Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Blow-up criteria for smooth solutions to the generalized 3D MHD equations

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Abstract
In this paper, we focus on the generalized 3D magnetohydrodynamic equations. Two logarithmically blow-up criteria of smooth solutions are established.

MSC: 76D03; 76W05

Keywords: generalized MHD equations; blow-up criteria

1 Introduction

We study blow up criteria of smooth solutions to the incompressible generalized magnetohydrodynamics (GMHD) equations in $\mathbb{R}^3$

$$\begin{cases} \partial_t u + u \cdot \nabla u - B \cdot \nabla B + (-\Delta)^\alpha u + \nabla(p + \frac{1}{2}|B|^2) = 0, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u + (-\Delta)^\beta B = 0, \\ \nabla \cdot u = 0, \qquad \nabla \cdot B = 0 \end{cases} \tag{1.1}$$

with the initial condition

$$t = 0: \quad u = u_0(x), B = B_0(x), \quad x \in \mathbb{R}^3. \tag{1.2}$$

Here $u = (u_1, u_2, u_3)$, $B = (B_1, B_2, B_3)$ and $P = p + \frac{1}{2}|B|^2$ are non-dimensional quantities corresponding to the flow velocity, the magnetic field and the total kinetic pressure at the point (x, t), while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic field with $\nabla \cdot u_0 = 0$ and $\nabla \cdot B_0 = 0$, respectively.

The GMHD equations is a generalized model of MHD equations. It has important physical background. Therefore, the GMHD equations are also mathematically significant. For 3D Navier-Stokes equations, whether there exists a global smooth solution to 3D impressible GMHD equations is still an open problem. In the absence of global well-posedness, the development of blow-up/ non blow-up theory is of major importance for both theoretical and practical purposes. Fundamental mathematical issues such as the global regularity of their solutions have generated extensive research and many interesting results have been established (see [1–5]).

When $\alpha = \beta = 1$, (1.1) reduces to MHD equations. There are numerous important progresses on the fundamental issue of the regularity for the weak solution to (1.1), (1.2) (see [6–18]). A criterion for the breakdown of classical solutions to (1.1) with zero viscosity and positive resistivity in $\mathbb{R}^3$ was derived in [9]. Some sufficient integrability conditions on two components or the gradient of two components of $u + B$ and $u - B$ in Morrey-Campanato spaces were obtained in [10]. A logarithmal improved blow-up criterion of smooth solutions in an appropriate homogeneous Besov space was obtained by Wang *et al.* [11]. Zhou and Fan [15] established various logarithmically improved regularity criteria for the 3D MHD equations in terms of the velocity field and pressure, respectively. These regularity criteria can be regarded as log in time improvements of the standard Serrin criteria established before. Two new regularity criteria for the 3D incompressible MHD equations involving partial components of the velocity and magnetic fields were obtained by Jia and Zhou [17].

When $\alpha = 1$, $B = 0$, (1.1) reduces to Navier-Stokes equations. Leray [19] and Hopf [20] constructed weak solutions to the Navier-Stokes equations, respectively. The solution is called the Leray-Hopf weak solution. Later on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equations. Different criteria for regularity of the weak solutions have been proposed and many interesting results have been obtained [21–25].

In the paper, we obtain two logarithmically blow-up criteria of smooth solutions to (1.1), (1.2) in Morrey-Campanato spaces. We hope that the study of equations (1.1) can improve the understanding of the problem of Navier-Stokes equations and MHD equations.

Now we state our results as follows.

Theorem 1.1 *Let* $u_0, B_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$, *with* $\nabla \cdot u_0 = 0$, $\nabla \cdot B_0 = 0$ *and* $\frac{3}{4} < \alpha = \beta \leq 1$. *Assume that* (u, B) *is a smooth solution to* (1.1), (1.2) *on* $[0, T)$. *If u satisfies*

$$\int_0^T \frac{\|u(t)\|_{\dot{M}_{p,q}}^r}{1 + \ln(e + \|u(t)\|_{L^\infty})}\,dt < \infty, \qquad \frac{2\alpha}{r} + \frac{3}{p} + 1 = 2\alpha, \frac{3}{2\alpha - 1} < p \leq \infty, 1 < p \leq q, \tag{1.3}$$

then the solution (u, B) *can be extended beyond* $t = T$.

We have the following corollary immediately.

