geq J_3 u \quad \text{for } u > l_3 \text{ and } t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Set $R_1 = \frac{l_3}{k}$ and $P_{R_1} = \{u \in P : \|u\|_0 < R_1\}$. Then $Q_{\underline{\lambda},\underline{\mu}} : \overline{P}_{R_1} \to P$ is completely continuous. It is easy to obtain

$$\begin{aligned}
(Q_{\underline{\lambda},\underline{\mu}} u)(t) &\geq (T_{\underline{\lambda},\underline{\mu}} u)(t) \geq \underline{\lambda} \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu) G_2(\nu,\tau) G_3(\tau,s) f\bigl(s,u(s)\bigr)\, ds\, d\tau\, d\nu \\
&\geq \underline{\lambda} \delta_1 \delta_2 \delta_3 C_{12} C_{23} G_1(t,t) \int_0^1 G_3(s,s) f\bigl(s,u(s)\bigr)\, ds \\
&\geq \underline{\lambda} \delta_1 \delta_2 \delta_3 C_{12} C_{23} G_1(t,t) \int_{\frac{1}{4}}^{\frac{3}{4}} G_3(s,s) f\bigl(s,u(s)\bigr)\, ds \\
&\geq \frac{1}{2} \underline{\lambda} \delta_1 \delta_2 \delta_3 C_{12} C_{23} m_1 C_3 J_3 u(t) \geq \frac{1}{2} \underline{\lambda} \delta_1 \delta_2 \delta_3 C_{12} C_{23} m_1 C_3 J_3 k \|u\|_0 > \|u\|_0
\end{aligned}$$

for $t \in [0,1]$ and $u \in \partial P_{R_1}$. Now $u(t) \geq k\|u\|_0 = kR_1 = l_3$, and so

$$\|Q_{\underline{\lambda},\underline{\mu}} u\|_0 > \|u\|_0.$$

In view of Lemma 4, $i(Q_{\underline{\lambda},\underline{\mu}}, P_{R_1}, P) = 0$. By the additivity of the fixed point index,

$$i(Q_{\underline{\lambda},\underline{\mu}}, P_{R_1} \setminus \overline{P \cap D}_{u_{\lambda^\star,\mu^\star}}, P) = i(Q_{\underline{\lambda},\underline{\mu}}, P_{R_1}, P) - i(Q_{\underline{\lambda},\underline{\mu}}, P \cap D_{u_{\lambda^\star,\mu^\star}}, P) = -1.$$

Thus $Q_{\underline{\lambda},\underline{\mu}}$ has a fixed point in $\{P \cap D_{u_{\lambda^\star,\mu^\star}}\} \setminus \{\theta\}$ and has another fixed point in $P_{R_1} \setminus P \cap D_{u_{\lambda^\star,\mu^\star}}$ by choosing $\lambda^\star = \widetilde{\lambda}$. □

Let us introduce the notation $\mu = 0$ in the equation of (13), then we have

$$\begin{aligned}&-u^{(6)} + A(t)u^{(4)} + B(t)u'' + C(t)u = \lambda f(t,u),\\ &u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0.\end{aligned} \tag{32}$$

In this case, we can prove the following theorem, which is similar to Theorem 1.

Theorem 2 *Suppose that* (H3)-(H5) *hold, and* $L < 1$. *Then there exists at* $\lambda^\star > 0$ *such that* (32) *has at least two, one and has no positive solutions for* $0 < \lambda < \lambda^\star$, $\lambda = \lambda^\star$ *for* $\lambda > \lambda^\star$, *respectively.*

We follow exactly the same procedure, described in detail in the proof of Theorem 1 for $\mu = 0$.

Let us introduce the following notations for $\mu = 0$ and $\lambda = 1$

$$\begin{aligned}&TFu(t) = \int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,s)G_3(s,\tau)f\bigl(\tau,u(\tau)\bigr)\,d\tau\,ds\,dv,\\ &Qu := HFu = TFu + (TG)TFu + (TG)^2TFu + \cdots + (TG)^nTFu + \cdots,\end{aligned} \tag{33}$$

i.e., $Qu = Q_{1,0}u = HFu$.

