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Symmetry of solutions to parabolic Monge-Ampère equations

Limei Dai*

*Correspondence: limeidai@yahoo.com.cn
School of Mathematics and Information Science, Weifang University, Weifang, Shandong 261061, China

Abstract
In this paper, we study the parabolic Monge-Ampère equation

$$-u_t \det(D^2 u) = f(t,u) \quad \text{in } \Omega \times (0,T].$$

Using the method of moving planes, we show that any parabolically convex solution is symmetric with respect to some hyperplane. We also give a counterexample in $\mathbb{R}^n \times (0,T]$ and an example in a cylinder to illustrate the results.

MSC: 35K96; 35B06

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1 Introduction

The Monge-Ampère equation has been of much importance in geometry, optics, stochastic theory, mass transfer problem, mathematical economics and mathematical finance theory. In optics, the reflector antenna system satisfies a partial differential equation of Monge-Ampère type. In [1, 2], Wang showed that the reflector antenna design problem was equivalent to an optimal transfer problem. An optimal transportation problem can be interpreted as providing a weak or generalized solution to the Monge-Ampère mapping problem [3]. More applications of the Monge-Ampère equation and the optimal transportation can be found in [3, 4]. In the meantime, the Monge-Ampère equation turned out to be the prototype for a class of questions arising in differential geometry.

For the study of elliptic Monge-Ampère equations, we can refer to the classical papers [5–7] and the study of parabolic Monge-Ampère equations; see the references [8–11] *etc.* The parabolic Monge-Ampère equation $-u_t \det(D^2u) = f$ was first introduced by Krylov [12] together with the other parabolic versions of elliptic Monge-Ampère equations; see [8] for a complete description and related results. It is also relevant in the study of deformation of surfaces by Gauss-Kronecker curvature [13, 14] and in a maximum principle for parabolic equations [15]. Tso [15] pointed out that the parabolic equation $-u_t \det(D^2u) = f$ is the most appropriate parabolic version of the elliptic Monge-Ampère equation $\det(D^2u) = f$ in the proof of Aleksandrov-Bakelman maximum principle of second-order parabolic equations. In this paper, we study the symmetry of solutions to the parabolic Monge-Ampère equation

$$-u_t \det\bigl(D^2 u\bigr) = f(t,u), \qquad (x,t) \in Q, \tag{1.1}$$

$$u = 0, \quad (x,t) \in SQ, \tag{1.2}$$

$$u = u_0(x), \quad (x,t) \in BQ, \tag{1.3}$$

where D^2u is the Hessian matrix of u in x, $Q = \Omega \times (0,T]$, Ω is a bounded and convex open subset in $\mathbb{R}^n$, $SQ = \partial\Omega \times (0,T)$ denotes the side of Q, $BQ = \overline{\Omega} \times \{0\}$ denotes the bottom of Q, and $\partial_p Q = SQ \cup BQ$ denotes the parabolic boundary of Q, f and u_0 are given functions.

There is vast literature on symmetry and monotonicity of positive solutions of elliptic equations. In 1979, Gidas *et al.* [16] first studied the symmetry of elliptic equations, and they proved that if $\Omega = \mathbb{R}^n$ or Ω is a smooth bounded domain in $\mathbb{R}^n$, convex in x_1 and symmetric with respect to the hyperplane $\{x \in \mathbb{R}^n : x_1 = 0\}$, then any positive solution of the Dirichlet problem

$$\Delta u + f(u) = 0, \quad x \in \Omega,$$
$$u = 0, \quad x \in \partial\Omega$$

satisfies the following symmetry and monotonicity properties:

$$u(-x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_n), \tag{1.4}$$

$$u_{x_1}(x_1, x_2, \ldots, x_n) < 0 \quad (x_1 > 0). \tag{1.5}$$

The basic technique they applied is the method of moving planes first introduced by Alexandrov [17] and then developed by Serrin [18]. Later the symmetry results of elliptic equations have been generalized and extended by many authors. Especially, Li [19] considered fully nonlinear elliptic equations on smooth domains, and Berestycki and Nirenberg [20] found a way to deal with general equations with nonsmooth domains using the maximum principles on domains with small measure. Recently, Zhang and Wang [21] investigated the symmetry of the elliptic Monge-Ampère equation $\det(D^2u) = e^{-u}$ and they got the following results.