Corollary 1.1 *Let* $u_0, B_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$, *with* $\nabla \cdot u_0 = 0$, $\nabla \cdot B_0 = 0$ *and* $\frac{3}{4} < \alpha = \beta \leq 1$. *Assume that* (u, B) *is a smooth solution to* (1.1), (1.2) *on* $[0, T)$. *Suppose that* T *is the maximal existence time, then*

$$\int_0^T \frac{\|u(t)\|_{\dot{M}_{p,q}}^r}{1 + \ln(e + \|u(t)\|_{L^\infty})}\,dt = \infty, \qquad \frac{2\alpha}{r} + \frac{3}{p} + 1 = 2\alpha, \frac{3}{2\alpha - 1} < p \leq \infty, 1 < p \leq q. \tag{1.4}$$

Theorem 1.2 *Let* $u_0, B_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$, *with* $\nabla \cdot u_0 = 0$, $\nabla \cdot B_0 = 0$ *and* $\frac{3}{4} < \alpha = \beta \leq 1$. *Assume that* (u, B) *is a smooth solution to* (1.1), (1.2) *on* $[0, T)$. *If*

$$\int_0^T \frac{\|\nabla u(t)\|_{\dot{M}_{p,q}}^r}{1 + \ln(e + \|\nabla u(t)\|_{L^\infty})}\,dt < \infty, \qquad \frac{2\alpha}{r} + \frac{3}{p} = 2\alpha, \frac{3}{2\alpha} < p \leq \infty, 1 < p \leq q, \tag{1.5}$$

then the solution (u, B) *can be extended beyond* $t = T$.

We have the following corollary immediately.

Corollary 1.2 *Let $u_0, B_0 \in H^m(\mathbb{R}^3)$, $m \geq 3$, with $\nabla \cdot u_0 = 0$, $\nabla \cdot B_0 = 0$ and $\frac{3}{4} < \alpha = \beta \leq 1$. Assume that (u, B) is a smooth solution to* (1.1), (1.2) *on* $[0, T)$. *Suppose that T is the maximal existence time, then*

$$\int_0^T \frac{\|\nabla u(t)\|^r_{\dot{M}_{p,q}}}{1 + \ln(e + \|\nabla u(t)\|_{L^\infty})}\,dt = \infty, \qquad \frac{2\alpha}{r} + \frac{3}{p} = 2\alpha, \frac{3}{2\alpha} < p \leq \infty, 1 < p \leq q. \tag{1.6}$$

The paper is organized as follows. We first state some preliminaries on function spaces and some important inequalities in Section 2. Then we prove main results in Section 3 and Section 4, respectively.

2 Preliminaries

Before stating our main results, we recall the definition and some properties of the homogeneous Morrey-Campanato space.

Definition 2.1 For $1 < p \leq q \leq +\infty$, the Morrey-Campanato space $\dot{M}_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{M}_{p,q} = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^3) : \|f\|_{\dot{M}_{p,q}} = \sup_{R>0} \sup_{x \in \mathbb{R}^3} R^{\frac{3}{q} - \frac{3}{p}} \|f\|_{L^p(B(x,R))} < \infty \right\},$$

where $B(x, R)$ denotes the ball of center x with radius R.

Let $1 \leq q' \leq p' < \infty$, we define the homogeneous space $\dot{N}_{p',q'}(\mathbb{R}^3)$ by

$$\dot{N}_{p',q'}(\mathbb{R}^3) = \begin{cases} f \in L^{p'}(\mathbb{R}^3) | f = \sum_{k \in \mathbb{N}} g_k, & \text{where } (g_k) \subset L^{p'}_{\mathrm{comp}}(\mathbb{R}^3) \text{ and} \\ \sum_{k \in \mathbb{R}^3} d_k^{3(\frac{1}{q'} - \frac{1}{p'})} \|g_k\|_{L^{p'}} < \infty, & \text{where for any } k, d_k = \mathrm{diam}(\mathrm{Supp}\, g_k) < \infty, \end{cases}$$

where $L^{p'}_{\mathrm{comp}}(\mathbb{R}^3)$ is the space of all $L^{p'}$ functions in $\mathbb{R}^3$ with compact support. $\dot{N}_{p',q'}(\mathbb{R}^3)$ is a Banach space when it is equipped with the norm

$$\|f\|_{\dot{N}_{p',q'}} = \inf\left\{ \sum_{k \in \mathbb{N}} d_k^{3(\frac{1}{q'} - \frac{1}{p'})} \|g_k\|_{L^{p'}} \right\},$$

where the infimum is taken over all possible decompositions.