Lemma 13 *Suppose that* (H3), (H4) *and* (H6) *hold, and* $L < 1$. *Then for any* $u \in C^+[0,1]\backslash\{\theta\}$, *there exist real numbers* $S_u \ge s_u > 0$ *such that*

$$s_u g(t) \le (Qu)(t) \le S_u g(t), \quad \text{for } t \in [0,1],$$

where $g(t) = \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,v)G_3(v,v)\,dv\,d\tau$.

Proof For any $u \in C^+[0,1]\backslash\{\theta\}$ from Lemma 6, we have

$$\begin{aligned}(Qu)(t) = (HFu)(t) &\le \frac{1}{1-L}\int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s)f\bigl(s,u(s)\bigr)\,ds\,d\tau\,dv\\ &\le \frac{C_3}{1-L}\max_{s\in[0,1]} f\bigl(s,u(s)\bigr)\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,v)G_3(v,v)\,dv\,d\tau\\ &= \frac{C_3}{1-L}\max_{s\in[0,1]} f\bigl(s,u(s)\bigr)g(t) = S_u g(t) \quad \text{for } t \in [0,1].\end{aligned}$$

Note that for any $u \in C^+[0,1]\backslash\{\theta\}$, there exists an interval $[a_1,b_1] \subset (0,1)$ and a number $p > 0$ such that $u(t) \ge p$ for $t \in [a_1,b_1]$. In addition, by (H6), there exists $s_0 > 0$ and $u^0 \in (0,\infty)$ such that $f(t,u^0) \ge s_0$ for $t \in [a_1,b_1]$. If $p \ge u^0$, then $f(t,u) \ge f(t,p) \ge f(t,u^0) \ge s_0$; if $p < u^0$, then $f(t,u) \ge f(t,p) \ge f(t,\frac{p}{u^0}p) \ge (\frac{p}{u^0})^\alpha s_0$. Hence

$$\begin{aligned}(Qu)(t) &\ge (TFu)(t)\\ &= \int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,s)f\bigl(s,u(s)\bigr)\,ds\,d\tau\,dv\\ &\ge \delta_3\int_0^1\int_0^1\int_0^1 G_1(t,v)G_2(v,\tau)G_3(\tau,\tau)G_3(s,s)f\bigl(s,u(s)\bigr)\,ds\,d\tau\,dv\end{aligned}$$

$$\begin{aligned}
&\geq \delta_3 g(t)\int_{a_1}^{b_1} G_3(s,s)f\bigl(s,u(s)\bigr)\,ds\,d\tau\,dv \\
&\geq (b_1-a_1)\delta_3 g(t) m_G\left(\frac{p}{u^0}\right)^{\alpha} = s_u g(t),
\end{aligned}$$

where $m_G = \min_{s\in[a_1,b_1]} G_3(s,s)$, $g(t) = \int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,\tau)\,d\tau\,dv$, $s_u = (b_1-a_1)\times \delta_3 m_G(\frac{p}{u^0})^{\alpha}$. □

Theorem 3 *Suppose that* (H3), (H4) *and* (H6) *hold*, $L < 1$ *and* $\lambda = 1$. *Then*

(i) (32) *has a unique positive solution* $u^\star \in C^+[0,1]\backslash\{\theta\}$ *satisfying*

$$m_u g(t) \leq u^\star(t) \leq M_u g(t) \quad \textit{for } t\in[0,1],$$

where $0 < m_u < M_u$ *are constants.*

(ii) *For any* $u_0(t)\in C^+[0,1]\backslash\{\theta\}$, *the sequence*

$$\begin{aligned}
u_n(t) &= (Qu_{n-1})(t) = (HFu_{n-1})(t) \\
&= TFu_{n-1} + (TG)TFu_{n-1} + (TG)^2TFu_{n-1} + \cdots + (TG)^nTFu_{n-1} + \cdots
\end{aligned}$$