Let Ω be a bounded convex domain in $\mathbb{R}^n$ with smooth boundary and symmetric with respect to the hyperplane $\{x \in \mathbb{R}^n : x_1 = 0\}$, then each solution of the Dirichlet problem

$$\det\left(D^2u\right) = e^{-u}, \quad x \in \Omega,$$
$$u = 0, \quad x \in \partial\Omega$$

has the above symmetry and monotonicity properties (1.4) and (1.5). Extensions in various directions including degenerate problems [22] or elliptic systems of equations [23] were studied by many authors.

For the symmetry results of parabolic equations on bounded and unbounded domains, the reader can be referred to [16, 24, 25] and the references therein. In particular, when $Q = \Omega \times J$, $J = (0,T]$, Gidas *et al.* [16] studied parabolic equations $-u_t + \Delta u + f(t,r,u) = 0$ and $-u_t + F(t,x,u,Du,D^2u) = 0$, and they proved that parabolic equations possessed the same symmetry as the above elliptic equations. When $J = (0,\infty)$, Hess and Poláčik [25] first studied the asymptotic symmetry results for classical, bounded, positive solutions of the problem

$$u_t - \Delta u = f(t,u), \quad (x,t) \in \Omega \times J, \tag{1.6}$$

$$u = 0, \quad (x,t) \in \partial\Omega \times J. \tag{1.7}$$

The symmetry of general positive solutions of parabolic equations was investigated in [24, 26, 27] and the references therein. A typical theorem of $J = \mathbb{R}$ is as follows.

Let Ω be convex and symmetric in x_1. If u is a bounded positive solution of (1.6) and (1.7) with $J = \mathbb{R}$ satisfying

$$\inf_{t\in\mathbb{R}} u(x,t) > 0 \quad (x \in \Omega, t \in J),$$

then u has the symmetry and monotonicity properties for each $t \in \mathbb{R}$:

$$u(-x_1, x', t) = u(x_1, x', t) \quad (x = (x_1, x') \in \Omega, t \in \mathbb{R}),$$
$$u_{x_1}(x,t) < 0 \quad (x \in \Omega, x_1 > 0, t \in \mathbb{R}).$$

The result of $J = (0, \infty)$ is as follows.

Assume that u is a bounded positive solution of (1.6) and (1.7) with $J = (0,\infty)$ such that for some sequence $t_n \to \infty$,

$$\liminf_{n\to\infty} u(x,t_n) > 0 \quad (x \in \Omega).$$

Then u is asymptotically symmetric in the sense that

$$\lim_{t\to\infty} \big(u(-x_1, x', t) - u(x_1, x', t)\big) = 0 \quad (x \in \Omega),$$
$$\limsup_{t\to\infty} u_{x_1}(x,t) \le 0 \quad (x \in \Omega, x_1 > 0).$$

In this paper, using the method of moving planes, we obtain the same symmetry of solutions to problem (1.1), (1.2) and (1.3) as elliptic equations.

2 Maximum principles

In this section, we prove some maximum principles. Let Ω be a bounded domain in $\mathbb{R}^n$, let $a^{ij}(x,t)$, $b(x,t)$, $c(x,t)$ be continuous functions in $\overline{Q}$, $Q = \Omega \times (0,T]$. Suppose that $b(x,t) < 0$, $c(x,t)$ is bounded and there exist positive constants λ_0 and Λ_0 such that

$$\lambda_0|\xi|^2 \le a^{ij}(x,t)\xi_i\xi_j \le \Lambda_0|\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Here and in the sequel, we always denote

$$D_i = \frac{\partial}{\partial x_i}, \qquad D_{ij} = \frac{\partial^2}{\partial x_i\,\partial x_j}.$$

We use the standard notation $C^{2k,k}(Q)$ to denote the class of functions u such that the derivatives $D_x^i D_t^j u$ are continuous in Q for $i + 2j \le 2k$.