Lemma 2.1 *Let $1 < q' \leq p' < \infty$ and p, q satisfy $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then $\dot{M}_{p,q}(\mathbb{R}^3)$ is the dual space of $\dot{N}_{p',q'}(\mathbb{R}^3)$.*

Lemma 2.2 *Let $1 < q' \leq p' < 2$, $m \geq 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Set*

$$\gamma = -\frac{3}{2} + \frac{3}{q} + \frac{3}{m} \in (0, 1].$$

Then there exists a constant $C > 0$ such that for any $f \in \dot{H}^\gamma$ and $g \in L^m$,

$$\|fg\|_{\dot{N}_{p',q'}} \leq C\|f\|_{\dot{H}^\gamma}\|g\|_{L^m}.$$

The following lemma comes from [21].

Lemma 2.3 *Assume that* $1 < p < \infty$. *For* $f, g \in W^{m,p}$, *and* $1 < q \leq \infty$, $1 < r < \infty$, *we have*

$$\|\nabla^\alpha (fg) - f\nabla^\alpha g\|_{L^p} \leq C\left(\|\nabla f\|_{L^{q_1}} \|\nabla^{\alpha-1} g\|_{L^{r_1}} + \|g\|_{L^{q_2}} \|\nabla^\alpha f\|_{L^{r_2}}\right), \tag{2.1}$$

where $1 \leq \alpha \leq m$ *and* $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{r_1} = \frac{1}{q_2} + \frac{1}{r_2}$.

The following inequality is the well-known Gagliardo-Nirenberg inequality.

Lemma 2.4 *Let* j, m *be any integers satisfying* $0 \leq j < m$, *and let* $1 \leq q, r \leq \infty$, *and* $p \in \mathbb{R}$, $\frac{j}{m} \leq \theta \leq 1$ *be such that*

$$\frac{1}{p} - \frac{j}{n} = \theta\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q}.$$

Then, for all $f \in L^q(\mathbb{R}^n) \cap W^{m,r}(\mathbb{R}^n)$, *there is a positive constant* C *depending only on* n, m, j, q, r, θ *such that the following inequality holds*:

$$\|\nabla^j f\|_{L^p} \leq C\|f\|_{L^q}^{1-\theta} \|\nabla^m f\|_{L^r}^\theta \tag{2.2}$$

with the following exception: *if* $1 < r < \infty$ *and* $m - j - \frac{n}{r}$ *is a nonnegative integer, then* (2.2) *holds only for a satisfying* $\frac{j}{m} \leq \theta < 1$.

3 Proof of Theorem 1.1

Proof Let $\wedge = (-\Delta)^{\frac{1}{2}}$. We multiply the first equation of (1.1) by $-\Delta u$ and use integration by parts. This yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\wedge^{1+\alpha} u(t)\|_{L^2}^2 = \int_{\mathbb{R}^3} u \cdot \nabla u \Delta u \, dx - \int_{\mathbb{R}^3} B \cdot \nabla B \Delta u \, dx. \tag{3.1}$$

Similarly, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla B(t)\|_{L^2}^2 + \|\wedge^{1+\alpha} B(t)\|_{L^2}^2 = \int_{\mathbb{R}^3} u \cdot \nabla B \Delta B \, dx - \int_{\mathbb{R}^3} B \cdot \nabla u \Delta B \, dx. \tag{3.2}$$

Summing up (3.1) and (3.2), we deduce that

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2\right) + \left(\|\wedge^{1+\alpha} u(t)\|_{L^2}^2 + \|\wedge^{1+\alpha} B(t)\|_{L^2}^2\right) \\ &\quad = \int_{\mathbb{R}^3} u \cdot \nabla u \Delta u \, dx - \int_{\mathbb{R}^3} B \cdot \nabla B \Delta u \, dx + \int_{\mathbb{R}^3} u \cdot \nabla B \Delta B \, dx - \int_{\mathbb{R}^3} B \cdot \nabla u \Delta B \, dx \\ &\quad \triangleq I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.3}$$

By using Lemmas 2.1, 2.2 and (2.2), we have

$$\begin{aligned} I_1 &\leq C\|u\|_{\dot{M}_{p,q}} \|\nabla u \Delta u\|_{\dot{N}_{p,q}} \\ &\leq C\|u\|_{\dot{M}_{p,q}} \|\nabla u\|_{\dot{H}^\alpha} \|\Delta u\|_{L^m} \end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|_{\dot{M}_{p,q}}\|\wedge^{1+\alpha}u\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}u\|_{L^2}^{\frac{5}{2\alpha}-\frac{3}{m\alpha}} \\
&\le C\|u\|_{\dot{M}_{p,q}}\|\nabla u\|_{L^2}^{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}u\|_{L^2}^{1+\frac{5}{2\alpha}-\frac{3}{m\alpha}} \\
&\le \frac{1}{8}\|\wedge^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{\dot{M}_{p,q}}^r\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.4}
$$

where $r = \frac{2}{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}$.