$(n = 1,2,\ldots)$ *converges uniformly to the unique solution* $u^\star$, *and the rate of convergence is determined by*

$$\bigl\|u_n(t) - u^\star(t)\bigr\| = O\bigl(1 - d^{\alpha^n}\bigr),$$

where $0 < d < 1$ *is a positive number.*

Proof In view of (H3), (H4) and (H6), $Q: C^+[0,1]\to C^+[0,1]$ is a nondecreasing operator and satisfies $Q(\rho u)\geq \rho^{\alpha}Q(u)$ for $t\in[0,1]$ and $u\in C^+[0,1]$. Indeed, let $u_\star(t)\leq u_{\star\star}(t)$, $u_\star, u_{\star\star}\in C^+[0,1]$, since $f(s,u)$ is nondecreasing in u, then by using $f(s,u_\star(s))\leq f(s,u_{\star\star}(s))$, for $t\in[0,1]$, it follows that

$$\begin{aligned}
TFu_\star(t) &= \int_0^1\int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,u_\star(s)\bigr)\,ds\,d\tau\,dv \\
&\leq \int_0^1\int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,u_{\star\star}(s)\bigr)\,ds\,d\tau\,dv = TFu_{\star\star}(t).
\end{aligned}$$

Moreover, from (25), it follows that for $TFu_\star(t)\leq TFu_{\star\star}(t)$

$$G(TFu_\star)(t)\leq G(TFu_{\star\star})(t) \quad \text{for } t\in[0,1]. \tag{34}$$

Finally, since $f(s,u)$ is nondecreasing in u, then by using form (34), $f(s,G(TFu_\star)(t))\leq f(s,G(TFu_{\star\star})(t))$, for $t\in[0,1]$, we have

$$\begin{aligned}
(TG)TF(u_\star) &= \int_0^1\int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,G(TFu_\star)(s)\bigr)\,ds\,d\tau\,dv \\
&\leq \int_0^1\int_0^1\int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,G(TFu_{\star\star})(s)\bigr)\,ds\,d\tau\,dv \\
&= (TG)TFu_{\star\star},
\end{aligned}$$

i.e.,

$$(TG)TF(u_\star) \le (TG)TFu_{\star\star}.$$

By induction, it is easy to see that

$$(TG)^n TF(u_\star) \le (TG)^n TFu_{\star\star}, \quad n = 1, 2, \ldots. \tag{35}$$

Hence, using (35), we have

$$\begin{aligned} Q(u_\star) &= TF(u_\star) + (TG)TF(u_\star) + (TG)^2 TF(u_\star) + \cdots + (TG)^n TF(u_\star) + \cdots \\ &\le TF(u_{\star\star}) + (TG)TF(u_{\star\star}) + (TG)^2 TF(u_{\star\star}) + \cdots + (TG)^n TF(u_{\star\star}) + \cdots \\ &= Q(u_{\star\star}). \end{aligned} \tag{36}$$

Now, we show that $Q : C^+[0,1] \to C^+[0,1]$ satisfies $Q(\rho u) \ge \rho^\alpha Q(u)$ for $t \in [0,1]$ and $u \in C^+[0,1]$. Note that

$$\begin{aligned} TF(\rho u) &= \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,\rho u(s)\bigr)\,ds\,d\tau\,d\nu \\ &\ge \rho^\alpha \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,u(s)\bigr)\,ds\,d\tau\,d\nu \\ &= \rho^\alpha TF(u). \end{aligned}$$

Moreover, from (25), it follows that for $TF(\rho u) \ge \rho^\alpha TF(u)$,

$$\begin{aligned} G(TF\rho u)(t) &\ge G\bigl(\rho^\alpha TF(u)\bigr)(t) \\ &= \rho^\alpha G\bigl(TF(u)\bigr)(t) \quad \text{for } t \in [0,1]. \end{aligned}$$

Finally, we have

$$\begin{aligned} (TG)TF(\rho u)(t) &= \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,G(TF\rho u)(s)\bigr)\,ds\,d\tau\,d\nu \\ &\ge \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,\rho^\alpha G\bigl(TF(u)\bigr)(s)\bigr)\,ds\,d\tau\,d\nu \\ &\ge \rho^{\alpha^2} \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,\tau)G_3(\tau,s)f\bigl(s,G\bigl(TF(u)\bigr)(s)\bigr)\,ds\,d\tau\,d\nu \\ &= \rho^{\alpha^2}(TG)TF(u)(t), \end{aligned}$$

i.e.,

$$(TG)(TF\rho u)(t) \ge \rho^{\alpha^2}(TG)TF(u)(t).$$

By induction, it is easy to see that

$$(TG)^n(TF\rho u)(t) \ge \rho^{\alpha^{n+1}}(TG)TF(\rho u)(t), \quad n = 1, 2, \ldots. \tag{37}$$