Theorem 2.1 *Let $\lambda(x,t)$ be a bounded continuous function on $\overline{Q}$, and let the positive function $\varphi \in C^{2,1}(\overline{Q})$ satisfy*

$$b(x,t)\varphi_t + a^{ij}(x,t)D_{ij}\varphi - \lambda(x,t)\varphi \le 0. \tag{2.1}$$

Suppose that $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ satisfies

$$b(x,t)u_t + a^{ij}(x,t)D_{ij}u - c(x,t)u \le 0, \quad (x,t) \in Q, \tag{2.2}$$

$$u \ge 0, \quad (x,t) \in \partial_p Q. \tag{2.3}$$

If

$$c(x,t) > \lambda(x,t), \quad (x,t) \in Q, \tag{2.4}$$

then $u \ge 0$ in Q.

Proof We argue by contradiction. Suppose there exists $(\overline{x},\overline{t}) \in Q$ such that $u(\overline{x},\overline{t}) < 0$. Let

$$v(x,t) = \frac{u(x,t)}{\varphi(x,t)}, \quad (x,t) \in Q.$$

Then $v(\overline{x},\overline{t}) < 0$. Set $v(x_0,t_0) = \min_{\overline{Q}} v(x,t)$, then $x_0 \in \Omega$ and $v(x_0,t_0) < 0$. Since $v(\cdot,t_0)$ attains its minimum at x_0, we have $Dv(x_0,t_0) = 0$, $D^2v(x_0,t_0) \ge 0$. In addition, we have $v_t(x_0,t_0) \le 0$. A direct calculation gives

$$v_t = \frac{u_t\varphi - u\varphi_t}{\varphi^2},$$
$$D_{ij}v = \frac{1}{\varphi}D_{ij}u - \frac{u}{\varphi^2}D_{ij}\varphi - \frac{1}{\varphi}D_ivD_j\varphi - \frac{1}{\varphi}D_jvD_i\varphi.$$

Taking into account $u(x_0,t_0) < 0$, we have at (x_0,t_0),

$$\begin{aligned}0 &\le \varphi a^{ij}D_{ij}v = a^{ij}D_{ij}u - \frac{a^{ij}D_{ij}\varphi}{\varphi}u \\ &\le a^{ij}D_{ij}u + \frac{u}{\varphi}(b\varphi_t - \lambda\varphi) \\ &\le a^{ij}D_{ij}u + \frac{b}{\varphi}u_t\varphi - \lambda u \\ &= a^{ij}D_{ij}u + bu_t - \lambda u \\ &< a^{ij}D_{ij}u + bu_t - cu \\ &\le 0.\end{aligned}$$

This is a contradiction and thus completes the proof of Theorem 2.1. □

Theorem 2.1 is also valid in unbounded domains if u is nonnegative at infinity. Thus we have the following corollary.

Corollary 2.2 *Suppose that* Ω *is unbounded,* $Q = \Omega \times (0, T]$. *Besides the conditions of Theorem* 2.1, *we assume*

$$\liminf_{|x|\to\infty} u(x,t) \geq 0. \tag{2.5}$$

Then $u \geq 0$ *in* Q.

Proof Still consider $v(x,t)$ in the proof of Theorem 2.1. Condition (2.5) shows that the minimum of $v(x,t)$ cannot be achieved at infinity. The rest of the proof is the same as the proof of Theorem 2.1. □

If Ω is a narrow region with width l,

$$\Omega = \{x \in \mathbb{R}^n | 0 < x_1 < l\},$$

then we have the following narrow region principle.

Corollary 2.3 (Narrow region principle) *Suppose that* $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ *satisfies* (2.2) *and* (2.3). *Let the width* l *of* Ω *be sufficiently small. If on* $\partial_p Q$, $u \geq 0$, *then we have* $u \geq 0$ *in* Q. *If* Ω *is unbounded, and* $\liminf_{|x|\to\infty} u(x,t) \geq 0$, *then the conclusion is also true.*

Proof Let $0 < \varepsilon < l$,

$$\varphi(x,t) = t + \sin\frac{x_1 + \varepsilon}{l}.$$

Then φ is positive and

$$\begin{aligned} &\varphi_t = 1, \\ &a^{ij} D_{ij}\varphi = -\left(\frac{1}{l}\right)^2 a^{11}\varphi. \end{aligned}$$