We apply integration by parts, $\nabla\cdot B = 0$, Lemmas 2.1, 2.2 and (2.2). This gives

$$
\begin{aligned}
I_2 + I_4 &= -\int_{\mathbb{R}^3}\partial_k^2 B_i\partial_i B_j u_j\,dx - 2\int_{\mathbb{R}^3}\partial_k B_i\partial_i\partial_k B_j u_j\,dx \\
&= C\|u\|_{\dot{M}_{p,q}}\|\nabla B\Delta B\|_{\dot{N}_{p,q}} \\
&\le C\|u\|_{\dot{M}_{p,q}}\|\nabla B\|_{\dot{H}^\alpha}\|\Delta B\|_{L^m} \\
&\le C\|u\|_{\dot{M}_{p,q}}\|\wedge^{1+\alpha}B\|_{L^2}\|\nabla B\|_{L^2}^{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}B\|_{L^2}^{\frac{5}{2\alpha}-\frac{3}{m\alpha}} \\
&\le C\|u\|_{\dot{M}_{p,q}}\|\nabla B\|_{L^2}^{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}B\|_{L^2}^{1+\frac{5}{2\alpha}-\frac{3}{m\alpha}} \\
&\le \frac{1}{8}\|\wedge^{1+\alpha}B\|_{L^2}^2 + C\|u\|_{\dot{M}_{p,q}}^r\|\nabla B\|_{L^2}^2,
\end{aligned} \tag{3.5}
$$

where $r = \frac{2}{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}$.

Similarly, we obtain

$$
I_3 \le \frac{1}{8}\|\wedge^{1+\alpha}B\|_{L^2}^2 + C\|u\|_{\dot{M}_{p,q}}^r\|\nabla B\|_{L^2}^2, \tag{3.6}
$$

where $r = \frac{2}{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}$.

Substituting (3.4)-(3.6) into (3.3) yields

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2\right) + \left(\|\wedge^{1+\alpha}u(t)\|_{L^2}^2 + \|\wedge^{1+\alpha}B(t)\|_{L^2}^2\right) \\
&\quad\le C\|u\|_{\dot{M}_{p,q}}^r\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) \\
&\quad\le C\frac{\|u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) \\
&\quad\le C\frac{\|u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\|u\|_{L^\infty}\right)\right] \\
&\quad\le C\frac{\|u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\|u\|_{H^3}\right)\right] \\
&\quad\le C\frac{\|u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\|u\|_{H^3}^2\right)\right] \\
&\quad\le C\frac{\|u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 B\|_{L^2}^2\right)\right].
\end{aligned} \tag{3.7}
$$

Owing to (1.3), we know that for any small constant $\varepsilon > 0$, there exists $T_0 < T$ such that

$$
\int_{T_0}^t \frac{\|u(\tau)\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|u\|_{L^\infty})}\,d\tau < \varepsilon \ll 1.
$$

For any $T_0 \le t < T$, let

$$\Theta(t) = \sup_{T_0 \le \tau \le t} \left(\left\| \nabla^3 u(\tau) \right\|_{L^2}^2 + \left\| \nabla^3 B(\tau) \right\|_{L^2}^2 \right).$$

By (3.7), we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\left\| \nabla u(t) \right\|_{L^2}^2 + \left\| \nabla B(t) \right\|_{L^2}^2 \right) + \left(\left\| \wedge^{1+\alpha} u(t) \right\|_{L^2}^2 + \left\| \nabla^{1+\alpha} B(t) \right\|_{L^2}^2 \right) \\ &\quad \le C \frac{\|u\|_{\dot{M}_{p,q}}^r}{1 + \ln(e + \|u\|_{L^\infty})} \left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 \right) \left[1 + \ln\left(e + \Theta(t)\right) \right]. \end{aligned} \tag{3.8}$$