Hence, using (35) and $\rho \in (0,1)$, $\alpha \in (0,1)$, we have

$$\begin{aligned} Q(\rho u) &= TF(\rho u) + (TG)TF(\rho u) + (TG)^2 TF(\rho u) + \cdots + (TG)^n TF(\rho u) + \cdots \\ &\geq \rho^\alpha TF(u) + \rho^{\alpha^2}(TG)TF(u) + \rho^{\alpha^3}(TG)^2 TF(u) + \cdots + \rho^{\alpha^{n+1}}(TG)^n TF(u) + \cdots \\ &\geq \rho^\alpha TF(u) + \rho^\alpha (TG)TF(u) + \rho^\alpha (TG)^2 TF(u) + \cdots + \rho^\alpha (TG)^n TF(u) + \cdots \\ &= \rho^\alpha \bigl(TF(u) + (TG)TF(u) + (TG)^2 TF(u) + \cdots + (TG)^n TF(u) + \cdots\bigr) \\ &= \rho^\alpha Q(u). \end{aligned} \tag{38}$$

By Lemma 13, there exists $0 < s_g \leq S_g$ such that

$$s_u g(t) \leq Qg(t) \leq S_u g(t).$$

Let

$$s = \sup\bigl\{s_g : s_u g(t) \leq Qg(t)\bigr\}, \qquad S = \inf\bigl\{S_g : Qg(t) \leq S_u g(t)\bigr\}.$$

Pick m_s and M_s such that

$$0 < m_s < \min\bigl\{1, s^{\frac{1}{1-\alpha}}\bigr\} \tag{39}$$

and

$$\max\bigl\{1, S^{\frac{1}{1-\alpha}}\bigr\} = M_s < \infty. \tag{40}$$

Set $u_0(t) = m_s g(t)$, $v_0(t) = M_s g(t)$, $u_n = Qu_{n-1}$, and $v_n = Qv_{n-1}$, $n = 1, 2, \ldots$. From (36) and (38), we have

$$m_s g(t) = u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \leq \cdots \leq v_n(t) \leq \cdots \leq v_1(t) \leq v_0(t) = M_s g(t). \tag{41}$$

Indeed, from (39) $m_s < 1$, and $m_s^{\alpha-1} s > 1$, we have

$$\begin{aligned} u_1(t) = Q(u_0) = Q\bigl(m_s g(t)\bigr) &\geq m_s^\alpha Q\bigl(g(t)\bigr) \geq m_s^\alpha s g(t) \\ &= m_s^{\alpha-1} s m_s g(t) = m_s^{\alpha-1} s u_0(t) \geq u_0(t), \end{aligned}$$

and by induction

$$u_{n+1}(t) = Q(u_n) \geq Q(u_{n-1}) = u_n(t).$$

From (40), $M_s > 1$, and $M_s^{\alpha-1} S < 1$, we have

$$\begin{aligned} v_1(t) = Q(v_0) &\leq M_s^\alpha Q\bigl(g(t)\bigr) = M_s^\alpha Q\left(\frac{1}{M_s} v_0\right) = M_s^\alpha Q(g) \\ &\leq M_s^\alpha S g \leq S M_s^{\alpha-1} M_s g = S M_s^{\alpha-1} v_0(t) \leq v_0(t), \end{aligned}$$

and by induction

$$v_{n+1}(t) = Q(v_n) \le Q(v_{n-1}) = v_n(t).$$

Let $d = \frac{m_s}{M_s}$. Then

$$u_n \ge d^{\alpha^n} v_n. \tag{42}$$

In fact $u_0 = dv_0$ is clear. Assume that (42) holds with $n = k$ (k is a positive integer), *i.e.*, $u_k \ge d^{\alpha^k} v_k$. Then

$$u_{k+1} = Q(u_k) \ge Q\big(d^{\alpha^k} v_k\big) \ge \big(d^{\alpha^k}\big)^\alpha Q(v_k) = d^{\alpha^{k+1}} Q(v_k) = d^{\alpha^{k+1}} v_{k+1}.$$