Choose $\lambda(x,t) = -\lambda_0 / l^2$. In virtue of the boundedness of $c(x,t)$, when l is sufficiently small, we have $c(x,t) > \lambda(x,t)$, and thus

$$\begin{aligned} &b\varphi_t + a^{ij} D_{ij}\varphi - \lambda\varphi \\ &\quad = b - \left(\frac{1}{l}\right)^2 a^{11}\varphi - \left(-\frac{\lambda_0}{l^2}\right)\varphi \\ &\quad = b - \left(\frac{1}{l}\right)^2 a^{11}\varphi + \frac{\lambda_0}{l^2}\varphi \\ &\quad \leq b < 0. \end{aligned}$$

From Theorem 2.1, we have $u \geq 0$. □

3 Main results

In this section, we prove that the solutions of (1.1), (1.2) and (1.3) are symmetric by the method of moving planes.

Definition 3.1 A function $u(x,t) : Q \to \mathbb{R}$ is called parabolically convex if it is continuous, convex in x and decreasing in t.

Suppose that the following conditions hold.

(A) $f_u(t,u)/f(t,u)$ is bounded in $[0,T] \times \mathbb{R}$.

(B) $\partial u_0/\partial x_1 < 0$ and

$$u_0(x) \leq u_0(x^\lambda), \quad x \in \Omega^\lambda, \tag{3.1}$$

where $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$, $\Omega^\lambda = \Omega \cap \{x \in \Omega : x_1 \leq \lambda\}$ $(\lambda < 0)$.

Theorem 3.1 *Let Ω be a strictly convex domain in $\mathbb{R}^n$ and symmetric with respect to the plane $\{x \in \Omega : x_1 = 0\}$, $Q = \Omega \times (0,T]$. Assume that conditions* (A) *and* (B) *hold and $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ is any parabolically convex solution of* (1.1), (1.2) *and* (1.3). *Then $u(x_1,x',t) = u(-x_1,x',t)$, where $(x,t) = (x_1,x',t) \in \mathbb{R}^{n+1}$, and when $x_1 \geq 0$, $\partial u(x,t)/\partial x_1 \leq 0$.*

Proof Let in $\Omega^\lambda \times (0,T]$, $u^\lambda(x,t) = u(x^\lambda,t)$, that is,

$$u^\lambda(x_1,x_2,\dots,x_n,t) = u(2\lambda - x_1,x_2,\dots,x_n,t), \quad (x,t) \in \Omega^\lambda \times (0,T].$$

Then

$$D^2u^\lambda(x_1,x_2,\dots,x_n,t) = P^T D^2u(2\lambda - x_1,x_2,\dots,x_n,t)P,$$

where $P = \operatorname{diag}(-1,1,\dots,1)$. Therefore,

$$\begin{aligned} -u_t^\lambda \det\bigl(D^2u^\lambda\bigr) &= -u_t(2\lambda - x_1,x_2,\dots,x_n,t)\det\bigl(D^2u(2\lambda - x_1,x_2,\dots,x_n,t)\bigr) \\ &= f\bigl(t,u(2\lambda - x_1,x_2,\dots,x_n,t)\bigr) \\ &= f\bigl(t,u^\lambda\bigr). \end{aligned} \tag{3.2}$$

We rewrite (3.2) in the form

$$\log\bigl(-u_t^\lambda\bigr) + \log\bigl(\det\bigl(D^2u^\lambda\bigr)\bigr) = \log f\bigl(t,u^\lambda\bigr). \tag{3.3}$$

On the other hand, from (1.1), we have

$$\log(-u_t) + \log\bigl(\det\bigl(D^2u\bigr)\bigr) = \log f(t,u). \tag{3.4}$$

According to (3.3) and (3.4), we have

$$\log(-u_t) - \log\bigl(-u_t^\lambda\bigr) + \log\bigl(\det\bigl(D^2u\bigr)\bigr) - \log\bigl(\det\bigl(D^2u^\lambda\bigr)\bigr) = \log f(t,u) - \log f\bigl(t,u^\lambda\bigr).$$