It follows from (3.8) and Gronwall's inequality that

$$\begin{aligned} &\left\| \nabla u(t) \right\|_{L^2}^2 + \left\| \nabla B(t) \right\|_{L^2}^2 + \int_{T_0}^t \left(\left\| \wedge^{1+\alpha} u(\tau) \right\|_{L^2}^2 + \left\| \wedge^{1+\alpha} B(\tau) \right\|_{L^2}^2 \right) d\tau \\ &\quad \le \left(\left\| \nabla u(T_0) \right\|_{L^2}^2 + \left\| \nabla B(T_0) \right\|_{L^2}^2 \right) \exp\left\{ C\left(1 + \ln\left(e + \Theta(t)\right)\right) \int_{T_0}^t \frac{\|u(\tau)\|_{\dot{M}_{p,q}}^r}{1 + \ln(e + \|u\|_{L^\infty})} \, d\tau \right\} \\ &\quad \le C_0 \exp\left\{ C\varepsilon\left[1 + \ln\left(e + \Theta(t)\right) \right] \right\} \\ &\quad \le C_0 \exp\left\{ 2C\varepsilon\left[\ln\left(e + \Theta(t)\right) \right] \right\} \\ &\quad \le C_0 \left(e + \Theta(t) \right)^{2C\varepsilon}, \end{aligned} \tag{3.9}$$

where $C_0 = \|\nabla u(T_0)\|_{L^2}^2 + \|\nabla B(T_0)\|_{L^2}^2$.

Applying ∇^m to the first equation of (1.1), then taking L^2 inner product of the resulting equation with $\nabla^m u$ and using integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\| \nabla^m u(t) \right\|_{L^2}^2 + \left\| \wedge^{m+\alpha} u(t) \right\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^3} \nabla^m (u \cdot \nabla u) \nabla^m u \, dx + \int_{\mathbb{R}^3} \nabla^m (B \cdot \nabla B) \nabla^m u \, dx. \end{aligned} \tag{3.10}$$

Similarly, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\| \nabla^m B(t) \right\|_{L^2}^2 + \left\| \wedge^{m+\alpha} B(t) \right\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^3} \nabla^m (u \cdot \nabla B) \nabla^m B \, dx + \int_{\mathbb{R}^3} \nabla^m (B \cdot \nabla u) \nabla^m B \, dx. \end{aligned} \tag{3.11}$$

Combining (3.10)-(3.11), using $\nabla \cdot u = 0$, $\nabla \cdot B = 0$ and integration by parts yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\| \nabla^m u(t) \right\|_{L^2}^2 + \left\| \nabla^m B(t) \right\|_{L^2}^2 \right) + \left\| \wedge^{m+\alpha} u(t) \right\|_{L^2}^2 + \left\| \wedge^{m+\alpha} B(t) \right\|_{L^2}^2 \\ &\quad = -\int_{\mathbb{R}^3} \left[\nabla^m (u \cdot \nabla u) - u\nabla \cdot \nabla\nabla^m u \right] \nabla^m u \, dx + \int_{\mathbb{R}^3} \left[\nabla^m (B \cdot \nabla B) - B \cdot \nabla\nabla^m B \right] \nabla^m u \, dx \\ &\quad = -\int_{\mathbb{R}^3} \left[\nabla^m (u \cdot \nabla B) - u\nabla \cdot \nabla\nabla^m B \right] \nabla^m B \, dx + \int_{\mathbb{R}^3} \left[\nabla^m (B \cdot \nabla u) - B \cdot \nabla\nabla^m u \right] \nabla^m B \, dx \\ &\quad \triangleq J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.12}$$

In what follows, for simplicity, we set $m = 3$.