By induction, it is easy to see that (42) holds. Furthermore, in view of (38), (41) and (42), we have

$$0 \le u_{n+z} - u_n \le v_n - u_n \le \big(1 - d^{\alpha^n}\big) v_0 = \big(1 - d^{\alpha^n}\big) M_s g(t)$$

and

$$\|u_{n+z} - u_n\| \le \|v_n - u_n\| \le \big(1 - d^{\alpha^n}\big) M_s \|g\|,$$

where z is a nonnegative integer. Thus, there exists $u^\star \in C^+[0,1]$ such that

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} v_n(t) = u^\star(t) \quad \text{for } t \in [0,1]$$

and $u^\star(t)$ is a fixed point of Q and satisfies

$$m_g g(t) \le u^\star(t) \le M_g g(t).$$

This means that $u^\star \in C_\star^+[0,1]$, where $C_\star^+[0,1] = \{u \in C^+[0,1], u(t) > 0 \text{ for } t \in (0,1)\}$.

Next we show that $u^\star$ is the unique fixed point of Q in $C_\star^+[0,1]$. Suppose, to the contrary, that there exists another $\overline{u} \in C_\star^+[0,1]$ such that $Q\overline{u} = \overline{u}$. We can suppose that

$$u^\star(t) \le \overline{u}(t), \qquad u^\star(t) \not\equiv \overline{u}(t) \quad \text{for } t \in [0,1].$$

Let $\widehat{\tau} = \sup\{0 < \tau < 1 : \tau u^\star \le \overline{u} \le \tau^{-1} u^\star\}$. Then $0 < \widehat{\tau} \le 1$ and $\widehat{\tau} u^\star \le \overline{u} \le \widehat{\tau}^{-1} u^\star$. We assert $\widehat{\tau} = 1$. Otherwise, $0 < \widehat{\tau} < 1$, and then

$$\overline{u} = Q\overline{u} \ge Q\big(\widehat{\tau} u^\star\big) \ge \widehat{\tau}^\alpha Q\big(u^\star\big) = \widehat{\tau}^\alpha u^\star,$$
$$u^\star = Qu^\star \ge Q(\widehat{\tau}\overline{u}) \ge \widehat{\tau}^\alpha Q(\overline{u}) = \widehat{\tau}^\alpha \overline{u}.$$

This means that $\widehat{\tau}^\alpha u^\star \le \overline{u} \le (\widehat{\tau}^\alpha)^{-1} u^\star$, which is a contradiction of the definition of $\widehat{\tau}$, because $\widehat{\tau} < \widehat{\tau}^\alpha$.

Let us introduce the following notations for $\mu = 0$

$$\begin{aligned} T_\lambda u(t) &:= TFu(t) = \lambda \int_0^1 \int_0^1 \int_0^1 G_1(t,\nu)G_2(\nu,s)G_3(s,\tau)f\big(\tau,u(\tau)\big)\,d\tau\,ds\,d\nu, \\ Q_\lambda u &:= HFu = TFu + (TG)TFu + (TG)^2TFu + \cdots + (TG)^nTFu + \cdots \\ &= T_\lambda u + (TG)T_\lambda u + (TG)^2T_\lambda u + \cdots + (TG)^nT_\lambda u + \cdots, \end{aligned}$$

i.e., $Q_\lambda u = \lambda Qu$, where Q is given by (33). □

Theorem 4 *Suppose that* (H3), (H4), (H6) *and* $L < 1$ *hold*. *Then* (32) *has a unique positive solution* $u_\lambda(t)$ *for any* $0 < \lambda \le 1$.

Proof Theorem 3 implies that for $\lambda = 1$, the operator Q_λ has a unique fixed point $u_1 \in C^+[0,1]$, that is $Q_1u_1 = u_1$. Then from Lemma 10, for every $\lambda_\star \in (0,1)$, there exists a function $u_\star \in P\backslash\{\theta\}$ such that $Q_{\lambda_\star}u_\star = u_\star$.