Therefore

$$\int_0^1 \frac{d}{ds}\log\bigl(-su_t-(1-s)u_t^\lambda\bigr)\,ds+\int_0^1 \frac{d}{ds}\log\det\bigl(sD^2u+(1-s)D^2u^\lambda\bigr)\,ds$$
$$=\int_0^1 \frac{d}{ds}\log f\bigl(t,su+(1-s)u^\lambda\bigr)\,ds.$$

As a result, we have

$$b(x,t)\bigl(u-u^\lambda\bigr)_t+a^{ij}(x,t)\bigl(u-u^\lambda\bigr)_{ij}-c(x,t)\bigl(u-u^\lambda\bigr)=0,\quad (x,t)\in\Omega^\lambda\times(0,T],\tag{3.5}$$

where

$$b(x,t)=\int_0^1 \frac{ds}{su_t+(1-s)u_t^\lambda},$$
$$a^{ij}(x,t)=\int_0^1 g_s^{ij}\,ds,$$
$$c(x,t)=\int_0^1 \frac{f_u}{f}\bigl(t,su+(1-s)u^\lambda\bigr)\,ds,$$

g_s^{ij} is the inverse matrix of $sD^2u+(1-s)D^2u^\lambda$. Then $b(x,t)<0$, $c(x,t)$ is bounded and by the *a priori* estimate [9] we know there exist positive constants λ_0 and Λ_0 such that

$$\lambda_0|\xi|^2\le a^{ij}\xi_i\xi_j\le\Lambda_0|\xi|^2,\quad \forall\xi\in\mathbb{R}^n.$$

Let

$$w^\lambda=u-u^\lambda,$$

then from (3.5),

$$b(x,t)w_t^\lambda+a^{ij}(x,t)w_{ij}^\lambda-c(x,t)w^\lambda=0,\quad (x,t)\in\Omega^\lambda\times(0,T].\tag{3.6}$$

Clearly,

$$w^\lambda(x,t)=0,\quad x\in\partial\Omega^\lambda\cap\{x_1=\lambda\},0<t\le T.\tag{3.7}$$

Because the image of $\partial\Omega\cap\partial\Omega^\lambda$ about the plane $\{x_1=\lambda\}$ lies in Ω, according to the maximum principle of parabolic Monge-Ampère equations,

$$u^\lambda(x,t)\le 0,\quad \forall x\in\partial\Omega\cap\partial\Omega^\lambda.$$

Thus

$$w^\lambda(x,t)=u-u^\lambda=0-u^\lambda\ge 0,\quad x\in\partial\Omega\cap\partial\Omega^\lambda,0<t\le T.\tag{3.8}$$

On the other hand, from (3.1),

$$w^{\lambda}(x,0) = u_0(x) - u_0\left(x^{\lambda}\right) \geq 0, \quad x \in \Omega^{\lambda}. \tag{3.9}$$

From Corollary 2.3, when the width of Ω^{λ} is sufficiently small, $w^{\lambda}(x,t) \geq 0$, $(x,t) \in \Omega^{\lambda} \times (0,T]$.

Now we start to move the plane to its right limit. Define

$$\Lambda = \sup\left\{\lambda < 0 | w^{\lambda}(x,t) \geq 0, x \in \Omega^{\lambda}, 0 < t \leq T\right\}.$$

We claim that

$$\Lambda = 0.$$

Otherwise, we will show that the plane can be further moved to the right by a small distance, and this would contradict with the definition of Λ.