Using Hölder's inequality and (2.1), (2.2), we have

$$\begin{aligned} J_1 &\le \left\|\nabla^3(u\cdot\nabla u) - u\cdot\nabla\nabla^3 u\right\|_{L^{\frac{3}{2}}} \left\|\nabla^3 u\right\|_{L^3} \\ &\le C\|\nabla u\|_{L^3}\left\|\nabla^3 u\right\|_{L^3}^2 \\ &\le C\|\nabla u\|_{L^2}^{\frac{6\alpha+1}{2(2+\alpha)}}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^{\frac{11}{2(2+\alpha)}} \\ &\le \frac{1}{8}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{2(6\alpha+1)}{4\alpha-3}} \\ &\le \frac{1}{8}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^2 + C\big(e+\Theta(t)\big)^{\frac{2(6\alpha+1)}{4\alpha-3}C\varepsilon}. \end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned} J_2 &\le \left\|\nabla^3(B\cdot\nabla B) - B\cdot\nabla\nabla^3 B\right\|_{L^{\frac{3}{2}}} \left\|\nabla^3 u\right\|_{L^3} \\ &\le C\|\nabla B\|_{L^3}\left\|\nabla^3 B\right\|_{L^3}\left\|\nabla^3 u\right\|_{L^3} \\ &\le C\|\nabla B\|_{L^2}^{\frac{4\alpha+2}{2(2+\alpha)}}\left\|\wedge^{3+\alpha}B\right\|_{L^2}^{\frac{6}{2(2+\alpha)}}\|\nabla u\|_{L^2}^{\frac{2\alpha-1}{2(2+\alpha)}}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^{\frac{5}{2(2+\alpha)}} \\ &\le \frac{1}{8}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^2 + C\|\nabla B\|_{L^2}^{\frac{(8\alpha+4)}{4\alpha+3}}\left\|\wedge^{3+\alpha}B\right\|_{L^2}^{\frac{12}{4\alpha+3}}\|\nabla u\|_{L^2}^{\frac{2(2\alpha-1)}{4\alpha+3}} \\ &\le \frac{1}{8}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^2 + \frac{1}{6}\left\|\wedge^{3+\alpha}B\right\|_{L^2}^2 + C\|\nabla B\|_{L^2}^{\frac{8\alpha+4}{4\alpha-3}}\|\nabla u\|_{L^2}^{\frac{2(2\alpha-1)}{4\alpha-3}} \\ &\le \frac{1}{8}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^2 + \frac{1}{6}\left\|\wedge^{3+\alpha}B\right\|_{L^2}^2 + C\big(e+\Theta(t)\big)^{\frac{2(6\alpha+1)}{4\alpha-3}C\varepsilon}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} J_3 &\le \left\|\nabla^3(u\cdot\nabla B) - u\cdot\nabla\nabla^3 B\right\|_{L^{\frac{3}{2}}} \left\|\nabla^3 B\right\|_{L^3} \\ &\le C\|\nabla u\|_{L^3}\left\|\nabla^3 B\right\|_{L^3}^2 + C\|\nabla B\|_{L^3}\left\|\nabla^3 B\right\|_{L^3}\left\|\nabla^3 u\right\|_{L^3} \\ &\le \frac{1}{8}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^2 + \frac{1}{6}\left\|\wedge^{3+\alpha}B\right\|_{L^2}^2 + C\big(e+\Theta(t)\big)^{\frac{2(6\alpha+1)}{4\alpha-3}C\varepsilon}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} J_4 &\le \left\|\nabla^3(B\cdot\nabla u) - B\cdot\nabla\nabla^3 u\right\|_{L^{\frac{3}{2}}} \left\|\nabla^3 B\right\|_{L^3} \\ &\le C\|\nabla u\|_{L^3}\left\|\nabla^3 B\right\|_{L^3}^2 + C\|\nabla B\|_{L^3}\left\|\nabla^3 B\right\|_{L^3}\left\|\nabla^3 u\right\|_{L^3} \\ &\le \frac{1}{8}\left\|\wedge^{3+\alpha}u\right\|_{L^2}^2 + \frac{1}{6}\left\|\wedge^{3+\alpha}B\right\|_{L^2}^2 + C\big(e+\Theta(t)\big)^{\frac{2(6\alpha+1)}{4\alpha-3}C\varepsilon}. \end{aligned} \tag{3.16}$$

Inserting (3.13)-(3.16) into (3.12) yields

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\nabla^3 u(t)\right\|_{L^2}^2 + \left\|\nabla^3 B(t)\right\|_{L^2}^2\right) + \left\|\wedge^{3+\alpha}u(t)\right\|_{L^2}^2 + \left\|\wedge^{3+\alpha}B(t)\right\|_{L^2}^2 \\ &\quad\le C\big(e+\Theta(t)\big)^{\frac{2(6\alpha+1)}{4\alpha-3}C\varepsilon}. \end{aligned}$$