Thus, u_λ is a unique positive solution of (32) for every $0 < \lambda \le 1$. □

4 Application

As an application of Theorem 1, consider the sixth-order boundary value problem

$$\begin{aligned} &-u^{(6)} + \big(1 - 0.5t^2\big)u^{(4)} + (4.5 - 0.5\sin\pi t)u'' + C(-5 + \cos 0.5\pi t)u \\ &\quad = \big(0.5t(1-t) + u\big)\varphi + \lambda\big(1 + \sin\pi t + u^2\big), \quad 0 < t < 1, \\ &-\varphi'' + 2\varphi = \mu u, \quad 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = u^{(4)}(0) = u^{(4)}(1) = 0, \\ &\varphi(0) = \varphi(1) = 0, \end{aligned} \tag{43}$$

for a fixed $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 1$ and $\varkappa = 2$. In this case, $a = \lambda_1 + \lambda_2 + \lambda_3 = 1$, $b = -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 = 4$, and $c = \lambda_1\lambda_2\lambda_3 = -4$. We have $A(t) = 1 - 0.5t^2$, $B(t) = 4.5 - 0.5\sin\pi t$, $C(t) = -5 + \cos 0.5\pi t$, $D(t) = 0.5t(1-t)$ and $f(t,u) = 1 + \sin\pi t + u^2$. It is easy to see that $\pi^6 + a\pi^4 - b\pi^2 + c = 1{,}015.3 > 0$, $a = \sup_{t\in[0,1]} A(t)$, $b = \inf_{t\in[0,1]} B(t)$ and $c = \sup_{t\in[0,1]} C(t)$. Note also that $K = \max_{0\le t\le 1}[-A(t) + B(t) - C(t) - (-a + b - c)] = 2$, $D_2 = \max_{t\in[0,1]} \int_0^1 G_2(t,\nu)\,d\nu = 0.15768$, $C = \max_{t\in[0,1]} D(t) = 0.125$, $d_1 = \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds = 0.10336$, $\mu^{**} = \frac{1-D_2K}{D_2Cd_1} = 336.1$ and $D_2K = 0.3153 < 1$. Thus, if $0 < \mu < 336.1$, then the conditions of Theorem 1 (note $L = D_2(K + \mu Cd_1) < 1$) are fulfilled (in particular, (H3)-(H5) are satisfied). As a result, Theorem 1 can be applied to (43).

Competing interests
The authors did not provide this information.

Authors' contributions
The authors did not provide this information.

Author details
[1]Department of Mathematics, Texas A&M University-Kingsville, 700 University Blvd., Kingsville, 78363-8202, USA.
[2]Department of Analysis, University of Miskolc, Egyetemvaros, 3515, Hungary. [3]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland.

Received: 20 May 2013 Accepted: 30 July 2013 Published: 14 August 2013

References

1. Evans, JD, Galactionov, VA, King, JR: Unstable sixth-order thin film equation: II. Global similarity patterns. Nonlinearity **20**, 1843-1881 (2007)
2. Yude, J, Yanping, G, Yukun, Y, Yingjie, F: Nodal solutions for sixth-order *m*-point boundary-value problems using bifurcation methods. Electron. J. Differ. Equ. **2012**, Article ID 217 (2012)
3. Liu, C, Liu, A, Tang, H: Time-periodic solutions for a driven sixth-order Cahn-Hilliard type equation. Bound. Value Probl. **2013**, Article ID 73 (2013)
4. Alvelid, M: Sixth order differential equation for sandwich beam deflection including transverse shear. Compos. Struct. **102**, 29-37 (2013)
5. Liu, X, Li, W: Positive solutions for the nonlinear fourth-order beam equation with three parameters. J. Math. Anal. Appl. **303**, 150-163 (2005)
6. Li, W: The existence and multiplicity of positive solutions of nonlinear sixth-order boundary value problem with three variable coefficients. Bound. Value Probl. **2012**, Article ID 22 (2012)
7. El-Gebeily, MA, O'Regan, D, Nieto, JJ: A monotone iterative technique for stationary and time dependent problems in Banach apaces. J. Comput. Appl. Math. **233**, 2395-2404 (2010)
8. Agarwal, RP, Kovacs, B, O'Regan, D: Existence of positive solution for a sixth-order differential system with variable parameters. J. Appl. Math. Comput. (2013). doi:10.1007/s12190-013-0701-1
9. Nieto, JJ: Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions. Proc. Am. Math. Soc. **125**, 2599-2604 (1997)
10. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, New York (1988)
11. Chai, G: Existence of positive solutions for fourth-order boundary value problem with variable parameters. Nonlinear Anal. **66**, 870-880 (2007)

doi:10.1186/1687-2770-2013-184
Cite this article as: Agarwal et al.: Positive solutions for a sixth-order boundary value problem with four parameters. *Boundary Value Problems* 2013 **2013**:184.