In fact, if $\Lambda < 0$, then the image of $\partial\Omega \cap \partial\Omega^{\Lambda}$ under the reflection about $\{x_1 = \Lambda\}$ lies inside Ω. According to the strong maximum principle of parabolic Monge-Ampère equations, for $x \in \Omega$, $u^{\Lambda} < 0$. Therefore, for $x \in \partial\Omega^{\Lambda} \cap \partial\Omega$, we have $w^{\Lambda} > 0$. On the other hand, by the definition of Λ, we have for $x \in \Omega^{\Lambda}$, $w^{\Lambda} \geq 0$. So, from the strong maximum principle [28] of linear parabolic equations and (3.6), we have for $(x,t) \in \Omega^{\Lambda} \times (0,T]$,

$$w^{\Lambda}(x,t) > 0. \tag{3.10}$$

Let d_0 be the maximum width of narrow regions so that we can apply the narrow region principle. Choose a small positive constant δ such that $\Lambda + \delta < 0$, $\delta \leq d_0/2 - \Lambda$. We consider the function $w^{\Lambda+\delta}(x,t)$ on the narrow region

$$\Sigma^{\Lambda+\delta} \times (0,T] = \left(\Omega^{\Lambda+\delta} \cap \left\{x_1 > \Lambda - \frac{d_0}{2}\right\}\right) \times (0,T].$$

Then $w^{\Lambda+\delta}(x,t)$ satisfies

$$b(x,t)w_t^{\Lambda+\delta} + a^{ij}(x,t)D_{ij}w^{\Lambda+\delta} - c(x,t)w^{\Lambda+\delta} = 0, \quad (x,t) \in \Sigma^{\Lambda+\delta} \times (0,T]. \tag{3.11}$$

Now we prove the boundary condition

$$w^{\Lambda+\delta}(x,t) \geq 0, \quad (x,t) \in \partial_p\left(\Sigma^{\Lambda+\delta} \times (0,T]\right). \tag{3.12}$$

Similar to boundary conditions (3.7), (3.8) and (3.9), boundary condition (3.12) is satisfied for $x \in \partial\Sigma^{\Lambda+\delta} \cap \partial\Omega$, $x \in \partial\Sigma^{\Lambda+\delta} \cap \{x_1 = \Lambda + \delta\}$ and for $t = 0$. In order to prove (3.12) is satisfied for $x \in \partial\Sigma^{\Lambda+\delta} \cap \{x_1 = \Lambda - d_0/2\}$, we apply the continuity argument. By (3.10) and the fact that $(\Lambda - d_0/2, x_2, \ldots, x_n)$ is inside Ω^{Λ}, there exists a positive constant c_0 such that

$$w^{\Lambda}\left(\Lambda - \frac{d_0}{2}, x_2, \ldots, x_n, t\right) \geq c_0.$$

Because w^λ is continuous in λ, then for small δ, we still have

$$w^{\Lambda+\delta}\left(\Lambda - \frac{d_0}{2}, x_2, \dots, x_n, t\right) \geq 0.$$

Therefore boundary condition (3.12) holds for small δ. From Corollary 2.3, we have

$$w^{\Lambda+\delta}(x,t) \geq 0, \quad x \in \Sigma^{\Lambda+\delta}, 0 < t \leq T. \tag{3.13}$$

Combining (3.10) and the fact that w^λ is continuous for λ, we know that $w^{\Lambda+\delta}(x,t) \geq 0$ for $x \in \Omega^\Lambda$ when δ is small. Then from (3.13), we know that

$$w^{\Lambda+\delta}(x,t) \geq 0, \quad x \in \Omega^{\Lambda+\delta}, 0 < t \leq T.$$

This contradicts with the definition of Λ, and so $\Lambda = 0$.

As a result, $w^0(x,t) \geq 0$ for $x \in \Omega^0$, which means that as $x_1 < 0$,

$$u(x_1, x_2, \dots, x_n, t) \geq u(-x_1, x_2, \dots, x_n, t).$$

Since Ω is symmetric about the plane $\{x_1 = 0\}$, then for $x_1 \geq 0$, $u(-x_1, x_2, \dots, x_n, t)$ also satisfies (1.1). Thus we can move the plane from the right towards the left and get the reverse inequality. Therefore

$$\begin{aligned} &\partial u(x,t)/\partial x_1 \leq 0, \quad x_1 \geq 0, \\ &u(x_1, x_2, \dots, x_n, t) = u(-x_1, x_2, \dots, x_n, t). \end{aligned} \tag{3.14}$$

Equation (3.14) means that u is symmetric about the plane $\{x_1 = 0\}$. Theorem 3.1 is proved. □

If we put the x_1 axis in any direction, from Theorem 3.1, we have the following.