Gronwall's inequality implies the boundedness of H^3-norm of u and B provided that $\frac{2(6\alpha+1)}{4\alpha-3}C\varepsilon \le 1$, which can be achieved by the absolute continuous property of integral (1.3). We have completed the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Proof Let $\wedge = (-\Delta)^{\frac{1}{2}}$. Multiplying the first equation of (1.1) by $-\Delta u$ and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\wedge^{1+\alpha}u(t)\|_{L^2}^2 = \int_{\mathbb{R}^3} u\cdot\nabla u\Delta u\,dx - \int_{\mathbb{R}^3} B\cdot\nabla B\Delta u\,dx. \tag{4.1}$$

Similarly, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla B(t)\|_{L^2}^2 + \|\wedge^{1+\alpha}B(t)\|_{L^2}^2 = \int_{\mathbb{R}^3} u\cdot\nabla B\Delta B\,dx - \int_{\mathbb{R}^3} B\cdot\nabla u\Delta B\,dx. \tag{4.2}$$

Summing up (4.1) and (4.2), we deduce that

$$\begin{aligned}
&\frac{d}{dt}\big(\|\nabla u(t)\|_{L^2}^2 + \|\nabla B(t)\|_{L^2}^2\big) + \big(\|\wedge^{1+\alpha}u(t)\|_{L^2}^2 + \|\wedge^{1+\alpha}B(t)\|_{L^2}^2\big)\\
&\quad= \int_{\mathbb{R}^3} u\cdot\nabla u\Delta u\,dx - \int_{\mathbb{R}^3} B\cdot\nabla B\Delta u\,dx + \int_{\mathbb{R}^3} u\cdot\nabla B\Delta B\,dx - \int_{\mathbb{R}^3} B\cdot\nabla u\Delta B\,dx\\
&\quad= \int_{\mathbb{R}^3} \nabla u\cdot\nabla u\nabla u\,dx - \int_{\mathbb{R}^3} \nabla B\cdot\nabla B\nabla u\,dx + \int_{\mathbb{R}^3} \nabla u\cdot\nabla B\nabla B\,dx - \int_{\mathbb{R}^3} \nabla B\cdot\nabla u\nabla B\,dx\\
&\quad\triangleq I_1' + I_2' + I_3' + I_4'.
\end{aligned} \tag{4.3}$$

Using Lemmas 2.1, 2.2 and (2.2), we obtain

$$\begin{aligned}
I_1' &\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla u\nabla u\|_{\dot{N}_{p,q}}\\
&\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla u\|_{\dot{H}^{\alpha}}\|\nabla u\|_{L^m}\\
&\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\wedge^{1+\alpha}u\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{3}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}u\|_{L^2}^{\frac{3}{2\alpha}-\frac{3}{m\alpha}}\\
&\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla u\|_{L^2}^{1-\frac{3}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}u\|_{L^2}^{1+\frac{3}{2\alpha}-\frac{3}{m\alpha}}\\
&\le \frac{1}{8}\|\wedge^{1+\alpha}u\|_{L^2}^2 + C\|u\|_{\dot{M}_{p,q}}^r\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{4.4}$$

where $r = \frac{2}{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}$.

Similarly, we obtain

$$\begin{aligned}
I_2' &\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla B\nabla B\|_{\dot{N}_{p,q}}\\
&\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla B\|_{\dot{H}^{\alpha}}\|\nabla B\|_{L^m}\\
&\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\wedge^{1+\alpha}B\|_{L^2}\|\nabla B\|_{L^2}^{1-\frac{3}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}B\|_{L^2}^{\frac{3}{2\alpha}-\frac{3}{m\alpha}}\\
&\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla B\|_{L^2}^{1-\frac{3}{2\alpha}+\frac{3}{m\alpha}}\|\wedge^{1+\alpha}B\|_{L^2}^{1+\frac{3}{2\alpha}-\frac{3}{m\alpha}}\\
&\le \frac{1}{8}\|\wedge^{1+\alpha}B\|_{L^2}^2 + C\|\nabla u\|_{\dot{M}_{p,q}}^r\|\nabla B\|_{L^2}^2,
\end{aligned} \tag{4.5}$$

where $r = \frac{2}{1-\frac{5}{2\alpha}+\frac{3}{m\alpha}}$.

$$I_3' \le \frac{1}{8}\|\wedge^{1+\alpha}B\|_{L^2}^2 + C\|\nabla u\|_{\dot{M}_{p,q}}^r\|\nabla B\|_{L^2}^2 \tag{4.6}$$

and

$$I_4' \leq \frac{1}{8}\left\|\wedge^{1+\alpha}B\right\|_{L^2}^2 + C\|\nabla u\|_{\dot{M}_{p,q}}^r\|\nabla B\|_{L^2}^2. \tag{4.7}$$

Combining (4.3)-(4.7) yields

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla u(t)\right\|_{L^2}^2 + \left\|\nabla B(t)\right\|_{L^2}^2\right) + \left(\left\|\wedge^{1+\alpha}u(t)\right\|_{L^2}^2 + \left\|\wedge^{1+\alpha}B(t)\right\|_{L^2}^2\right) \\
&\quad\leq C\|\nabla u\|_{\dot{M}_{p,q}}^r\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) \\
&\quad\leq C\frac{\|\nabla u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) \\
&\quad\leq C\frac{\|\nabla u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\|\nabla u\|_{L^\infty}\right)\right] \\
&\quad\leq C\frac{\|\nabla u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\|u\|_{H^3}\right)\right] \\
&\quad\leq C\frac{\|\nabla u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\|u\|_{H^3}^2\right)\right] \\
&\quad\leq C\frac{\|\nabla u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) \\
&\qquad\times\left[1+\ln\left(e+\left\|\nabla^3 u\right\|_{L^2}^2 + \left\|\nabla^3 B\right\|_{L^2}^2\right)\right].
\end{aligned} \tag{4.8}$$

Thanks to (1.5), we know that for any small constant $\varepsilon > 0$, there exists $T_0 < T$ such that

$$\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\,d\tau < \varepsilon \ll 1.$$

For any $T_0 \leq t < T$, set

$$\Theta(t) = \sup_{T_0\leq\tau\leq t}\left(\left\|\nabla^3 u(\tau)\right\|_{L^2}^2 + \left\|\nabla^3 B(\tau)\right\|_{L^2}^2\right). \tag{4.9}$$

By (4.8) and (4.9), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla u(t)\right\|_{L^2}^2 + \left\|\nabla B(t)\right\|_{L^2}^2\right) + \left(\left\|\wedge^{1+\alpha}u(t)\right\|_{L^2}^2 + \left\|\wedge^{1+\alpha}B(t)\right\|_{L^2}^2\right) \\
&\quad\leq C\frac{\|\nabla u\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\left(\|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right)\left[1+\ln\left(e+\Theta(t)\right)\right].
\end{aligned} \tag{4.10}$$

Equation (4.10) and Gronwall's inequality give the estimate

$$\begin{aligned}
&\left\|\nabla u(t)\right\|_{L^2}^2 + \left\|\nabla B(t)\right\|_{L^2}^2 + \int_{T_0}^t\left(\left\|\wedge^{1+\alpha}u(\tau)\right\|_{L^2}^2 + \left\|\wedge^{1+\alpha}B(\tau)\right\|_{L^2}^2\right)d\tau \\
&\quad\leq \left(\left\|\nabla u(T_0)\right\|_{L^2}^2 + \left\|\nabla B(T_0)\right\|_{L^2}^2\right) \\
&\qquad\times\exp\left\{C\left(1+\ln\left(e+\Theta(t)\right)\right)\int_{T_0}^t \frac{\|\nabla u(\tau)\|_{\dot{M}_{p,q}}^r}{1+\ln(e+\|\nabla u\|_{L^\infty})}\,d\tau\right\}
\end{aligned}$$

$$\begin{aligned}&\le C_0\exp\bigl\{C\varepsilon\bigl[1+\ln\bigl(e+\Theta(t)\bigr)\bigr]\bigr\}\\&\le C_0\exp\bigl\{2C\varepsilon\bigl[\ln\bigl(e+\Theta(t)\bigr)\bigr]\bigr\}\\&\le C_0\bigl(e+\Theta(t)\bigr)^{2C\varepsilon},\end{aligned}\tag{4.11}$$

where $C_0 = \|\nabla u(T_0)\|_{L^2}^2 + \|\nabla B(T_0)\|_{L^2}^2$.

From (4.11), H^3 estimate for this case is the same as that for Theorem 1.1. Thus, Theorem 1.2 is proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LH and YW carried out the proof of the main part of this article. All authors have read and approved the final manuscript.

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Received: 22 April 2013 Accepted: 5 August 2013 Published: 21 August 2013

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doi:10.1186/1687-2770-2013-187
Cite this article as: Hu and Wang: Blow-up criteria for smooth solutions to the generalized 3D MHD equations. *Boundary Value Problems* 2013 **2013**:187.