Corollary 3.2 *If Ω is a ball, $Q = \Omega \times (0,T]$, then any parabolically convex solution $u \in C^{2,1}(\overline{Q})$ of* (1.1), (1.2) *and* (1.3) *is radially symmetric about the origin.*

Remark 3.1 Solutions of (1.1) in $\mathbb{R}^n \times (0,T]$ may not be radially symmetric. For example,

$$-u_t \det(D^2 u) = e^{-u}, \quad (x,t) \in \mathbb{R}^n \times (0,T] \tag{3.15}$$

has a non-radially symmetric solution. In fact, we know that $f(x) = 2\log(1 + e^{\sqrt{2}x}) - \sqrt{2}x - \log 4$ $(x > 0)$ satisfies $f'' = e^{-f}$ in $\mathbb{R}^1$, and $f(x) = f(-x)$, $x < 0$. Define

$$u(x,t) = \log(T-t) + f(x_1) + f(x_2) + \cdots + f(x_n),$$

then u is a solution of (3.15) but not radially symmetric.

We conclude this paper with a brief examination of Theorem 3.1. Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^n$, and let radially symmetric function $u_0(x) = u_0(r)$, $r = |x|$ satisfy

$$\frac{u_0(r)(u_0'(r))^{n-1}u_0''(r)}{r^{n-1}} = -1, \quad 0 < r < 1, \tag{3.16}$$

$$u_0(1) = u_0'(0) = 0. \tag{3.17}$$

Example 3.1 Let u_0 satisfy (3.16) and (3.17). Then any solution of

$$-u_t \det(D^2u) = 1, \quad (x,t) \in B \times (0,T], \tag{3.18}$$

$$u = 0, \quad (x,t) \in \partial B \times (0,T), \tag{3.19}$$

$$u = u_0, \quad (x,t) \in \overline{B} \times \{0\} \tag{3.20}$$

is of the form

$$u = -\big[(n+1)t + 1\big]^{\frac{1}{n+1}} u_0(r), \tag{3.21}$$

where $r = |x|$.

Proof According to Corollary 3.2, the solution is symmetric. Let

$$u(x,t) = u(r,t), \qquad r = |x|.$$

Then

$$u_i = \frac{\partial u(r,t)}{\partial r}\frac{x_i}{r},$$
$$u_{ij} = \frac{\partial^2 u(r,t)}{\partial r^2}\frac{x_ix_j}{r^2} + \frac{\partial u(r,t)}{\partial r}\left(\frac{\delta_{ij}}{r} - \frac{x_ix_j}{r^3}\right),$$
$$\det(D^2u) = \left(\frac{\partial u/\partial r}{r}\right)^{n-1}\frac{\partial^2 u}{\partial r^2}.$$

Therefore (3.18) is

$$-\frac{\partial u}{\partial t}\left(\frac{\partial u/\partial r}{r}\right)^{n-1}\frac{\partial^2 u}{\partial r^2} = 1. \tag{3.22}$$

We seek the solution of the form

$$u(r,t) = T(t)u_0(r).$$

Then

$$-u_0(r)T'(t)\frac{(u_0'(r)T(t))^{n-1}}{r^{n-1}}u_0''(r)T(t) = 1.$$

That is,

$$\frac{u_0(r)(u_0'(r))^{n-1}u_0''(r)}{r^{n-1}} = -\frac{1}{T'(t)(T(t))^n}. \tag{3.23}$$

Therefore

$$T'(t)\big(T(t)\big)^n = 1. \tag{3.24}$$

By (3.20), we know that

$$T(0) = 1. \tag{3.25}$$

From (3.24) and (3.25), we have

$$T(t) = \big[(n+1)t + 1\big]^{\frac{1}{n+1}}.$$

As a result,

$$u(r,t) = -\big[(n+1)t + 1\big]^{\frac{1}{n+1}} u_0(r).$$

From the maximum principle, we know that the solution of (3.18)-(3.20) is unique. Thus any solution of (3.18), (3.19) and (3.20) is of the form of (3.21). □

Competing interests
The author declares that they have no competing interests.

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